Positive Solutions for a Singular Three-Point Boundary Value Problem of Second-Order Dynamic Equation on Time Scales

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Abstract—**This paper is concerned with a singular three-point boundary value problem of second-order dynamic equation on time scales of the following form:**

$$\begin{cases} u^{\Delta\Delta}(x)+\eta q(x)z(x,u(x))=0, x\in(0,1)_{\mathbb{T}}, \\ u(0)=\alpha u^{\Delta}(0), u(1)=\beta u(\zeta), \end{cases}$$

where $\eta>0$ is a parameter, $\alpha>0, 0<\zeta<1, 0<\beta\zeta<1$ and $(1-\beta\zeta)+\alpha(1-\beta)>0$. Applying the fixed point index theory, sufficient conditions for the existence of at least one or two positive solutions of the problem are established. The interesting point of the obtained results is that $q(x)$ may be singular at $x=0$ and/or 1, $z(x,u)$ may be singular at $u=0$.

Index Terms—**Positive solution; Singular boundary value problem; Second-order dynamic equation; Time scale.**

I. Introduction

In the past few decades, boundary value problems have been widely studied by many scholars, because boundary value problems can describe many dynamic phenomena in nature and society; see, for example [1-12].

Consider the following second-order boundary value problem

$$\begin{cases} u''(x)=g(x,u(x),u'(x))+f(x), x\in(0,1), \\ u'(0)=0, u(1)=\sum_{j=1}^{n-2}a_jb_j. \end{cases} \tag{1}$$

In [5], Feng studied (1), where g is defined on $[0,1]\times\mathbb{R}\times\mathbb{R}$, and g is continuous and satisfies the nonlinear growth; $a_j\in\mathbb{R}, j=1,2,\cdots,n-2$ are constants and have the same sign, $b_j\in(0,1), j=1,2,\cdots,n-2$.

In [6], Gupta also studied (1), where $a_j>0, j=1,2,\cdots,n-2$ are positive constants, $b_j\in(0,1), j=1,2,\cdots,n-2$, and $0<b_1<b_2<\cdots<b_{n-2}<1$; $|g(x,u_1,u_2)|\le a_1(t)|_1|+a_2(t)|u_2|+a_3(t)$, and $\kappa_1\|a_1\|_1+\kappa_2\|a_2\|_1\le 1$, where $a_i(t)\in L^1(0,1), i=1,2,3$, κ_1 and κ_2 are constants.

From the above works, we can see that the nonlinear term g satisfies the monotone and growth conditions, but the conditions are more stronger. So, the first aim of this paper is to study a boundary value problem under more general conditions.

On the other hand, the theory of dynamic equations on time scales has been developed rapidly in recent years. This is because the dynamic equations on time scales can not only accurately describe the dynamic processes of many systems in the real world, but also obtain some new qualitative phenomena of the systems. So, the second aim of this paper is to study a boundary value problem on time scales in order to obtain more general results.

In this paper, we shall study a singular three-point boundary value problem of second-order dynamic equation on time scales of the following form:

$$\begin{cases} u^{\Delta\Delta}(x)+\eta q(x)z(x,u(x))=0, x\in(0,1)_{\mathbb{T}}, \\ u(0)=\alpha u^{\Delta}(0), u(1)=\beta u(\zeta), \end{cases} \tag{2}$$

where $\eta>0$ is a parameter, $\alpha>0, 0<\zeta<1, 0<\beta\zeta<1$ and $(1-\beta\zeta)+\alpha(1-\beta)>0$; $q\in C((0,1)_{\mathbb{T}},(0,+\infty))$, $q(x)$ may be singular at $x=0$ and/or 1; $z\in C([0,1]_{\mathbb{T}}\times(0,+\infty),(0,+\infty))$, $z(x,u)$ may be singular at $u=0$.

Applying the fixed point index theory, sufficient conditions for the existence of at least one or two positive solutions of (2) will be established. Throughout of this paper, $[x_1,x_2]_{\mathbb{T}}$ and $(x_1,x_2)_{\mathbb{T}}$ denote $[x_1,x_2]\cap\mathbb{T}$ and $(x_1,x_2)\cap\mathbb{T}$, respectively.

II. Preliminaries

A comprehensive review on the basic theory of calculus on time scales, see [13].

Let $\mathbb{X}=C[0,1]_{\mathbb{T}}$ is a Banach space with the norm $\|u\|=\sup\limits_{x\in[0,1]_{\mathbb{T}}}|u(x)|$, $\mathbb{A}$ is a positive cone in $C[0,1]_{\mathbb{T}}$, and

$$\mathbb{A}=\{u\in\mathbb{X}: u(x)\ge 0, x\in[0,1]_{\mathbb{T}}\}.$$

Let

$$\begin{aligned}\mathbb{B}=\{u\in\mathbb{A}: u(x) \text{ is a concave function}, \\ x\in[0,1]_{\mathbb{T}}, \inf_{x\in[\zeta,1]_{\mathbb{T}}}u(x)\ge\delta_0\|u\|\},\end{aligned} \tag{3}$$

where $\delta_0=\min\{\zeta,\beta\zeta,\frac{\beta(1-\zeta)}{1-\beta\zeta}\}$.

