Quartic Catmull-Rom Spline Function with Local Parameters and Its Optimal Interpolation

Juncheng Li, Chengzhi Liu and Ge Ding

Abstract—**When the data points remain unchanged, the shape of the cubic Catmull-Rom spline function cannot be adjusted. A quartic Catmull-Rom spline function with local parameters is given. The spline function not only inherits the interpolation and continuity of the cubic Catmull-Rom spline function but also can adjust the interpolation curve's shape locally or globally by modifying the values of the contained parameters. To obtain the optimal quartic Catmull-Rom spline function interpolating a given function, a scheme for determining the optimal values of the contained parameters is given. Some numerical examples show that the optimal quartic Catmull-Rom spline function has a better interpolation effect than the cubic Catmull-Rom spline function.**

Index Terms—**Catmull-Rom spline function, local parameter, shape adjustment, optimal interpolation**

I. Introduction

Data interpolation has always been an important research topic in computer-aided geometric design and computer graphics. The cubic Catmull-Rom spline [1] has been successfully applied to geometric design [2, 3] and engineering applications [4, 5] as the automatic interpolation of the data points without solving the equation system. Usually, by modifying some data points, the shape of the cubic Catmull-Rom spline will change. However, in many interpolation problems, the data points are often inconvenient to modify, and the shape of the cubic Catmull-Rom spline cannot be adjusted, which limits the application of the cubic Catmull-Rom spline in practical engineering problems to a certain extent.

Recently, some curves with parameters have been proposed to meet the needs of shape design in practical engineering problems. When the control points are fixed, these curves can be freely adjusted by modifying the values of the parameters, which enriches the performance of the curves. Such as Bézier-like curves with parameters [6-9], B-spline-like curves with parameters [10-12], and interpolation curves with parameters [13-15].

To overcome the shortcoming that the cubic Catmull-Rom spline cannot adjust its shape when the data points are fixed, a quartic Catmull-Rom spline with parameters was proposed in Ref. [16]. The spline not only inherits the features of the cubic Catmull-Rom spline but also can realize local adjustment or global adjustment by altering the values of parameters when the data points are fixed. It is a practical method that can be used to construct interpolation curves. It is noted that Ref. [16] only studied the parametric curve of the quartic Catmull-Rom spline but did not discuss the function of the quartic Catmull-Rom spline. Functional splines are also widely applied in practical engineering problems, especially in function interpolation. The primary purpose of this paper is to discuss the quartic Catmull-Rom spline function with local parameters.

The remainder of this paper is organized as follows. Section II gives the definition and properties of the quartic Catmull-Rom spline function. Section III presents the method of constructing the optimal spline function in function interpolation and gives some numerical examples. A short conclusion is presented in Section IV.

II. The Quartic Catmull-Rom Spline Function

To define the quartic Catmull-Rom spline function with local parameters, we first introduce the basis of the quartic Catmull-Rom spline.

Definition 1. *The basis of the quartic Catmull-Rom spline is defined by [16]*

$$\begin{cases} f_{i,0}(t)=\frac{1}{2}\left(-t+2(1+\alpha_i)t^2-(1+4\alpha_i)t^3+2\alpha_i t^4\right),\\ f_{i,1}(t)=\frac{1}{2}\left(2-(5+2\alpha_i)t^2+(3+4\alpha_i)t^3-2\alpha_i t^4\right),\\ f_{i,2}(t)=\frac{1}{2}\left(t+2(2-\alpha_i)t^2-(3-4\alpha_i)t^3-2\alpha_i t^4\right),\\ f_{i,3}(t)=\frac{1}{2}\left(-(1-2\alpha_i)t^2+(1-4\alpha_i)t^3+2\alpha_i t^4\right), \end{cases} \tag{1}$$

where $0\le t\le 1$, α_i *is an arbitrary real number.*

For any α_i, the basis of the quartic Catmull-Rom spline has the following properties [16]

(A) Partition of unity: $f_{i,0}(t)+f_{i,1}(t)+f_{i,2}(t)+f_{i,3}(t)\equiv 1$.

(B) Symmetry: $f_{i,j}(1-t)=f_{i,3-j}(t)$, $j=0,1,2,3$.

(C) Properties at the endpoints:

Manuscript received February 25, 2023; revised June 3, 2023.

This work was supported by the Hunan Provincial Natural Science Foundation of China under Grant 2021JJ30373, and the National Natural Science Foundation of China under Grant 12101225.

Juncheng Li is a professor of the College of Mathematics and Finance, Hunan University of Humanities, Science and Technology, Loudi, 417000 PR China (phone: 86-738-8325585; fax: 86-738-8325585; e-mail: lijuncheng82@126.com).

