Fermatean Fuzzy Interactive Aggregation Operators and their Application to Decision-making

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***Abstract*—This article explores some interactive aggregation operators for Fermatean fuzzy sets (FFS). These operators are generalisations of those defined for intuitionistic fuzzy sets. It revises the existing operational rules for Fermatean Fuzzy Numbers (FFNs) to construct interactive aggregation operators for FFSs. The improved operational rules incorporate the effects of the membership and non-membership values of the operand FFNs on the resulting FFN. Based on the weighted arithmetic mean and weighted geometric mean, this article proposes two aggregation operators, namely, the Fermatean Fuzzy Interactive Weighted Averaging (FFIWA) operator and the Fermatean Fuzzy Interactive Weighted Geometric (FFIWG) operator. It discusses some properties of the proposed operators and develops an approach for Multi-Attribute Decision making (MADM) using these operators. A numerical example illustrates the developed approach, which is tested for validity using some evaluation criteria.**

***Index Terms*—Fermatean fuzzy sets, Fermatean fuzzy numbers, interaction aggregation, decision-making.**

I. Introduction

MULTI-attribute decision-making (MADM), an essential component of decision theory, selects the best alternative from a set of alternatives while allowing the decision-maker to evaluate each alternative based on some criteria or attributes. It is often difficult for a decision-maker to accurately evaluate the attributes because of the uncertainty in the information available for decision-making. For this reason, several methods have been developed for managing uncertainty in the decision-making process. The fuzzy sets introduced by L. A Zadeh [1] with a membership value in [0, 1] are very suitable for decision-making in uncertain environments. Atanassov [2] designed an intuitionistic fuzzy set (IFS) with membership and non-membership values. Compared to fuzzy sets the IFSs are more effective in handling uncertain information. The researchers investigated different aggregation operators [3], [4], [5], [6], [7] of IFSs to use them effectively in MADM. In all these articles, the membership value of the aggregate depends only on the membership values of the collection being aggregated, and analogously, the non-membership values of the aggregate depend only on the non-membership values of the collection. Such aggregation operators may produce incorrect output in some situations. To address this shortcoming, authors [5], [6], [8] derived aggregation operators that accommodate the impacts of both membership and non-membership values. Pythagorean fuzzy set (PFS) [9] extended the framework for depicting uncertain information. Many authors [10], [11], [12] concentrated on interactive aggregation operators for PFSs for solving Pythagorean fuzzy MADM problems. The Fermatean fuzzy set (FFS) [13] is an extension of IFS. FFSs are advantageous in representing the uncertainty of objective things because they assign a degree of membership, a degree of non-membership, and a degree of hesitation to each element in a universe. As a result, more and more researchers are using FFSs to characterise imprecise or ambiguous decision information and to deal with uncertainty in decision-making in many contexts. In the literature [14], [15], [16] several aggregation operations have been created to combine Fermatean fuzzy information in different contexts.

The sections of this article are as follows. Section II gives an overview of FFSs. Section III introduces two aggregation operators for FFNs that consider the effects of membership and non-membership values of the aggregated FFNs on the resulting FFN. It also lists some properties that these aggregation operators satisfy. Section IV analyses the suitability of the constructed aggregation operators for MADM problems.

II. Preliminaries

Definition 1: [13] Consider a universal set $\mathfrak{X}$, a Fermatean fuzzy set on $\mathfrak{X}$ is expressed as

$$\mathbf{A} = \{(x, (\mathsf{u}_\mathbf{A}(x), \mathsf{v}_\mathbf{A}(x))) \mid x \in \mathfrak{X}\}$$

wherein $\mathsf{u}_\mathbf{A}$ and $\mathsf{v}_\mathbf{A}$ are functions from $\mathfrak{X}$ to the interval $[0, 1]$, comprising $(\mathsf{u}_\mathbf{A}(x))^3 + (\mathsf{v}_\mathbf{A}(x))^3 \leq 1$. $\mathsf{u}_\mathbf{A}$ suggests the membership value and, $\mathsf{v}_\mathbf{A}$ expresses the non-membership value of x to $\mathbf{A}$.

Each $(\mathsf{u}_\mathbf{A}(x), \mathsf{v}_\mathbf{A}(x))$ is termed as a Fermatean fuzzy number (FFN), documented as $\varrho = (\mathsf{u}, \mathsf{v})$. The set of all FFNs is denoted as $\mathcal{F}$. By calculating two values: the score value and the accuracy degree of an FFN, any two FFNs can be ordered.