Let r, R are two positive constants, and $0<r<R<+\infty$. Define

$$\begin{aligned}\mathbb{B}_r&=\{u\in\mathbb{B}:\|u\|<r\},\\ \partial\mathbb{B}_r&=\{u\in\mathbb{B}:\|u\|=r\},\\ \bar{\mathbb{B}}_{r,R}&=\{u\in\mathbb{B}:r\le\|u\|\le R\}.\end{aligned}$$

We first make the following assumptions:

(H_1) $\alpha>0$, $0<\zeta<1$, $0<\beta<\frac{1+\alpha}{\zeta+\alpha}(\le\frac{1}{\zeta})$, and $\Gamma=(1-\beta\zeta)+\alpha(1-\beta)>0$;

(H_2) $q(x)\not\equiv 0, \forall x\in(0,1)_{\mathbb{T}}$, and

$$0<\int_0^1 q(y)(y+\alpha)(1-y)\Delta y<+\infty;$$

Manuscript received January 21, 2023; revised May 12, 2023.

This work is supported by the Key Scientific Research Project of Colleges and Universities of Henan Province (No.21A110001).

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(H_3) Let $E(i)=[0,\frac{1}{i}]_{\mathbb{T}}\cup[\frac{i-1}{i},1]_{\mathbb{T}}$, and

$$\lim_{i\to+\infty}\sup_{u\in\bar{\mathbb{B}}_{r,R}}\int_{E[i]}(y+\alpha)(1-y)q(y)z(y,u(y))\Delta y=0.$$

Lemma 1. *Assume that* (H_1) *holds. Furthermore, if*

(H_4) $u\in C((0,1)_{\mathbb{T}},[0,+\infty))$*, and*

$$0<\int_0^1(\alpha+y)(1-y)u(y)\Delta y<+\infty;$$

holds, then the following boundary value problem

$$\begin{cases} u^{\Delta\Delta}(x)+v(x)=0, x\in[0,1]_{\mathbb{T}},\\ u(0)=\alpha u^{\Delta}(0), u(1)=\beta u(\zeta),\end{cases}\tag{4}$$

has a unique solution

$$u(x)=\int_0^1 G(x,y)v(y)\Delta y,\tag{5}$$

where $G(x,y):[0,1]_{\mathbb{T}}\times[0,1]_{\mathbb{T}}\to[0,+\infty)$*, and*

$$G(x,y)=\begin{cases}\frac{1}{\Gamma}(\sigma(y)+\alpha)((1-x)+\beta(x-\zeta)),\\ \qquad 0\le\sigma(y)\le x\le1, 0\le\sigma(y)\le\zeta<1;\\ \frac{1}{\Gamma}(\sigma(y)+\alpha)((1-x)+\beta(x-\sigma(y))(\zeta+\alpha)),\\ \qquad 0<\zeta\le\sigma(y)\le x\le1;\\ \frac{1}{\Gamma}(x+\alpha)((1-\sigma(y))+\beta(\sigma(y)-\zeta)),\\ \qquad 0\le x\le\sigma(y)\le\zeta<1;\\ \frac{1}{\Gamma}(x+\alpha)(1-\sigma(y)), 0\le x\le\sigma(y)\le1,\\ \qquad 0<\zeta\le\sigma(y)\le1.\end{cases}\tag{6}$$

Proof: Integrating the equation in (4), we have

$$u^{\Delta}(x)=-\int_0^x v(y)\Delta y+u^{\Delta}(0).$$

Since

$$\begin{aligned}&\int_0^x\left(\int_0^t v(y)\Delta y\right)\Delta t\\ =\;& t\int_0^t v(y)\Delta y\Big|_0^x-\int_0^x\sigma(t)v(t)\Delta t\\ =\;& x\int_0^x v(y)\Delta y-\int_0^x\sigma(y)v(y)\Delta y\\ =\;&\int_0^x(x-\sigma(y))v(y)\Delta y,\end{aligned}$$

then

$$\begin{aligned}&u(x)\\ =\;&-\int_0^x(x-\sigma(y))v(y)\Delta y+u^{\Delta}(0)x+u(0)\\ =\;&-\int_0^x(x-\sigma(y))v(y)\Delta y+(x+\alpha)u^{\Delta}(0).\end{aligned}\tag{7}$$

Take $x=1$ in (7), by (4), then

$$\begin{aligned}&u^{\Delta}(0)\\ =\;&\frac{1}{(1-\beta\zeta)+\alpha(1-\beta)}\int_0^1(1-\sigma(y))v(y)\Delta y\\ &-\frac{\beta}{(1-\beta\zeta)+\alpha(1-\beta)}\int_0^{\zeta}(\zeta-\sigma(y))v(y)\Delta y,\end{aligned}$$

and then

$$\begin{aligned}&u(x)\\ =\;&-\int_0^x(x-\sigma(y))v(y)\Delta y\\ &+\frac{x+\alpha}{(1-\beta\zeta)+\alpha(1-\beta)}\int_0^1(1-\sigma(y))v(y)\Delta y\\ &-\frac{\beta(x+\alpha)}{(1-\beta\zeta)+\alpha(1-\beta)}\int_0^{\zeta}(\zeta-\sigma(y))v(y)\Delta y.\end{aligned}$$

If $x\le\zeta$,

$$\begin{aligned}&u(x)\\ =\;&\int_0^x\frac{(\sigma(y)+\alpha)((1-x)+\beta(x-\zeta))}{(1-\beta\zeta)+\alpha(1-\beta)}v(y)\Delta y\\ &+\int_t^{\zeta}\frac{(x+\alpha)((1-\sigma(y))+\beta(\sigma(y)-\zeta))}{(1-\beta\zeta)+\alpha(1-\beta)}\\ &\times v(y)\Delta y\\ &+\int_{\zeta}^1\frac{(x+\alpha)(1-\sigma(y))}{(1-\beta\zeta)+\alpha(1-\beta)}v(y)\Delta y\\ =\;&\int_0^1 G(x,y)u(y)\Delta y.\end{aligned}$$

If $x\ge\zeta$,

$$\begin{aligned}&u(x)\\ =\;&\int_0^{\zeta}\frac{(\sigma(y)+\alpha)((1-x)+\beta(x-\zeta))}{(1-\beta\zeta)+\alpha(1-\beta)}v(y)\Delta y\\ &+\int_{\zeta}^x\frac{(\sigma(y)+\alpha)(1-x)+\beta(x-\sigma(y))(\zeta+\alpha)}{(1-\beta\zeta)+\alpha(1-\beta)}\\ &\times v(y)\Delta y\\ &+\int_t^1\frac{(x+\alpha)(1-\sigma(y))}{(1-\beta\zeta)+\alpha(1-\beta)}v(y)\Delta y\\ =\;&\int_0^1 G(x,y)u(y)\Delta y.\end{aligned}$$

The proof is completed.