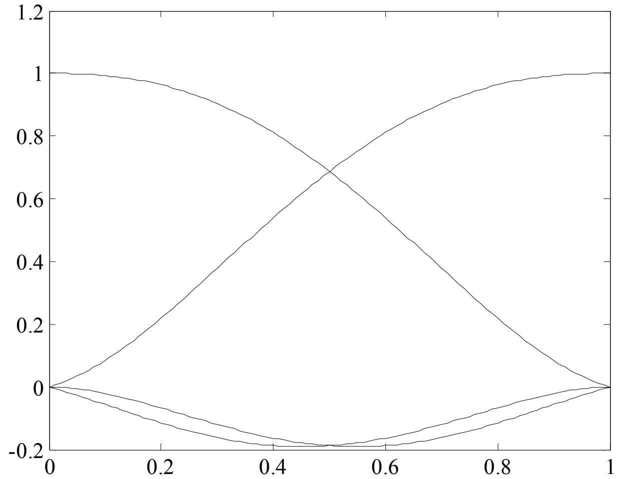
Chengzhi Liu is an associate professor of the College of Mathematics and Finance, Hunan University of Humanities, Science and Technology, Loudi, 417000 PR China (e-mail: it-rocket@163.com).

Ge Ding is an undergraduate of the College of Mathematics and Finance, Hunan University of Humanities, Science and Technology, Loudi, 417000 PR China (e-mail: 2721416315@qq.com).

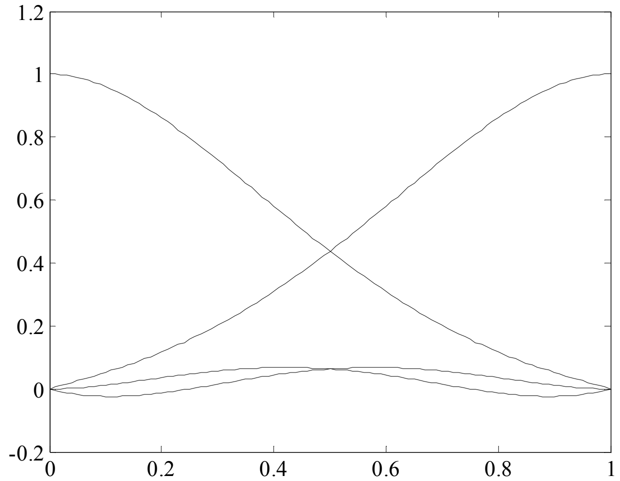
$$\begin{cases} f_{i,0}(0) = f_{i,2}(0) = f_{i,3}(0) = 0, \quad f_{i,1}(0) = 1, \\ f_{i,0}(1) = f_{i,1}(1) = f_{i,3}(1) = 0, \quad f_{i,2}(1) = 1. \end{cases} \tag{2}$$

$$\begin{cases} f'_{i,0}(0) = -1/2, \quad f'_{i,1}(0) = f'_{i,3}(0) = 0, \quad f'_{i,2}(0) = 1/2, \\ f'_{i,0}(1) = f'_{i,2}(1) = 0, \quad f'_{i,1}(1) = -1/2, \quad f'_{i,3}(1) = 1/2. \end{cases} \tag{3}$$

Remark 1. The basis of the quartic Catmull-Rom spline inherits the properties of the basis of the cubic Catmull-Rom spline [1]. In particular, Eq. (1) becomes the basis of the cubic Catmull-Rom spline when $\alpha_i = 0$. Due to the presence of α_i, different graphs of the basis of the quartic Catmull-Rom spline will be obtained when α_i is taken different values, as shown in Fig. 1.



(a) $\alpha_i = -2$



(b) $\alpha_i = 2$

Fig. 1. Graphs of the basis with different parameters

Based on the basis of the quartic Catmull-Rom spline, the corresponding quartic Catmull-Rom spline function can be defined as follows.

Definition 2. *Given the data points* (x_i, y_i) $(i = 0,1,\cdots,n)$, *set* $h = x_{i+1} - x_i$ $(i = 0,1,\cdots,n-1)$. *Take*

$$x_{-1} = 2x_0 - x_1,\ x_{n+1} = 2x_n - x_{n-1},$$

$$y_{-1} = 2y_0 - y_1,\ y_{n+1} = 2y_n - y_{n-1}.$$

For $x_i \le x \le x_{i+1}$ $(i = 0,1,\cdots,n-1)$, *let* $t = (x - x_i)/h$, *the quartic Catmull-Rom spline function is expressed by*

$$s_i(x) = \sum_{j=0}^{3} f_{i,j}(t) y_{i+j-1},\ i = 0,1,\cdots,n-1, \tag{4}$$

where $f_{i,j}(t)$ $(j = 0,1,2,3)$ *is the basis expressed in Eq. (1).*

Theorem 1. *The quartic Catmull-Rom spline function has the following properties,*

(A) Interpolation: For any α_i, *the spline interpolates the data points* (x_i, y_i) $(i = 0,1,\cdots,n)$.