Definition 2: [13] Consider an FFN $\varrho = (\mathsf{u}, \mathsf{v})$, score value of ϱ is calculated as $score(\varrho) = \mathsf{u}^3 - \mathsf{v}^3$, and $score(\varrho) \in [-1, 1]$.

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Definition 3: [13] Consider an FFN $\varrho = (\mathsf{u}, \mathsf{v})$, accuracy degree of ϱ is calculated as $accuracy(\varrho) = \mathsf{u}^3 + \mathsf{v}^3$, and $accuracy(\varrho) \in [0, 1]$.

Definition 4: [13] Consider two FFNs $\varrho_1 = (\mathsf{u}_1, \mathsf{v}_1)$ and $\varrho_2 = (\mathsf{u}_2, \mathsf{v}_2)$, if $score(\varrho_1) < score(\varrho_2)$, then $\varrho_1 < \varrho_2$; suppose $score(\varrho_1) = score(\varrho_2)$, then in that context if:

1) $accuracy(\varrho_1) = accuracy(\varrho_2)$, then it suggests that ϱ_1 and ϱ_2 depict the same information, i.e., $\varrho_1 = \varrho_2$
2) $accuracy(\varrho_1) < accuracy(\varrho_2)$, then $\varrho_1 < \varrho_2$.

Due to the current focus on the aggregation of Fermatean fuzzy data, several aggregation operators are evolving. This article lists the existing operations on FFNs in Definition 5.

Definition 5: [13] Consider FFNs $\varrho_1 = (\mathsf{u}_1, \mathsf{v}_1), \varrho_2 = (\mathsf{u}_2, \mathsf{v}_2)$, and $\varrho = (\mathsf{u}, \mathsf{v})$, below are some operations defined on them:

1) $\varrho_1 \boxplus \varrho_2 = \left(\sqrt[3]{\mathsf{u}_1^3 + \mathsf{u}_2^3 - \mathsf{u}_1^3\mathsf{u}_2^3}, \mathsf{v}_1\mathsf{v}_2\right)$
2) $\varrho_1 \boxtimes \varrho_2 = \left(\mathsf{u}_1\mathsf{u}_2, \sqrt[3]{\mathsf{v}_1^3 + \mathsf{v}_2^3 - \mathsf{v}_1^3\mathsf{v}_2^3}\right)$
3) $a\varrho = \left(\sqrt[3]{1 - (1 - \mathsf{u}^3)^a}, \mathsf{v}^a\right), a > 0$
4) $\varrho^a = \left(\mathsf{u}^a, \sqrt[3]{1 - (1 - \mathsf{v}^3)^a}\right), a > 0$

III. Fermatean fuzzy interactive aggregation operators

The aggregation of FFNs using the operations defined in [13] has several shortcomings, including:

1) The membership value of the result is unaffected by the non-membership values of the operands, and vice versa. For example for two FFNs $\varrho_1 = (0.50, 0.60)$ and $\varrho_1 = (0.30, 0.90)$, $\varrho_1 \boxplus \varrho_2 = (0.3856, 0.54)$. Changes in the non-membership values of the operands do not affect the membership value of the resultant. Considering a different set of FFNs $\varrho_1 = (0.50, 0.70)$ and $\varrho_2 = (0.30, 0.80)$, $\varrho_1 \boxplus \varrho_2 = (0.3856, 0.56)$. This also occurs in other operations mentioned in [13].
2) Non-zero membership (or non-membership) values of operands are not significant if any of them is zero. For example for two FFNs $\varrho_1 = (0.50, 0.60)$ and $\varrho_1 = (0.00, 0.90)$, $\varrho_1 \boxplus \varrho_2 = (0.00, 0.54)$ and $\varrho_1 \boxtimes \varrho_2 = (0.00, 0.8874)$.

The above deficiencies show the importance of simultaneously including the effects of operand membership and non-membership in operational rules. To address the deficiencies noted above, the operations are modified to consider the effects of the operands' membership and non-membership values on the resultant. They are referred to as interactive operational rules and are essential for defining interactive aggregation operators for Fermatean fuzzy information.