Lemma 2. *Assume that* (H_1) *and* (H_4) *hold, then* $G(x,y)$ *satisfies*

(i) $G(x,y)$ *is continuous on* $[0,1]_{\mathbb{T}}\times[0,1]_{\mathbb{T}}$*;*

(ii) $G(x,y)\ge0$, $\forall x,y\in[0,1]_{\mathbb{T}}$*;*

(iii) $k_1(x)G(y,y)\le G(x,y)\le k_2(\sigma(y)+\alpha)(1-\sigma(y))$, $\forall(x,y)\in[0,1]_{\mathbb{T}}\times[0,1]_{\mathbb{T}}$*, where*

$$k_1(x)=\min\{1,\beta(1-\zeta),x,1-x\},$$
$$k_2=\frac{\max\left\{1+\beta,\frac{\beta(1-\zeta)}{1-\beta\zeta}\right\}}{(1-\beta\zeta)+\alpha(1-\beta)}.$$

Let

$$l=\eta\int_{\zeta}^1 k_1(y)G(y,y)q(y)\Delta y,$$
$$L=\eta k_2\int_0^1(\sigma(y)+\alpha)(1-\sigma(y))q(y)\Delta y.$$

Remark 1. *By* (H_2)*, we have*

$$0<k_2\int_0^1(\sigma(y)+\alpha)(1-\sigma(y))q(y)\Delta y<+\infty,$$

and then

$$\begin{aligned} 0 &< \min_{x\in[\zeta,1]_{\mathbb{T}}} k_1(x)\int_\zeta^1 G(x,y)q(y)\Delta y \\ &\le \min_{x\in[\zeta,1]_{\mathbb{T}}} k_1(x)\int_0^1 G(x,y)q(y)\Delta y < +\infty. \end{aligned}$$

Define $\Phi : \mathbb{B}\setminus\{0\} \to \mathbb{B}$, and

$$(\Phi u)(x) = \eta\int_0^1 G(x,y)q(y)z(y,u(y))\Delta y, x\in[0,1]_{\mathbb{T}}. \quad (8)$$

Lemma 3. *Assume that (H_1)-(H_3) hold, then $\Phi : \bar{\mathbb{B}}_{r,R} \to \mathbb{B}$ is completely continuous, and the positive fixed point u of Φ is a positive solution of (2).*

Proof: $(\Phi u)(x)$ is a nonnegative concave function, by the properties of concave function and the Ascoli-Arzela theorem, it is easy to show that $\Phi : \bar{\mathbb{B}}_{r,R} \to \mathbb{B}$ is completely continuous. Furthermore, one can see that if Φ exists a fixed point $u^* \neq 0$, then $u^* \neq 0$ is a solution of (2); by the maximum principle, $u(x) > 0, t \in (0,1)_{\mathbb{T}}$, that is, u^* is a positive solution of (2). This completes the proof.

Let

$$(\Psi u)(x) = \int_0^1 G(x,y)q(y)u(y)\Delta y, x\in[0,1]_{\mathbb{T}}. \quad (9)$$

Lemma 4. *Assume that (H_1), (H_2) and (H_4) hold, then $\Psi : \mathbb{A} \to \mathbb{A}$ is completely continuous and $\Psi(\mathbb{A}) \subset \mathbb{A}$; the spectral radius $r(\Psi) \neq 0$, and $\omega = \lambda_1\Psi\omega$, $\lambda_1 = (r(\Psi))^{-1}$, $\omega > 0$ is the eigenfunction.*

Proof: From Lemma 3, $\Psi : \mathbb{A} \to \mathbb{A}$ is completely continuous, and $\Psi(\mathbb{A}) \subset \mathbb{A}$. From $(H_1)-(H_2)$, there exists a constant $y_0 \in (0,1)_{\mathbb{T}}$ and $G(y_0,y_0)q(y_0) > 0$. Choose $a_1, a_2 \in [0,1]_{\mathbb{T}}$, and $y_0 \in (a_1,a_2) \subset [a_1,a_2] \subset (0,1)_{\mathbb{T}}$, and $G(x,y)q(y) > 0$, $\forall x,y \in [a_1,a_2]$. Let $g \in C[0,1]_{\mathbb{T}}$ and $g(x) > 0, \forall x \in (a_1,a_2)$, then

$$\begin{aligned} (\Psi g)(x) &= \int_0^1 G(x,y)q(y)g(y)\Delta y \\ &\ge \int_{a_1}^{a_2} G(x,y)q(y)g(y)\Delta y > 0, \end{aligned}$$

and then there exists a positive constant $a_3 > 0$ and $a_3(\Psi g)(x) \ge g(x), \forall x \in [0,1]_{\mathbb{T}}$. By the Krein-Rutmann theorem, Lemma 4 holds. The proof is completed.

The following lemmas, see [14,15].