(B) Continuity: For any α_i, *the spline function reaches* C^l *continuity.*

(C) Shape flexibility: The shape of the quartic Catmull-Rom spline function can be locally or globally adjusted by altering the value of α_i, *while all the data points* (x_i, y_i) $(i = -1,0,1,\cdots,n+1)$ *are kept unchanged.*

Proof. (A) For any α_i, from Eqs. (2) and (4) we have

$$s_i(x_i) = y_i,\ s_i(x_{i+1}) = y_{i+1}, \tag{5}$$

where $i = 0,1,\cdots,n-1$. Eq. (5) shows that the quartic Catmull-Rom spline function interpolates the data points (x_i, y_i) ($i = 0,1,\cdots,n$).

(B) For any α_i, from Eqs. (3) and (4) we have

$$s'_i(x_i) = \frac{1}{2h}(y_{i+1} - y_{i-1}),\ s'_i(x_{i+1}) = \frac{1}{2h}(y_{i+2} - y_i), \tag{6}$$

where $i = 0,1,\cdots,n-1$. From Eqs. (5) and (6) we have

$$s_i(x_{i+1}) = s_{i+1}(x_{i+1}),\ s'_i(x_{i+1}) = s'_{i+1}(x_{i+1}), \tag{7}$$

where $i = 0,1,\cdots,n-2$. Eq. (7) shows that the quartic Catmull-Rom spline function reaches C^1 continuity.

(C) From Eq. (4), the *i*th spline function contains a local parameter. The modified value of the local parameter will only affect the shape of the *i*th spline function when all the data points are fixed. That is, the local adjustability of the spline function is realized. If all the local parameters are unified into the same parameter, the global adjustment can be achieved by modifying the unified parameter.

Remark 2. Theorem 1 shows that the quartic Catmull-Rom spline function inherits the interpolation and continuity of the cubic Catmull-Rom spline function [1] and has local or global adjustability when all data points remain unchanged. In particular, it is easy to know from Eq. (4) that the quartic Catmull-Rom spline function is the cubic Catmull-Rom spline function when $\alpha_i = 0$, so the quartic Catmull-Rom spline function is an extension of the cubic Catmull-Rom spline function.

Example 1. Let

$$x_i = i\pi/2,\ y_i = \cos(x_i),\ i = 0,1,2,3,4.$$

Fig. 2 shows the local adjustment of the quartic Catmull-Rom spline function interpolating the data points, where the parameters are taken as $(\alpha_0, \alpha_1, \alpha_3) = (1,2,1)$, and $\alpha_2 = -2$

(short dotted line), $\alpha_2 = 0$ (solid line), $\alpha_2 = 2$ (long dotted line).