Definition 6: Consider FFNs $\varrho_1 = (\mathsf{u}_1, \mathsf{v}_1)$, $\varrho_2 = (\mathsf{u}_2, \mathsf{v}_2)$ and $\varrho = (\mathsf{u}, \mathsf{v})$,

1) $\varrho_1 \boxplus \varrho_2 = \left(\sqrt[3]{\mathsf{u}_1^3 + \mathsf{u}_2^3 - \mathsf{u}_1^3\mathsf{u}_2^3}, \sqrt[3]{\mathsf{v}_1^3 + \mathsf{v}_2^3 - \mathsf{v}_1^3\mathsf{v}_2^3 - \mathsf{v}_1^3\mathsf{u}_2^3 - \mathsf{u}_1^3\mathsf{v}_2^3}\right)$
2) $\varrho_1 \boxtimes \varrho_2 = \left(\sqrt[3]{\mathsf{u}_1^3 + \mathsf{u}_2^3 - \mathsf{u}_1^3\mathsf{u}_2^3 - \mathsf{u}_1^3\mathsf{v}_2^3 - \mathsf{v}_1^3\mathsf{u}_2^3}, \sqrt[3]{\mathsf{v}_1^3 + \mathsf{v}_2^3 - \mathsf{v}_1^3\mathsf{v}_2^3}\right)$
3) $a\varrho = \left(\sqrt[3]{1 - (1 - \mathsf{u}^3)^a}, \sqrt[3]{\begin{array}{l}(1 - \mathsf{u}^3)^a - \\ (1 - (\mathsf{u}^3 + \mathsf{v}^3))^a\end{array}}\right)$, $a > 0$.
4) $\varrho^a = \left(\sqrt[3]{\begin{array}{l}(1 - \mathsf{v}^3)^a - \\ (1 - (\mathsf{u}^3 + \mathsf{v}^3))^a\end{array}}, \sqrt[3]{1 - (1 - \mathsf{v}^3)^a}\right)$, $a > 0$.

Theorem 1: Let $\varrho_1 = (\mathsf{u}_1, \mathsf{v}_1)$ and $\varrho_2 = (\mathsf{u}_2, \mathsf{v}_2)$ be two FFNs and $a > 0$, then

1) $(\varrho_1 \boxplus \varrho_2)^c = \varrho_1^c \boxtimes \varrho_2^c$
2) $(\varrho_1 \boxtimes \varrho_2)^c = \varrho_1^c \boxplus \varrho_2^c$
3) $(\varrho_1^c)^a = (a\varrho_1)^c$
4) $(a\varrho_1^c)^a = (a\varrho_1)^c$

Proof: Proofs follow directly from the operational rules defined in Definition 6. ■

This article discusses the concepts of weighted arithmetic mean and the weighted geometric mean of a collection of FFNs using the interactive operations defined in Definition 6.

Definition 7: Consider a collection $\{\varrho_\iota : \varrho_\iota = (\mathsf{u}_\iota, \mathsf{v}_\iota), \iota = 1, 2, \cdots, n\}$ of FFNs and a weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ specifying the weights for each ϱ_ι such that $\mathsf{q}_\iota > 0$ and $\sum_{\iota=1}^{n} \mathsf{q}_\iota = 1$. The mapping FFIWA : $F^n \to F$

$$\text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \boxplus_{\iota=1}^{n} (\mathsf{q}_\iota \varrho_\iota)$$

is called a Fermatean fuzzy interactive weighted averaging (FFIWA) operator.

Theorem 2: Consider a collection $\{\varrho_\iota : \varrho_\iota = (\mathsf{u}_\iota, \mathsf{v}_\iota), \iota = 1, 2, \cdots, n\}$ of FFNs and a weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ specifying the weights for each ϱ_ι such that $\mathsf{q}_\iota > 0$ and $\sum_{\iota=1}^{n} \mathsf{q}_\iota = 1$. Then the aggregate obtained by employing the FFIWA operator is an FFN, and

$$\text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \left(\begin{array}{l}\sqrt[3]{1 - \prod_{\iota=1}^{n} (1 - \mathsf{u}_\iota^3)^{\mathsf{q}_\iota}}, \\ \sqrt[3]{\prod_{\iota=1}^{n} (1 - \mathsf{u}_\iota^3)^{\mathsf{q}_\iota} - \prod_{\iota=1}^{n} (1 - (\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3))^{\mathsf{q}_\iota}}\end{array}\right)$$

Proof: Proof by mathematical induction on n. ■

1) Properties of FFIWA operator:

Theorem 3: Consider a collection $\varrho_\iota, \iota = 1, 2, \cdots, n$ of FFNs with a weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ specifying the weights for each ϱ_ι such that $\mathsf{q}_\iota > 0$ and $\sum_{\iota=1}^{n} \mathsf{q}_\iota = 1$, then