Let $\mathbb{X}$ is a Banach space, $\mathbb{A} \subset \mathbb{X}$ and $\mathbb{B} \subset \mathbb{X}$ are cones, $D_0(\mathbb{B}) \subset \mathbb{B}$ is a bounded open set, the operator $\Phi : \bar{D}_0(\mathbb{B}) \to \mathbb{B}$ is completely continuous.

Lemma 5. ([14]) *If $\Phi u \neq bu, \forall u \in \partial D_0(\mathbb{B})$. Assume that $\psi,\phi,\varphi : \mathbb{X} \to \mathbb{X}$, and $\psi(\mathbb{B}) \subset \mathbb{B}, \phi(\mathbb{B}) \subset \mathbb{A}, \varphi(\mathbb{A}) \subset \mathbb{B}$, for $u_0 \in \mathbb{B}\setminus\{\theta\}$, and*

(i) $\phi\psi^n u_0 \ge \phi u_0$, $n = 1,2,3,\ldots$;
(ii) $\phi\psi u = \varphi\phi u, \forall u \in \partial D_0(\mathbb{B})$;
(iii) $\phi\Phi u \ge \phi\psi u, \forall u \in \partial D_0(\mathbb{B})$;
then $i(\Phi, D_0(\mathbb{B}), \mathbb{B}) = 0$.

Lemma 6. ([15]) *If $\Phi u \neq bu, \forall u \in \partial D_0(\mathbb{B}), b \ge 1$, then $i(\Phi, D_0(\mathbb{B}), \mathbb{B}) = 1$.*

Lemma 7. ([15]) *The operator Φ satisfies*

(i) If $\|\Phi u\| > \|u\|, \forall u \in \partial D_0(\mathbb{B})$, then $i(\Phi, D_0(\mathbb{B}), \mathbb{B}) = 0$;
(ii) If $\theta \in D_0(\mathbb{B})$ and $\|\Phi u\| < \|u\|, \forall u \in \partial D_0(\mathbb{B})$; then $i(\Phi, D_0(\mathbb{B}), \mathbb{B}) = 1$.

Let

$$z_\ell = \liminf_{u\to\ell}\inf_{x\in[0,1]_{\mathbb{T}}}\frac{z(x,u)}{u}, z^\ell = \limsup_{u\to\ell}\sup_{x\in[0,1]_{\mathbb{T}}}\frac{z(x,u)}{u},$$

where ℓ denotes 0 or ∞.

III. Existence of solutions

Theorem 1. *Assume that (H_1)-(H_3) hold, and*

$$0 \le z^\infty < z_0 \le +\infty.$$

If

$$\eta \in \left(\frac{\lambda_1}{z_0}, \frac{\lambda_1}{z^\infty}\right), \quad (10)$$

where λ_1 is the first eigenvalue of Ψ which has been defined by (9), then (2) exists at least one positive solution.

Proof: By (10), there exists positive constants $r > 0$, $R_0 > r$ and $0 < \xi < 1$, and

$$z(x,u) \ge \frac{\lambda_1}{\eta}u, \forall 0 \le u \le r, 0 \le x \le 1, \quad (11)$$

$$z(x,u) \le \frac{\xi}{\eta}\lambda_1 u, \forall u \ge R_0, 0 \le x \le 1. \quad (12)$$

Let

$$(\Psi_1 u)(x) = \xi\lambda_1(\Psi u)(x), \forall x \in [0,1]_{\mathbb{T}}, u \in C[0,1]_{\mathbb{T}},$$

then $\Psi_1 : \mathbb{A} \to \mathbb{A}$ is completely continuous, and $\Psi_1(\mathbb{A}) \subset \mathbb{A}$. By Lemma 4, $r(\Psi_1) \neq 0$. Because of λ_1 is the first eigenvalue of Ψ, and $0 < \xi < 1$, then $0 < r(\Psi_1) < 1$. Take $d_0 = \frac{1}{6}(1 - r(\Psi_1)) > 0$, by the Gelfand's formula,

$$\|\Psi_1^n\| \le (r(\Psi_1) + d_0)^n, \forall n \ge Q, \quad (13)$$

where Q is a natural number.

Let $\Psi_1^0 = I$ is the identity operator, and

$$\|u\|_1 = \sum_{p=1}^{Q}(r(\Psi_1) + d_0)^{Q-p}\|\Psi_1^{p-1}u\|, u \in C[0,1]_{\mathbb{T}}, \quad (14)$$

then

$$\begin{aligned} &(r(\Psi_1) + d_0)^{Q-1}\|u\| \le \|u\|_1 \\ &= \sum_{p=1}^{Q}(r(\Psi_1) + d_0)^{Q-p}\|\Psi_1^{p-1}\|\|u\|. \end{aligned} \quad (15)$$

Let

$$\begin{aligned} W = \sup_{u\in\partial P_{R_0}} \eta k_2 \int_0^1 &(\sigma(y) + \alpha)(1 - \sigma(y)) \\ &\times q(y)z(y,u(y)))\Delta y. \end{aligned} \quad (16)$$

It is easy to show that $W < +\infty$.

Take $R_1 > \max\{R_0, 2\|W\|_1 d_0^{-1}\}$. From (15), there exists a positive constant R, $R > R_1 > 0$, and $\|u\|_1 > R_1, \forall\|u\| > R$. Extend Φ, that is, $\Phi : \bar{\mathbb{B}}_R \to \mathbb{B}$, then Φ is completely continuous. If Φ exists a fixed point on $\partial\mathbb{B}_r$, Theorem 1 holds. Suppose that there is no fixed point on $\partial\mathbb{B}_r$. Let u_1

is a positive eigenfunction of Ψ corresponding to the first eigenvalue λ_1,

$$u_1(x) = \lambda_1((B\Psi u_1)(x) = \lambda_1 \int_0^1 G(x,y)q(y)u_1(y)\Delta y,$$

then

$$\|u_1\| \leq \lambda_1 k_2 \int_0^1 (\sigma(y)+\alpha)(1-\sigma(y))q(y)u_1(y)\Delta y.$$

Because of $u_1^{\Delta\Delta}\lambda_1 q(x)u_1(x) \leq 0, x \in (0,1)_{\mathbb{T}}$, then u_1 is a nonnegative continuous concave function. Let $\|u_1\| = u_1(x_0), x_0 \in (0,1)_{\mathbb{T}}$.