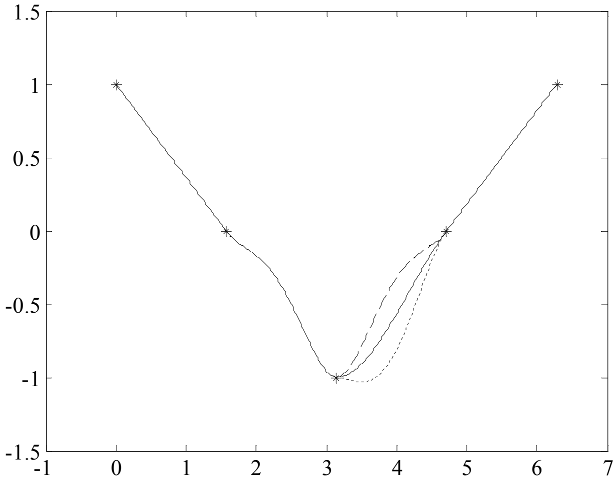


Fig. 2. Local adjustment of the quartic Catmull-Rom spline function in example 1

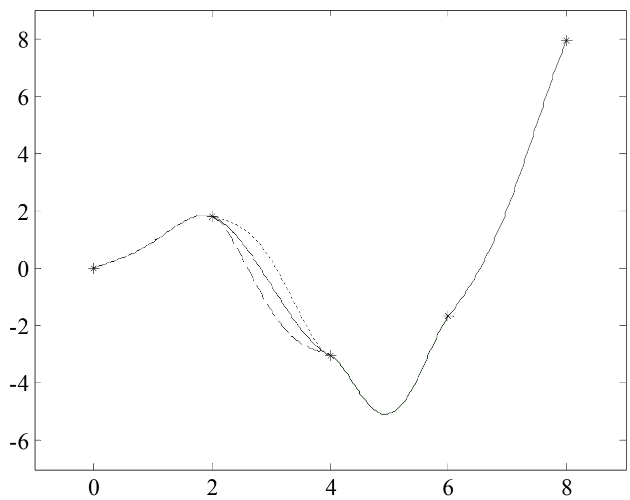


Fig. 3. Local adjustment of the quartic Catmull-Rom spline function in example 2

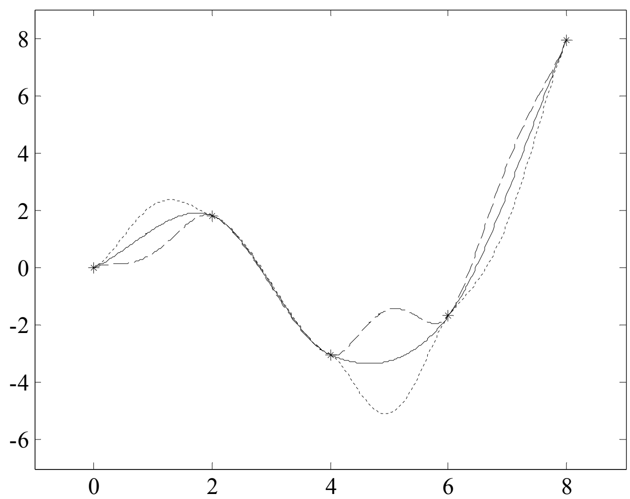


Fig. 4. Global adjustment of the quartic Catmull-Rom spline function in example 2

Example 2. Let $x_i = 2i$, $y_i = x_i \sin x_i$, $i = 0,1,2,3,4$. The local adjustment of the quartic Catmull-Rom spline function interpolating the data points is shown in Fig. 3, where the parameters are taken as $(\alpha_0, \alpha_2, \alpha_3) = (1, -2, -1)$, and $\alpha_1 = -30$ (short dotted line), $\alpha_1 = 0$ (solid line), $\alpha_1 = 30$ (long dotted line). The global adjustment of the quartic Catmull-Rom spline function interpolating the data points is shown in Fig. 4, where the unified parameter is taken $\alpha = -2$ (short dotted line), $\alpha = 0$ (solid line) and $\alpha = 2$ (long dotted line).

III. THE OPTIMAL QUARTIC CATMULL-ROM SPLINE FUNCTION

It can be seen from Definition 2 that if the data points (x_i, y_i) $(i = 0,1,\cdots,n)$ are taken from a function, the quartic Catmull-Rom spline function interpolating the function can be defined as follows.

Definition 3. *Given a continuous function* $y = g(x)$ $(a \le x \le b)$, *the function values at the node* $x_i = a + hi$ $(i = 0,1,\cdots,n)$ *are* $y_i = g(x_i)$, *here* $h = (b-a)/n$. *Take* $x_{-1} = 2x_0 - x_1$, $x_{n+1} = 2x_n - x_{n-1}$, *assume* $y_{-1} = g(x_{-1})$ *and* $y_{n+1} = g(x_{n+1})$ *exist. For* $x_i \le x \le x_{i+1}$ $(i = 0,1,\cdots,n-1)$, *let* $t = (x - x_i)/h$, *the quartic Catmull-Rom spline function interpolating* $y = g(x)$ *is expressed by*

$$s_i(x) = \sum_{j=0}^{3} f_{i,j}(t) y_{i+j-1} , \ i = 0,1,\cdots,n-1 , \tag{8}$$

where $f_{i,j}(t)$ $(j = 0,1,2,3)$ *is the quartic Catmull-Rom spline basis function defined in Eq. (1).*

Theorem 1 shows that the shape of the quartic Catmull-Rom spline function can be locally or globally adjusted by modifying the values of the parameter α_i when all the data points are fixed. Although this facilitates the shape adjustment of the interpolation curves, it should be noted that the effect of the quartic Catmull-Rom spline function interpolating a function may not be ideal once the values of the parameters are not appropriate. For example, Fig. 5 is the result of continuing to draw the interpolated function (dash-dotted line) based on Fig. 3.