1) Idempotency: $\text{FFIWA}(\varrho, \varrho, \cdots, \varrho) = \varrho$
2) Boundedness: For $\varrho^- = (\min_\iota \mathsf{u}_\iota, \max_\iota \mathsf{v}_\iota)$, $\varrho^+ = (\max_\iota \mathsf{u}_\iota, \min_\iota \mathsf{v}_\iota)$, $\varrho^- \le \text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n) \le \varrho^+$.
3) Homogeneity: $\text{FFIWA}(a\varrho_1, a\varrho_2, \cdots, a\varrho_n) = a\text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n)$
4) Monotonicity: For a set of FFNs $\varrho'_\iota = (\mathsf{u}'_\iota, \mathsf{v}'_\iota)$ with the same weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ such that $\mathsf{u}_\iota \le \mathsf{u}'_\iota$ and $\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3 \ge (\mathsf{u}'_\iota)^3 + (\mathsf{v}'_\iota)^3$ for all $\iota = 1, 2, \cdots, n$, $\text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n) \le \text{FFIWA}(\varrho'_1, \varrho'_2, \cdots, \varrho'_n)$.

Proof: For $\varrho_\iota = \varrho = (\mathsf{u}, \mathsf{v})$ with weight vector q

satisfying $\mathsf{q}_\iota > 0$ and $\sum_{\iota=1}^{n} \mathsf{q}_\iota = 1$.

$$\text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \begin{pmatrix} \sqrt[3]{1 - \prod_{\iota=1}^{n} (1-\mathsf{u}_\iota^3)^{\mathsf{q}_\iota}}, \\ \sqrt[3]{\prod_{\iota=1}^{n} (1-\mathsf{u}_\iota^3)^{\mathsf{q}_\iota} - \prod_{\iota=1}^{n} (1-(\mathsf{u}_\iota^3+\mathsf{v}_\iota^3))^{\mathsf{q}_\iota}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[3]{1-(1-\mathsf{u}^3)^{\sum_{\iota=1}^{n} \mathsf{q}_\iota}}, \\ \sqrt[3]{\begin{array}{l}(1-\mathsf{u}^3)^{\sum_{\iota=1}^{n} \mathsf{q}_\iota} - \\ \left(1-\left(\mathsf{u}^3+\mathsf{v}^3\right)\right)^{\sum_{\iota=1}^{n} \mathsf{q}_\iota}\end{array}} \end{pmatrix} = (\mathsf{u}, \mathsf{v})$$

$\min_\iota \mathsf{u}_\iota^3 \le \mathsf{u}_\iota^3 \le \max_\iota \mathsf{u}_\iota^3$

$1 - \max_\iota \mathsf{u}_\iota^3 \le 1 - \mathsf{u}_\iota^3 \le 1 - \min_\iota \mathsf{u}_\iota^3$

$(1 - \max_\iota \mathsf{u}_\iota^3)^{\mathsf{q}_\iota} \le (1 - \mathsf{u}_\iota^3)^{\mathsf{q}_\iota} \le (1 - \min_\iota \mathsf{u}_\iota^3)^{\mathsf{q}_\iota}$

$(1 - \max_\iota \mathsf{u}_\iota^3) \le \prod_{\iota=1}^{n} \left(1 - \mathsf{u}_\iota^3\right)^{\mathsf{q}_\iota} \le (1 - \min_\iota \mathsf{u}_\iota^3)$

$\min_\iota \mathsf{u}_\iota^3 \le 1 - \prod_{\iota=1}^{n} \left(1 - \mathsf{u}_\iota^3\right)^{\mathsf{q}_\iota} \le \max_\iota \mathsf{u}_\iota^3 \quad \ldots(1)$

$\min_\iota \mathsf{u}_\iota \le \sqrt[3]{1 - \prod_{\iota=1}^{n} (1 - \mathsf{u}_\iota^3)^{\mathsf{q}_\iota}} \le \max_\iota \mathsf{u}_\iota \quad \ldots(2)$

$\min_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\} \le (\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3) \le \max_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\}$

$1 - \max_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\} \le 1 - (\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3) \le$
$1 - \min_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\}$

$1 - \max_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\} \le \prod_{\iota=1}^{n} \left(1 - (\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3)\right)^{\mathsf{q}_\iota} \le$
$1 - \min_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\}$

$\min_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\} \le 1 - \prod_{\iota=1}^{n} \left(1 - (\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3)\right)^{\mathsf{q}_\iota} \le$
$\max_\iota\{\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\} \quad \ldots(3)$

From (1) and (3)

$\min_\iota \mathsf{v}_\iota^3 \le \prod_{\iota=1}^{n} \left(1 - \mathsf{u}_\iota^3\right)^{\mathsf{q}_\iota} - \prod_{\iota=1}^{n} \left(1 - (\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3)\right)^{\mathsf{q}_\iota}$
$\le \max_\iota \mathsf{v}_\iota^3$