Suppose that $0 < \beta < 1$, then $\min_{\zeta \leq x \leq 1} u_1(x) = u_1(1)$. Since u_1 is concavity, then

$$\begin{aligned}
\|u_1\| &= u_1(x_0) \leq u_1(1) + \frac{u_1(\zeta)-u_1(1)}{\zeta-1}(x_0-1) \\
&\leq \frac{1-\beta\zeta}{\beta(1-\zeta)}u_1(1), \forall 0 \leq x_0 \leq \zeta < 1; \\
\|u_1\| &= u_1(x_0) \leq u_1(0) + \frac{u_1(\zeta)-u_1(0)}{\zeta}x_0 \\
&\leq \frac{u_1(\zeta)}{\zeta} = \frac{1}{\beta\zeta}u_1(1), \forall \zeta \leq x_0 < 1.
\end{aligned}$$

Suppose that $1 \leq \beta < \frac{1+\alpha}{\zeta+\alpha}(\leq \frac{1}{\zeta})$, then $\min_{\zeta \leq x \leq 1} u_1(x) = u_1(\zeta)$, and

$$\|u_1\| = u_1(x_0) \leq \frac{u_1(\zeta)}{\zeta}x_0 < \frac{u_1(\zeta)}{\zeta}.$$

From the above analysis, we have

$$\min_{\zeta \leq x \leq 1} u_1(x) \geq \min\left\{\beta, \beta\zeta, \frac{\beta(1-\zeta)}{1-\beta\zeta}\right\}\|u_1\| = \delta_0\|u_1\|,$$

that is $u_1 \in \mathbb{B} \setminus \{\theta\}$.

Let $(\tilde{\Psi}_1 u)(x) = \lambda_1(\Psi u)(x), u \in \mathbb{A}$, then $\tilde{\Psi}_1 : \mathbb{A} \to \mathbb{A}$ is completely continuous, and $\tilde{\Psi}_1(\mathbb{A}) \subset \mathbb{B}, \tilde{\Psi}_1 u_1 = \lambda_1 \Psi u_1 = u_1$.

By (11), if $u \in \partial\mathbb{B}_r$, then

$$\begin{aligned}
(\Phi u)(x) &= \eta \int_0^1 G(x,y)q(y)z(y,u(y))\Delta y \\
&\geq \eta\frac{\lambda_1}{\eta}\int_0^1 G(x,y)q(y)u(y)\Delta y \\
&= \lambda_1(\Psi u)(x) = (\tilde{\Psi}_1 u)(x), x \in [0,1]_{\mathbb{T}}.
\end{aligned}$$

Let $D_0(v) = \mathbb{B}_r, \psi = \varphi = \tilde{\Psi}_1, \phi = I$ and $n = 1$. By Lemma 5,

$$i(\Phi, \mathbb{B}_r, \mathbb{B}) = 0. \tag{17}$$

Next, we show that

$$\Phi u \neq bu, b \geq 1, \forall u \in \partial\mathbb{B}_R. \tag{18}$$

If not, there exist $u_0 \in \partial\mathbb{B}_R$ and $b_0 \geq 1$, and

$$\Phi u_0 = b_0 u_0. \tag{19}$$

Let $u_0(x) = \min\{u_0(x), R_0\}$, then $\tilde{y}_0 \in \partial\mathbb{B}_{R_0}$. By (12),

$$\begin{aligned}
&(\Phi u_0)(x) \\
&= \eta \int_0^1 G(x,y)q(y)z(y,u_0(y))\Delta y \\
&= \eta \int_{E[u_0>R_0]} G(x,y)q(y)z(y,u_0(y))\Delta y \\
&\quad + \eta \int_{[0,1]_{\mathbb{T}}\setminus E[u_0>R_0]} G(x,y)q(y)z(y,u_0(y))\Delta y \\
&\leq \eta\frac{\xi}{\eta}\lambda_1 \int_{E[u_0>R_0]} G(x,y)q(y)z(y,u_0(y))\Delta y \\
&\quad + \eta \int_{[0,1]_{\mathbb{T}}\setminus E[u_0>R_0]} G(x,y)q(y)z(y,u_0(y))\Delta y \\
&\leq \xi\lambda_1 \int_0^1 G(x,y)q(y)z(y,u_0(y))\Delta y \\
&\quad + \eta k_2 \int_0^1 (\sigma(y)+\alpha)(1-\sigma(y))q(y)z(y,\tilde{y}_0(y))\Delta y \\
&\leq (\Psi_1 u_0)(x) + W,
\end{aligned}$$

where $E[u_0 > R_0] = \{x : u_0(x) > R_0, x \in [0,1]_{\mathbb{T}}\}$ and W is defined by (16), then

$$\begin{aligned}
0 &\leq b_0 u_0(x) = (\Phi u_0)(x) \\
&\leq (\Psi_1 u_0)(x) + W, x \in [0,1]_{\mathbb{T}}.
\end{aligned} \tag{20}$$