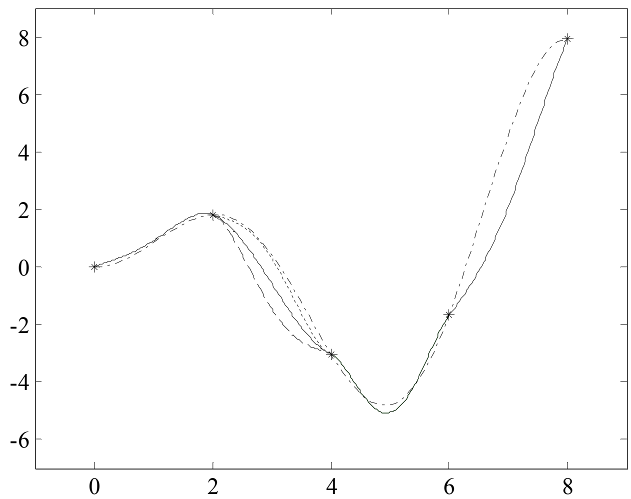


Fig. 5. The quartic Catmull-Rom spline function and the interpolated function

Fig. 5 shows that the interpolation effect of the second segment is better when the parameter is taken as $\alpha_1 = -30$ (short dotted line) but is not ideal when the parameters are taken as $\alpha_1 = 0$ (solid line) and $\alpha_1 = 30$ (long dotted line). In addition, it can also be found from Fig. 5 that the interpolation effect of the first and third segments is better due to α_0 and α_2 are taken as appropriate values. Still, the interpolation effect of the fourth segment is poor because α_3 is taken as an improper value.

Since the shape of the quartic Catmull-Rom spline function can be fine-tuned by modifying the values of the local parameters, we can optimize the local parameters so that the quartic Catmull-Rom spline function achieves the best interpolation effect. To this end, a criterion for evaluating the interpolation effect of the quartic Catmull-Rom spline function is described as follows.

Definition 4. *For $x_i \le x \le x_{i+1}$, the error of the ith quartic Catmull-Rom spline function $s_i(x)$ approximating the interpolated function $y = g(x)$ is expressed by*

$$e_i = \int_{x_i}^{x_{i+1}} \left(s_i(x) - g(x)\right)^2 \mathrm{d}x . \tag{9}$$

The smaller the e_i, the better the interpolation effect of the ith quartic Catmull-Rom spline function.

It is easy to find that when the interpolated function $y = g(x)$ is given, only α_i is variable in Eq. (9). To obtain the quartic Catmull-Rom spline function $s_i(x)$ with the best interpolation effect on the interval $[x_i, x_{i+1}]$, we can obtain the following model,

$$\min_{\alpha_i \in \mathbb{R}} \quad e_i(\alpha_i) = \int_{x_i}^{x_{i+1}} \left(s_i(x) - g(x)\right)^2 \mathrm{d}x . \tag{10}$$

Theorem 2. *Given a continuous function $y = g(x)$ $(a \le x \le b)$, the function values at the node $x_i = a + hi$ $(i = 0,1,\cdots,n)$ are $y_i = g(x_i)$, here $h = (b-a)/n$. Take $x_{-1} = 2x_0 - x_1$, $x_{n+1} = 2x_n - x_{n-1}$, assume $y_{-1} = g(x_{-1})$ and $y_{n+1} = g(x_{n+1})$ exist. When $y_{i-1} - y_i - y_{i+1} + y_{i+2} \ne 0$ $(i = 0,1,\cdots, n-1)$, to obtain the quartic Catmull-Rom spline function $s_i(x)$ with the best interpolation effect on the interval $[x_i, x_{i+1}]$, the best parameter should be taken as*

$$\alpha_i = -\frac{630\int_{x_i}^{x_{i+1}} \left(A_i(x)(B_i(x) - g(x))\right)\mathrm{d}x}{h(y_{i-1} - y_i - y_{i+1} + y_{i+2})^2}, \tag{11}$$

where

$$A_i(x) = (t^2 - 2t^3 + t^4)(y_{i-1} - y_i - y_{i+1} + y_{i+2}),$$
$$t = (x - x_i)/h,$$
$$B_i(x) = \frac{1}{2}\Big((-t + 2t^2 - t^3)y_{i-1} + (2 - 5t^2 + 3t^3)y_i + (t + 4t^2 - 3t^3)y_{i+1} + (-t^2 + t^3)y_{i+2}\Big).$$

Proof. From Eq. (1), we can rewrite Eq. (8) as

$$s_i(x) = A_i(x)\alpha_i + B_i(x) . \tag{12}$$

By substituting Eq. (12) into Eq. (10), we have

$$e_i(\alpha_i) = \alpha_i^2 \int_{x_i}^{x_{i+1}} \left(A_i(x)\right)^2 \mathrm{d}x + 2\alpha_i \int_{x_i}^{x_{i+1}} \left(A_i(x)(B_i(x) - g(x))\right)\mathrm{d}x + \int_{x_i}^{x_{i+1}} \left(B_i(x) - g(x)\right)^2 \mathrm{d}x . \tag{13}$$

From Eq. (13), we have

$$\frac{\mathrm{d}e_i(\alpha_i)}{\mathrm{d}\alpha_i} = 2\alpha_i \int_{x_i}^{x_{i+1}} \left(A_i(x)\right)^2 \mathrm{d}x + 2\int_{x_i}^{x_{i+1}} \left(A_i(x)(B_i(x) - g(x))\right)\mathrm{d}x . \tag{14}$$