$\min_\iota \mathsf{v}_\iota \le \sqrt[3]{\prod_{\iota=1}^{n} (1 - \mathsf{u}_\iota^3)^{\mathsf{q}_\iota} - \prod_{\iota=1}^{n} (1 - (\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3))^{\mathsf{q}_\iota}}$
$\le \max_\iota \mathsf{v}_\iota \quad \ldots(4)$

From (2) and (4)

$\varrho^- \le \text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n) \le \varrho^+$

$$a\varrho_1 = \left(\sqrt[3]{1-(1-\mathsf{u}_1^3)^a}, \sqrt[3]{\begin{array}{l}\left(1-\mathsf{u}_1^3\right)^a \\ -\left(1-\left(\mathsf{u}_1^3+\mathsf{v}_1^3\right)\right)^a\end{array}} \right)$$

$$\vdots$$

$$a\varrho_n = \left(\sqrt[3]{1-(1-\mathsf{u}_n^3)^a}, \sqrt[3]{\begin{array}{l}\left(1-\mathsf{u}_n^3\right)^a \\ -\left(1-\left(\mathsf{u}_n^3+\mathsf{v}_n^3\right)\right)^a\end{array}} \right)$$

$$\text{FFIWA}(a\varrho_1, a\varrho_2, \cdots, a\varrho_n) = \begin{pmatrix} \sqrt[3]{1 - \prod_{\iota=1}^{n} (1-\mathsf{u}_\iota^3)^{a\mathsf{q}_\iota}}, \\ \sqrt[3]{\prod_{\iota=1}^{n} (1-\mathsf{u}_\iota^3)^{a\mathsf{q}_\iota} - \prod_{\iota=1}^{n} (1-(\mathsf{u}_\iota^3+\mathsf{v}_\iota^3))^{a\mathsf{q}_\iota}} \end{pmatrix} = a\text{FFIWA}$$

Since $\mathsf{u}_\iota \le \mathsf{u}_\iota'$,

$$\sqrt[3]{1 - \prod_{\iota=1}^{n} (1-\mathsf{u}_\iota^3)^{\mathsf{q}_\iota}} \le \sqrt[3]{1 - \prod_{\iota=1}^{n} (1-(\mathsf{u}_\iota')^3)^{\mathsf{q}_\iota}}.$$

Since $\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3 \ge (\mathsf{u}_\iota')^3 + (\mathsf{v}_\iota')^3$,

$$\prod_{\iota=1}^{n} \left(1 - \left(\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3\right)\right)^{\mathsf{q}_\iota} \le \prod_{\iota=1}^{n} \left(1 - \left((\mathsf{u}_\iota')^3 + (\mathsf{v}_\iota')^3\right)\right)^{\mathsf{q}_\iota}$$

which gives

$$\sqrt[3]{\prod_{\iota=1}^{n} (1-\mathsf{u}_\iota^3)^{\mathsf{q}_\iota} - \prod_{\iota=1}^{n} (1-(\mathsf{u}_\iota^3+\mathsf{v}_\iota^3))^{\mathsf{q}_\iota}} \ge \sqrt[3]{\prod_{\iota=1}^{n} (1-(\mathsf{u}_\iota')^3)^{\mathsf{q}_\iota} - \prod_{\iota=1}^{n} (1-((\mathsf{u}_\iota')^3+(\mathsf{v}_\iota')^3))^{\mathsf{q}_\iota}}.$$

Thus the ordering defined for FFNs gives $\text{FFIWA}(\varrho_1, \varrho_2, \cdots, \varrho_n) \le \text{FFIWA}(\varrho_1', \varrho_2', \cdots, \varrho_n')$. ■

Definition 8: Consider a collection $\{\varrho_\iota : \varrho_\iota = (\mathsf{u}_\iota, \mathsf{v}_\iota), \iota = 1, 2, \cdots, n\}$ of FFNs and a weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ specifying the weights for each ϱ_ι such that $\mathsf{q}_\iota > 0$ and $\sum_{\iota=1}^{n} \mathsf{q}_\iota = 1$. The mapping $\text{FFIWG} : F^n \to F$

$$\text{FFIWG}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \boxtimes_{\iota=1}^{n} (\varrho_\iota)^{\mathsf{q}_\iota}$$

is called a Fermatean fuzzy interactive weighted geometric (FFIWG) operator.