Since $\Psi_1(\mathbb{B}) \subset \mathbb{B}$, then $0 \leq (\Psi_1^p(\Phi u_0))(x) \leq (\Psi_1^p(\Psi_1 u_0 + W))(x)$, $\forall x \in [0,1]_{\mathbb{T}}$. Hence,

$$\begin{aligned}
\|\Phi u_0\|_1 &= \sum_{p=1}^{Q}(r(\Psi_1)+d_0)^{Q-p}\|\Psi_1^{p-1}(\Phi u_0)\| \\
&\leq \sum_{p=1}^{Q}(r(\Psi_1)+d_0)^{Q-p}\|\Psi_1^{p-1}(\Psi_1 u_0 + W)\| \\
&= \|\Psi_1 u_0 + W\|_1.
\end{aligned} \tag{21}$$

Since $\|u_0\| \geq R$, then $\|u_0\|_1 > R_1$. It follows from (13), (14) and (21) that

$$\begin{aligned}
b_0\|u_0\|_1 &= \|\Phi u_0\|_1 \leq \|\Psi_1 u_0\|_1 + \|W\|_1 \\
&= \sum_{p=1}^{Q}(r(\Psi_1)+d_0)^{Q-p}\|\Psi_1^p u_0\| + \|W\|_1 \\
&= (r(\Psi_1)+d_0)\sum_{p=1}^{Q-1}(r(\Psi_1)+d_0)^{Q-p-1} \\
&\quad \times\|\Psi_1^k u_0\| + (r(\Psi_1)+d_0)^N\|u_0\| + \|W\|_1 \\
&= (r(\Psi_1)+d_0)\sum_{p=1}^{Q-1}(r(\Psi_1)+d_0)^{Q-p} \\
&\quad \times\|\Psi_1^{p-1}u_0\| + \|W\|_1 \\
&= (r(\Psi_1)+d_0)\|u_0\|_1 + \|W\|_1 \\
&\leq (r(\Psi_1)+d_0)\|u_0\|_1 + \frac{d_0}{2}R_1 \\
&\leq (r(\Psi_1)+d_0)\|u_0\|_1 + \frac{d_0}{2}\|u_0\|_1 \\
&\leq (r(\Psi_1)+\frac{3}{2}d_0)\|u_0\|_1 \\
&= \frac{1}{4}(1+3r(\Psi_1))\|u_0\|_1.
\end{aligned} \tag{22}$$

Since $b_0 \geq 1$, by (22), then $r(\Psi_1) \geq 1$, which is a contradiction to $r(\Psi_1) < 1$. So (18) holds. By Lemma 6,

$$i(\Phi, \mathbb{B}_R, \mathbb{B}) = 1. \tag{23}$$

It follows from (17) and (23) that

$$i(\Phi, \mathbb{B}_{R,r}, \mathbb{B}) = i(\Phi, \mathbb{B}_R, \mathbb{B}) - i(\Phi, \mathbb{B}_r, \mathbb{B}) = 1,$$

that is, (2) exists at least one positive solution. This completes the proof.

Theorem 2. *Assume that (H_1)-(H_3) hold, and*

$$0 \leq z^0 < z_\infty \leq +\infty.$$

If

$$\eta \in \left(\frac{\lambda_1}{z_\infty}, \frac{\lambda_1}{z^0}\right), \tag{24}$$

where λ_1 is the first eigenvalue of Ψ which has been defined by (9), then (2) exists at least one positive solution.

Proof: By (24), there exists a positive constant $r > 0$, and

$$z(x,u) \leq \frac{\lambda_1}{\eta} u, \forall 0 \leq u \leq r, 0 \leq x \leq 1. \tag{25}$$

Let

$$\Psi_2 y = \lambda_1 \Psi y, u \in C[0,1]_{\mathbb{T}},$$

then $\Psi_2 : \mathbb{B} \to \mathbb{B}$ and

$$\Psi_2(\mathbb{B}) \subset P, r(\Psi_2) = 1. \tag{26}$$

By (25), if $u \in \partial\mathbb{B}_r$, then

$$\begin{aligned}(\Phi u)(x) &\leq \frac{\lambda_1 \eta}{\eta} \int_0^1 G(x,y) q(y) u(y) \Delta y \\ &= (\Psi_2 u)(x), x \in [0,1]_{\mathbb{T}},\end{aligned}$$

that is, $\Phi u \leq \Psi_2 y, \forall u \in \partial\mathbb{B}_r$.

If there exists a fixed point on $\partial\mathbb{B}_r$, Theorem 2 holds. Suppose that there is no fixed point on $\partial\mathbb{B}_r$. Next we show that

$$\Phi u \neq bu, \forall u \in \partial\mathbb{B}_r, b \geq 1. \tag{27}$$

If not, there exist $u \in \partial\mathbb{B}_r$ and $b_0 \geq 1$, and

$$Tu_0 = b_0 u_0.$$

Since $b_0 > 1$ and $b_0 u_0 = Tu_0 \leq \Psi_2 y$, then $b_0^n u_0 \leq \Psi_2^n u_0 (n = 1, 2, \cdots)$, and

$$b_0^n u_0(x) \leq \Psi_2^n u_0(x) \leq \|\Psi_2^n\| \|u_0\|, x \in [0,1]_{\mathbb{T}}. \tag{28}$$