To obtain the quartic Catmull-Rom spline function $s_i(x)$ with the best interpolation effect on the interval $[x_i, x_{i+1}]$, there must be $\mathrm{d}e_i(\alpha_i)/\mathrm{d}\alpha_i = 0$. Because

$$\int_{x_i}^{x_{i+1}} \left(A_i(x)\right)^2 \mathrm{d}x = \frac{h}{630}(y_{i-1} - y_i - y_{i+1} + y_{i+2})^2, \tag{15}$$

we can get Eq. (11) from Eqs. (14) and (15) when $y_{i-1} - y_i - y_{i+1} + y_{i+2} \ne 0$. Since the solution of $\mathrm{d}e_i(\alpha_i)/\mathrm{d}\alpha_i = 0$ is unique, Eq. (11) must be the optimal solution of Eq. (10).

After all the optimal values of the parameters are obtained one by one from Eq. (11), the optimal quartic Catmull-Rom spline function interpolating the function $y = g(x)$ $(a \le x \le b)$ can be obtained from Eq. (8).

Example 3. Let $y = g(x) = x\sin x$ $(0 \le x \le 8)$, take

$$x_i = 2i, \; y_i = x_i \sin x_i, \; i = 0,1,2,3,4 .$$

It can be calculated from Eq. (11) that the optimal values of the parameters are $\alpha_0 = 1.4947$, $\alpha_1 = -37.6772$, $\alpha_2 = -1.8879$, $\alpha_3 = -0.6772$. The optimal quartic Catmull-Rom spline function (solid line) and the interpolated function (short dotted line) are shown in Fig. 6.

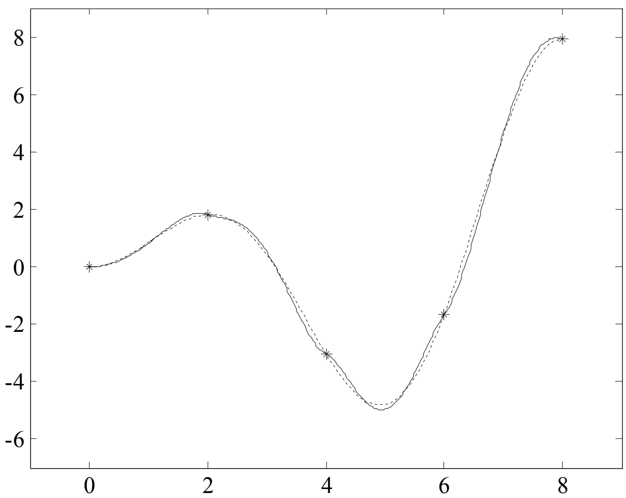


Fig. 6. The optimal quartic Catmull-Rom spline function and the interpolated function

Fig. 6 shows that the optimal quartic Catmull-Rom spline function approximates the interpolated function well.

From Definition 4, the global approximation error of the quartic Catmull-Rom spline function $s_i(x)$ $(i=0,1,\cdots,n-1)$ interpolating the function $y=g(x)$ can be expressed by

$$e=\sum_{i=0}^{n-1}e_i=\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\left(s_i(x)-g(x)\right)^2\mathrm{d}x\,. \tag{16}$$

The following numerical examples are presented to compare the global approximation error of the optimal quartic Catmull-Rom spline function and the cubic Catmull-Rom spline function interpolating the given functions.

Example 4. The interpolation conditions are given as follows,

$$y=g(x)=x\sin x\ \ (0\le x\le 8)\,,$$

$$x_i=\frac{8}{n}i\,,\ i=0,1,\cdots,n\,.$$

When n takes different values, the global approximation errors of the two methods are shown in Table I.

TABLE I
GLOBAL APPROXIMATION ERROR OF THE TWO METHODS IN EXAMPLE 4

n	Cubic Catmull-Rom spline function	Optimal quartic Catmull-Rom spline function
4	0.3964×10^{1}	0.2390×10^{0}
8	0.4241×10^{-1}	0.1166×10^{-1}
12	0.2614×10^{-2}	0.1254×10^{-2}

Fig. 7 shows the optimal quartic Catmull-Rom spline function for $n=4$ (solid line), the cubic Catmull-Rom spline function (short dotted line) and interpolated function (long dotted line).