Theorem 4: Consider a collection $\{\varrho_\iota : \varrho_\iota = (\mathsf{u}_\iota, \mathsf{v}_\iota), \iota = 1, 2, \cdots, n\}$ of FFNs and a weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ specifying the weights for each ϱ_ι such that $\mathsf{q}_\iota > 0$ and $\sum_{\iota=1}^{n} \mathsf{q}_\iota = 1$, then the aggregate obtained through employing the FFIWG operator is an FFN, also

$$\text{FFIWG}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \begin{pmatrix} \sqrt[3]{\prod_{\iota=1}^{n} (1-\mathsf{v}_\iota^3)^{\mathsf{q}_\iota} - \prod_{\iota=1}^{n} (1-(\mathsf{u}_\iota^3+\mathsf{v}_\iota^3))^{\mathsf{q}_\iota}}, \\ \sqrt[3]{1 - \prod_{\iota=1}^{n} (1-\mathsf{v}_\iota^3)^{\mathsf{q}_\iota}} \end{pmatrix}$$

Proof: Proof by mathematical induction on n. ■

2) Properties of FFIWG operator:

Theorem 5: Consider a collection ϱ_ι, $\iota = 1, 2, \cdots, n$ of FFNs with a weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ specifying the weights for each ϱ_ι such that $\mathsf{q}_\iota > 0$ and $\sum_{\iota=1}^{n} \mathsf{q}_\iota = 1$, then

1) Idempotency: $\text{FFIWG}(\varrho, \varrho, \cdots, \varrho) = \varrho$
2) Boundedness: For $\varrho^- = (\min_\iota \mathsf{u}_\iota, \max_\iota \mathsf{v}_\iota)$, $\varrho^+ = (\max_\iota \mathsf{u}_\iota, \min_\iota \mathsf{v}_\iota)$, $\varrho^- \le \text{FFIWG}(\varrho_1, \varrho_2, \cdots, \varrho_n) \le \varrho^+$.

3) Homogeneity: FFIWG $(a\varrho_1, a\varrho_2, \cdots, a\varrho_n) = a$FFIWG $(\varrho_1, \varrho_2, \cdots, \varrho_n)$
4) Monotonicity: For a set of FFNs $\varrho'_\iota = (\mathsf{u}'_\iota, \mathsf{v}'_\iota)$ with the same weight vector $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2, \cdots, \mathsf{q}_n)^T$ such that $\mathsf{v}_\iota \geq \mathsf{v}'_\iota$ and $\mathsf{u}_\iota^3 + \mathsf{v}_\iota^3 \leq (\mathsf{u}'_\iota)^3 + (\mathsf{v}'_\iota)^3$ for all $\iota = 1, 2, \cdots, n$, FFIWG $(\varrho_1, \varrho_2, \cdots, \varrho_n) \leq$FFIWG $(\varrho'_1, \varrho'_2, \cdots, \varrho'_n)$.

Proof: Similar to proof of Theorem 3. ■

IV. Decision making by applying the proposed aggregation operators

Multi-attribute decision-making (MADM) is a decision-making method widely used in science, engineering and business. An MADM problem includes numerous attributes that represent the different aspects under which the alternatives are examined. The attributes are assigned weights that are normalised so that they sum to one. The problem is to rank the alternatives or find the best alternative(s) from a given collection of alternatives based on some related decision criteria (attributes). This section focuses on Fermatean fuzzy MADM (FF-MADM) problems, which can be mathematically represented as follows: Let $A = \{a_1, a_2, ..., a_m\}$ be a set of m alternatives, and let $G = \{g_1, g_2, ..., g_n\}$ be the universe of discourse with n attributes. The vector $w = (w_1, w_2, ..., w_n)^T$ is the weight vector, where each w_ι denotes the relevance of each attribute g_ι. If the decision-maker specifies the rating of each alternative a_i for each attribute g_ι as an FFN $(\mathsf{u}_{\mathsf{i}\iota}, \mathsf{v}_{\mathsf{i}\iota})$, where $\mathsf{u}_{\mathsf{i}\iota}$ denotes the degree to which alternative a_i fulfils attribute g_ι, and $\mathsf{v}_{\mathsf{i}\iota}$ denotes the degree to which alternative a_i does not satisfy attribute g_ι. An FF-MADM is briefly represented in the form of a matrix called a decision matrix, where each entry is an FFN. This article proposes an approach to solve an FF-MADM from the perspective of information fusion. The aggregation operators for FFNs are used as tools to combine the Fermatean fuzzy information and the attribute weights. The following steps are included in this approach.