Considering the supremum of (28) on $[0,1]_{\mathbb{T}}$, there is $b_0^n \leq \|\Psi_2^n\|$. By the Gelfand's formula, then $r(\Psi_2) = \lim_{n\to+\infty} \sqrt[n]{\|\Psi_2^n\|} \geq b_0 > 1$, which is a contradiction to $r(\Psi_2) = 1$. By Lemma 6,

$$i(\Phi, \mathbb{B}_r, \mathbb{B}) = 1. \tag{29}$$

From (24), there exists $R > r > 0$, and

$$z(x,u) \geq \frac{\lambda_1}{\eta} u, \forall u \geq R, 0 \leq x \leq 1.$$

Extend Φ, that is, $\Phi : \bar{\mathbb{B}}_R \to \mathbb{B}$, then Φ is completely continuous. If Φ exists a fixed point on $\partial\mathbb{B}_r$, Theorem 2 holds. Suppose that Φ has no fixed point on $\partial\mathbb{B}_R$. Similarly to the proof in Theorem 1, then

$$i(\Phi, \mathbb{B}_R, \mathbb{B}) = 0. \tag{30}$$

By (29) and (30),

$$i(\Phi, \mathbb{B}_{R,r}, \mathbb{B}) = i(\Phi, \mathbb{B}_R, \mathbb{B}) - i(\Phi, \mathbb{B}_r, \mathbb{B}) = -1,$$

that is, (2) exists at least one positive solution. This completes the proof.

Define

$$(\Psi_\zeta u)(x) = \int_\zeta^1 G(x,y) q(y) u(y) \Delta y, x \in [0,1]_{\mathbb{T}}. \tag{31}$$

Lemma 8. *Assume that (H_1), (H_2), (H_4) hold, then $\Psi_\zeta : \mathbb{A} \to \mathbb{A}$ is completely continuous and $\Psi_\zeta(\mathbb{A}) \subset \mathbb{A}$; the spectral radius $r(\Psi_\zeta) \neq 0$, and $h_1 = \lambda_\zeta \Psi_\zeta h_1$, $\lambda_\zeta = (r(\Psi_\zeta))^{-1}$, $h_1 > 0$ is the eigenfunction.*

Theorem 3. *Assume that (H_1)-(H_3) hold. Furthermore,*

$$\eta > \frac{\lambda_1}{z_0}, \eta > \frac{\lambda_\zeta}{z_\infty}, \tag{32}$$

and

$$z(x,u) \leq \frac{x(1-x)}{u}, 0 < u \leq r^*, 0 < x < 1, \tag{33}$$

where $r^* > \sqrt{L_1}$ is a constant, $L_1 = \frac{L}{\zeta(1-\zeta)}$, λ_1 and λ_ζ are the first eigenvalues of Ψ and Ψ_ζ, which have been defined by (9) and (31), respectively. Then (2) exists at least two positive solutions $u_1, u_2 \in \mathbb{B}$.

Proof: From (32), there exist positive constants $r_1 > 0$, $r_3 > r^*$, and $0 < r_1 \leq \sqrt{L_1}$, then

$$z(x,u) \leq \frac{\lambda_1}{\eta} u, 0 < u \leq r_1, 0 \leq x \leq 1, \tag{34}$$

and

$$z(x,u) \leq \frac{\lambda_\zeta}{\eta} u, u \geq \delta_0 r_3, 0 \leq x \leq 1.$$

Hence, for $u \in \partial\mathbb{B}_{r_3}$,

$$z(x,u) \leq \frac{\lambda_\zeta}{\eta} u(x), u(x) \geq \delta_0 r_3, x \in [\zeta, 1]_{\mathbb{T}}. \tag{35}$$

Extend Φ, that is, $\Phi : \bar{\mathbb{B}}_{r_3} \to \mathbb{B}$, then Φ is completely continuous. Suppose that Φ has no fixed point on $\partial\mathbb{B}_{r_1}$ and $\partial\mathbb{B}_{r_3}$. Similarly to the proof of Theorem 1,

$$i(\Phi, \mathbb{B}_{r_1}, \mathbb{B}) = 0. \tag{36}$$

Take $\sqrt{L_1} < r2r^*$. Since $u(x)$ on $(0,1)_{\mathbb{T}}$ is concavity, if $u \in \partial\mathbb{B}_{r_2}$, then $u(x) \geq \|u\| \min\{t, 1-t, \zeta, 1-\zeta\}$. So $u(x) \geq \|u\| t(1-x)\zeta(1-\zeta)$, $\forall x \in (0,1)_{\mathbb{T}}$ and $0 < u(x) \leq r_2 \leq r^*$, $\forall x \in (0,1)_{\mathbb{T}}$. From (33), if $u \in \partial\mathbb{B}_{r_2}$, then

$$\begin{aligned}z(x,u(x)) &\leq \frac{x(1-x)}{u(x)} \leq \frac{x(1-x)}{\|u\| x(1-x)\zeta(1-\zeta)} \\ &= \frac{1}{r_2 \zeta(1-\zeta)}, x \in (0,1)_{\mathbb{T}}.\end{aligned}$$

Thus, if $u \in \partial\mathbb{B}_{r_2}$, then

$$\begin{aligned} & \|\Phi u\| \\ \leq\ & \eta k_2 \int_0^1 (\sigma(y)+\alpha)(1-\sigma(y))q(y)z(y,u(y))\Delta y \\ \leq\ & \frac{1}{r_2\zeta(1-\zeta)}\eta k_2 \int_0^1 (\sigma(y)+\alpha)(1-\sigma(y))q(y)\Delta y \\ \leq\ & \frac{L}{r_2\zeta(1-\zeta)} = \frac{L_1}{r_2} < r_2 = \|u\|. \end{aligned} \tag{37}$$

By Lemma 7,

$$i(\Phi, \mathbb{B}_{r_2}, \mathbb{B}) = 1. \tag{38}$$

Suppose that u_1 is a positive eigenvalue function of Ψ_ζ corresponding to the first eigenvalue λ_ζ, and

$$u_1(x) = \lambda_\zeta(\Psi_\zeta u_1)(x) = \lambda_\zeta \int_\zeta^1 G(x,y)q(y)u_1(y)\Delta y,$$

that is, $u_1 \in \mathbb{B} \setminus \{\theta\}$.