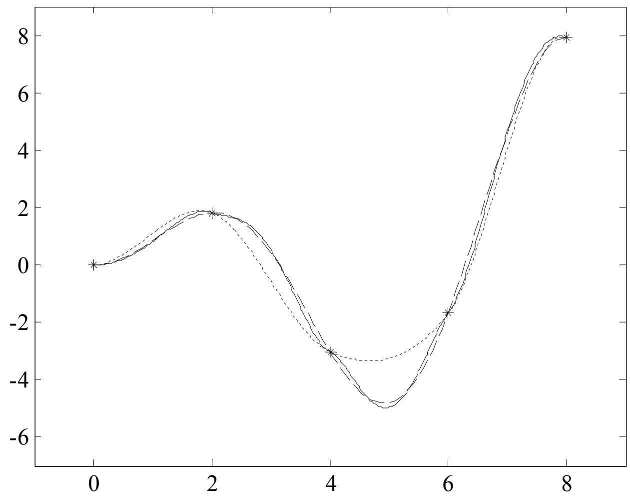


Fig. 7. The optimal quartic Catmull-Rom spline function interpolating $x\sin x$

Example 5. The interpolation conditions are given as follows,

$$y=g(x)=1/x^2\ \ (0.5\le x\le 1.5)\,,$$

$$x_i=0.5+\frac{1}{n}i\,,\ i=0,1,\cdots,n\,.$$

When n takes different values, the global approximation errors of the two methods are shown in Table II.

TABLE II
GLOBAL APPROXIMATION ERROR OF THE TWO METHODS IN EXAMPLE 5

n	Cubic Catmull-Rom spline function	Optimal quartic Catmull-Rom spline function
4	0.2194×10^{-1}	0.4190×10^{-2}
8	0.7052×10^{-4}	0.2391×10^{-4}
12	0.3471×10^{-5}	0.1720×10^{-5}

Fig. 8 shows the optimal quartic Catmull-Rom spline function for $n=4$ (solid line), the cubic Catmull-Rom spline function (short dotted line) and interpolated function (long dotted line).

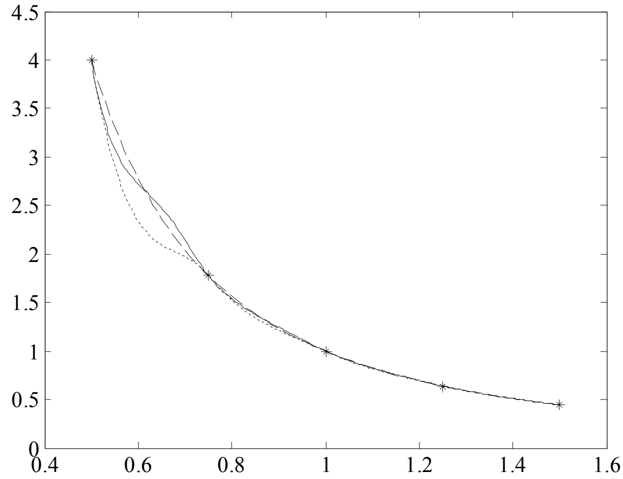


Fig. 8. The optimal quartic Catmull-Rom spline function interpolating $1/x^2$

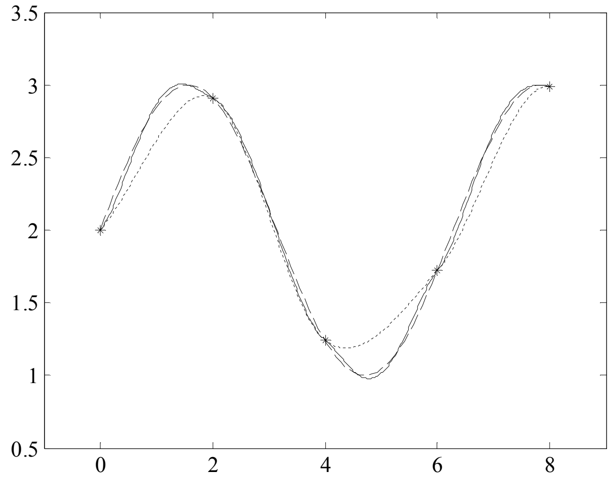


Fig. 9. The optimal quartic Catmull-Rom spline function interpolating $2+\sin x$

Example 6. The interpolation conditions are given as follows,

$$y=g(x)=2+\sin x\ \ (0\le x\le 8)\,,$$

$$x_i=\frac{8}{n}i\,,\ i=0,1,\cdots,n\,.$$

When n takes different values, the global approximation errors of the two methods are shown in Table III.

TABLE III

Global approximation error of the two methods in Example 6

n	Cubic Catmull-Rom spline function	Optimal quartic Catmull-Rom spline function
4	0.1591×10^{0}	0.1005×10^{-1}
8	0.1336×10^{-2}	0.3853×10^{-3}
12	0.8064×10^{-4}	0.3996×10^{-4}

Fig. 9 shows the optimal quartic Catmull-Rom spline function for $n=4$ (solid line), the cubic Catmull-Rom spline function (short dotted line) and interpolated function (long dotted line).