Step 1. The assessments of the decision-maker are depicted as a matrix, called the decision matrix $O = (\varrho_{\mathsf{i}\iota})_{m\times n}$ where $\varrho_{\mathsf{i}\iota}$ is an FFN $(\mathsf{u}_{\mathsf{i}\iota}, \mathsf{v}_{\mathsf{i}\iota})$.

$$O = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{cccc} g_1 & g_2 & \cdots\cdots & g_n \end{array} \\ \begin{pmatrix} \varrho_{11} & \varrho_{12} & \cdots & \varrho_{1n} \\ \varrho_{21} & \varrho_{22} & \cdots & \varrho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{m1} & \varrho_{m2} & \cdots & \varrho_{mn} \end{pmatrix} \end{array}$$

Step 2. Normalise the decision matrix, transforming the preference values of the cost type attributes of O into the benefit type attributes by applying (5).

$$\varrho_{i\iota} = \begin{cases} (\mathsf{u}_{i\iota}, \mathsf{v}_{i\iota}) & ; \text{ if } g_\iota \text{ is a benefit type criterion} \\ (\mathsf{v}_{i\iota}, \mathsf{u}_{i\iota}) & ; \text{ if } g_\iota \text{ is a cost type criterion} \end{cases} \quad \ldots (5)$$

TABLE I
Ordering of alternatives for aggregation operators

	FFWA	FFWG	FFIWA	FFIWG
a_1	0.1434	0.0616	0.3554	0.3040
a_2	−0.2245	−0.1848	−0.2830	−0.3237
a_3	0.0651	0.0455	0.0776	0.0777

The normalised matrix is represented as $\check{O}$

$$\check{O} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{cccc} g_1 & g_2 & \cdots\cdots & g_n \end{array} \\ \begin{pmatrix} \check{\varrho}_{11} & \check{\varrho}_{12} & \cdots & \check{\varrho}_{1n} \\ \check{\varrho}_{21} & \check{\varrho}_{22} & \cdots & \check{\varrho}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \check{\varrho}_{m1} & \check{\varrho}_{m2} & \cdots & \check{\varrho}_{mn} \end{pmatrix} \end{array}$$

Step 3. Compute the aggregate value $\check{\varrho}_\mathsf{i}$ for the alternatives a_i, $\mathsf{i} = 1, 2, ..., m$ using the FFIWA (or FFIWG) operator.
Step 4. Determine the $score(\check{\varrho}_\mathsf{i})$ of each $\check{\varrho}_\mathsf{i}$.
Step 5. Order the alternatives a_i $(\mathsf{i} = 1, 2, ..., m)$ in accordance with the $score(\check{\varrho}_\mathsf{i})$ [$accuracy(\check{\varrho}_\mathsf{i})$ if necessary] and select the alternative(s) with the highest order as the optimal one.

A. Numerical Example

Choosing a suitable job is a crucial decision in one's life. An individual's happiness, confidence, well-being and health depend on finding the right job. A person applies for a position at different companies in his field of expertise. Eventually, he received offers for this job from three companies. Now he evaluates each offer based on five characteristics: salary, benefits and perks, working hours, commuting costs, and growth opportunities. To represent this MADM problem mathematically, label the job offers as a_1, a_2, and a_3 and the attributes as g_1, g_2, g_3, g_4, and g_5, respectively. The weights for each attribute are 0.30, 0.25, 0.12, 0.15, and 0.18, respectively.
Step 1. The person evaluates each job in accordance with the criteria and makes the decision matrix.

$$O = \begin{bmatrix} (0.90, 0.20) & (0.50, 0.30) & (0.40, 0.60) & (0.40, 0.80) & (0.10, 0.60) \\ (0.10, 0.60) & (0.20, 0.80) & (0.60, 0.20) & (0.90, 0.20) & (0.60, 0.50) \\ (0.40, 0.10) & (0.60, 0.05) & (0.50, 0.40) & (0.50, 0.50) & (0.10, 0.30) \end{bmatrix}$$

Step 2. Since g_4 is a cost type attribute, normalise the decision matrix O using (5).

$$\check{O} = \begin{bmatrix} (0.90, 0.20) & (0.50, 0.30) & (0.40, 0.60) & (0.80, 0.40) & (0.10, 0.60) \\ (0.10, 0.60) & (0.20, 0.80) & (0.60, 0.20) & (0.20, 0.90) & (0.60, 0.50) \\ (0.40, 0.10) & (0.60, 0.05) & (0.50, 0.40) & (0.50, 0.50) & (0.10, 0.30) \end{bmatrix}$$