Define

$$(\Psi_3 u)(x) = \lambda_\zeta(\Psi_\zeta u)(x), u \in C[0,1]_{\mathbb{T}}.$$

By Lemma 8, $\Psi_3 : \mathbb{A} \to \mathbb{A}$ is a linear operator and completely continuous and $\Psi_3(\mathbb{A}) \subset \mathbb{A}, \Psi_3 u_1 = \lambda_\zeta \Psi_\zeta u_1 = u_1$. From (35), $u \in \partial\mathbb{B}_{r_3}$, then

$$\begin{aligned} (\Phi u)(x) &= \eta \int_0^1 G(x,y)q(y)z(y,u(y))\Delta y \\ &\geq \lambda_\zeta \int_\zeta^1 G(x,y)q(y)u(y)\Delta y \\ &= (\Psi_3 u)(x), x \in [0,1]_{\mathbb{T}}. \end{aligned}$$

Let $D_0(\mathbb{B}) = \mathbb{B}_{r_3}, \psi = \varphi = \Psi_3, \phi = I, n = 1$. From Lemma 5,

$$i(\Phi, \mathbb{B}_{r_3}, \mathbb{B}) = 0. \tag{39}$$

By (36), (38) and (39),

$$\begin{aligned} i(\Phi, \mathbb{B}_{r_2,r_1}, \mathbb{B}) = i(\Phi, \mathbb{B}_{r_2}, \mathbb{B}) - i(\Phi, \mathbb{B}_{r_1}, \mathbb{B}) = 1, \\ i(\Phi, \mathbb{B}_{r_3,r_2}, \mathbb{B}) = i(\Phi, \mathbb{B}_{r_3}, \mathbb{B}) - i(\Phi, \mathbb{B}_{r_2}, \mathbb{B}) = -1, \end{aligned}$$

that is, (2) exists at least two positive solutions. This completes the proof.

Theorem 4. *Assume that (H_1)-(H_3) hold. Furthermore,*

$$\eta \leq \frac{\lambda_1}{z^0}, \eta \leq \frac{\lambda_1}{z^\infty}, \tag{40}$$

and

$$z(x,u) \geq \frac{\tilde{r}^*}{l}, 0 < u \leq \tilde{r}^*, 0 \leq x \leq 1, \tag{41}$$

where $\tilde{r}^* > 0$, and λ_1 is the first eigenvalue of Ψ which has been defined by (9). Then (2) exists at least two positive solutions.

Proof: By (40), there exist positive constants $r_1' > 0$ and $0 < r_1' < \tilde{r}^*$, $r_2' > 0$ and $r_2' > \tilde{r}^*$, choose $0 < \varepsilon < 1$, then

$$z(x,u) \leq \frac{\lambda_1}{\eta}u, 0 \leq u \leq r_1', 0 \leq x \leq 1, \tag{42}$$

and

$$z(x,u) \leq \varepsilon\frac{\lambda_1}{\eta}u, u \geq r_2', 0 \leq x \leq 1. \tag{43}$$

Suppose that Φ has no fixed point on $\partial\mathbb{B}_{r_1'}$ and $\partial\mathbb{B}_{r_2'}$. It follows from (42), (43) and the permanence property of fixed point index that

$$i(\Phi, \mathbb{B}_{r_1'}, \mathbb{B}) = 1, \tag{44}$$

and

$$i(\Phi, \mathbb{B}_{r_2'}, \mathbb{B}) = -1. \tag{45}$$

Since $u(x)$ is concavity on $[0,1]_{\mathbb{T}}$, $u \in \partial\mathbb{B}_{r_1'}$, then $0 < \delta_0\|u\| \leq u(x) \leq \|u\| = \tilde{r}^*, x \in [\zeta,1]_{\mathbb{T}}$. By (41), $u \in \partial\mathbb{B}_{\tilde{r}^*}$, then

$$\begin{aligned} (\Phi u)(x) &= \eta \int_0^1 G(x,y)q(y)z(y,u(y))\Delta y \\ &\geq \frac{\tilde{r}^*}{l}\eta \int_\zeta^1 k_1(y)G(y,y)q(y)\Delta y = \tilde{r}^*, \end{aligned}$$

that is, $\|\Phi u\| \geq \|u\|$. By Lemma 7,

$$i(\Phi, \mathbb{B}_{\tilde{r}^*}, \mathbb{B}) = 0. \tag{46}$$

It follows from (44)-(46) that

$$\begin{aligned} i(\Phi, \mathbb{B}_{\tilde{r}^*,r_1'}, \mathbb{B}) = i(\Phi, \mathbb{B}_{\tilde{r}^*}, \mathbb{B}) - i(\Phi, \mathbb{B}_{r_1'}, \mathbb{B}) = -1, \\ i(\Phi, \mathbb{B}_{r_2',\tilde{r}^*}, \mathbb{B}) = i(\Phi, \mathbb{B}_{r_2'}, \mathbb{B}) - i(\Phi, \mathbb{B}_{\tilde{r}^*}, \mathbb{B}) = 1, \end{aligned}$$

that is, (2) exists at least two positive solutions. This completes the proof.

IV. Conclusion

A singular boundary value problem on time scales is studied in this paper under weaker conditions via the fixed point index theory. The nonlinear term is not required to be monotone or growth, this new approach is different from the works in [4,5,6,8,9,11,12], and the existing results are developed even if $\mathbb{T} = \mathbb{R}$. Furthermore, we may bring many other boundary value problems under investigation on time scales to obtain more general results; see [16-18].

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