Step 3. The values aggregated using FFIWA operator for each job are $\check{\varrho}_1 = (0.7475, 0.3965)$, $\check{\varrho}_2 = (0.4192, 0.7092)$ and $\check{\varrho}_3 = (0.4796, 0.3200)$. Using the FFIWG operator: $\check{\varrho}_1 = (0.7319, 0.4499)$, $\check{\varrho}_2 = (0.3764, 0.7224)$ and $\check{\varrho}_3 = (0.4798, 0.3197)$.
Step 4. Calculate the score values for each $\check{\varrho}_\mathsf{i}$.
Step 5. Using the FFIWA operator, we obtain the following ranking: $a_1 > a_3 > a_2$ and a_1 is the optimal solution (job). The same is obtained using the FFIWG operator. Using the same MADM approach with the aggregation operators from [14], we obtain a ranking for the alternatives similar to that of the illustrated numerical example (summarised in Table I).

B. Evaluation of the proposed MADM approach

The goal of MADM approaches is to improve decision quality by making decision-making precise, more logical, and more efficient. Although enormous efforts and significant progress have been made in developing several MADM models to address different types of decision problems, no model can be considered the best for a general MADM problem. Different MADM methods can lead to different solutions (rankings) for the same problem (numerical data). In the MADM approach discussed in this article, the ranking of alternatives depends on the aggregation operator used in merging the given information. To illustrate the viability of a typical MADM method, Wang and Triantaphyllou [17] established some evaluation criteria. An effective MADM should meet the following criteria:

1) Evaluation Criterion I: The method should not affect the specification of the best alternative if the membership and non-membership values of a non-optimal alternative and the worst alternative are swapped (provided that the weights of the attributes remain unchanged).
2) Evaluation Criterion II: The method must be transitive.
3) Evaluation Criterion III: If the given problem is decomposed into subproblems and solved independently by the same method, the cumulative ranking of the alternatives should be the same as the ranking of the undecomposed problem.

Using the illustrated numerical example, this article examines the feasibility of the MADM techniques proposed in this article and in [14].

1) Testing using the Evaluation Criterion I: Swap the membership and non-membership values of the assessments of the worst alternative (a_2) and the non-optimal one (a_3) and apply the method. The values aggregated with the FFWA operator are $\breve{\varrho}_1 = (0.5810, 0.3750)$, $\breve{\varrho}_2 = (0.6290, 0.2900)$, $\breve{\varrho}_3 = (0.2195, 0.4230)$ and those aggregated with the FFWG operator are $\breve{\varrho}_1 = (0.4664, 0.3415)$, $\breve{\varrho}_2 = (0.5812, 0.2259)$, $\breve{\varrho}_3 = (0.1541, 0.3663)$. Moreover, in both cases, the ranking of alternatives is $a_2 > a_1 > a_3$ and the optimal alternative is a_2, which is different from the optimal alternative for the original problem. The first evaluation criterion is not met, indicating that the MADM approach using FFWA and FFWG operators are not effective. The aggregate values using the proposed FFIWA operator are $\breve{\varrho}_1 = (0.7475, 0.3965)$, $\breve{\varrho}_2 = (0.7224, 0.3764)$, $\breve{\varrho}_3 = (0.3197, 0.4798)$, and using the proposed FFIWG operator are $\breve{\varrho}_1 = (0.7319, 0.4499)$, $\breve{\varrho}_2 = (0.7092, 0.4192)$, $\breve{\varrho}_3 = (0.3200, 0.4796)$ and in both approaches, the ranking of alternatives is $a_1 > a_2 > a_3$, with the optimal alternative a_1, identical to the original optimal alternative. Therefore, the proposed approaches satisfy the first evaluation criterion.

2) Testing using the Evaluation Criteria II and III: Split the given problem into smaller problems that is, problems with alternatives $\{a_1, a_2\}$, $\{a_1, a_3\}$ and $\{a_2, a_3\}$ and find the ranking using the proposed approaches. The ranking obtained in sub-problems is: $a_1 > a_2$, $a_1 > a_3$ and $a_3 > a_2$. Thus, the combined ranking of all the smaller problems is $a_1 > a_3 > a_2$, consistent with the original ranking. The transitivity of the approach for the given problem follows directly from the test for the Evaluation Criterion III. The MADM approach proposed in this article satisfies all three evaluation criteria and is thus validated as an effective MADM approach.

V. Conclusion

The article defines a few operations of FFNs that consider the effect of membership and non-membership on one of these values of the resulting FFN. In addition, the article defines the FFIWA and FFIWG operators for FFSs using these operations. It then provides a numerical example of an MADM approach generated for decision problems where the inputs are FFNs. Future work in this direction includes the construction of several aggregation operators for FFSs with potential applications in decision-making.

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