Example 7. The interpolation conditions are given as follows,

$$y=g(x)=(5-x)\cos x\ \ (-1\le x\le 11),$$

$$x_i=-1+\frac{12}{n}i,\ i=0,1,\cdots,n.$$

When n takes different values, the global approximation errors of the two methods are shown in Table IV.

TABLE IV

Global approximation error of the two methods in Example 7

n	Cubic Catmull-Rom spline function	Optimal quartic Catmull-Rom spline function
4	0.4172×10^{2}	0.2037×10^{1}
8	0.9389×10^{0}	0.7009×10^{-1}
12	0.5245×10^{-1}	0.9726×10^{-2}

Fig. 10 shows the optimal quartic Catmull-Rom spline function for $n=4$ (solid line), the cubic Catmull-Rom spline function (short dotted line) and interpolated function (long dotted line).

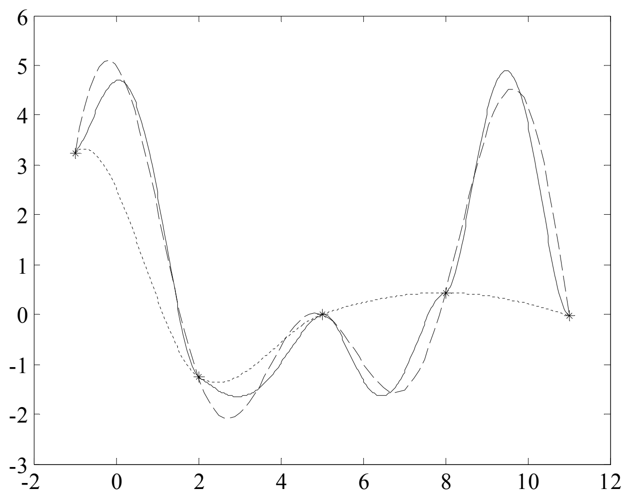


Fig. 10. The optimal quartic Catmull-Rom spline function interpolating $(5-x)\cos x$

Examples 4-7 show that the global approximation error of the optimal quartic Catmull-Rom spline function is significantly smaller than that of the cubic Catmull-Rom spline function, which means the optimal quartic Catmull-Rom spline function has a better interpolation effect than the cubic Catmull-Rom spline function.

IV. Conclusion

Although there is no need to solve equation systems when using the cubic Catmull-Rom spline function to construct interpolation curves, the shapes of the interpolation curves cannot be adjusted when the data points are fixed. This paper proposes a quartic Catmull-Rom spline function that not only inherits the main characteristics of the cubic Catmull-Rom spline function but also can adjust the shape of the interpolation curves locally or globally by the parameters. By determining the optimal values of the parameters, the generated quartic Catmull-Rom spline function can better approximate the given function than the cubic Catmull-Rom spline function.

References

[1] T. D. Derose and B. A. Barsky, "Geometric continuity, shape parameters, and geometric constructions for Catmull-Rom splines", *ACM Transactions on Graphics*, vol.7, no. 1, pp. 1-41, 1988.

[2] M. Maggini, S. Melacci and L. Sarti, "Representation of facial features by Catmull-Rom splines", *Lecture Notes in Computer Science*, vol. 4674, pp. 408-415, 2007.

[3] C. Yuksel, S. Schaefer and J. Keyser, "Parameterization and applications of Catmull-Rom curves", *Computer-Aided Design*, vol. 43, no. 7, pp. 747-755, 2011.

[4] A. Uncini and F. Piazza, "Blind signal processing by complex domain adaptive spline neural networks", *IEEE Transactions on Neural Networks*, vol. 14, no. 2, pp. 399-412, 2003.

[5] M. Scarpiniti, D. Comminiello, R. Parisi and A.Uncini, "Nonlinear spline adaptive filtering", *Signal Processing*, vol. 93, no. 4, pp. 772-783, 2013.

[6] L. Y. Peng and Y. P. Zhu, "A class of trigonometric Bézier basis functions with six shape parameters over triangular domain", *IAENG International Journal of Applied Mathematics*, vol. 49, no. 4, pp. 618-624, 2019.

[7] S. BiBi, M. Abbas, K. Miura, M. Misro, "Geometric modeling of novel generalized hybrid trigonometric Bézier-like curve with shape parameters and its applications", *Mathematics*, vol. 8, 2020.

[8] M. Hussain, S. Zafar, M. Hussain, "Energy-minimizing cubic H-Bézier curve", *Journal of Testing and Evaluation*, vol. 49, 2021.

[9] M. Ameer, M. Abbas, T. Abdeljawad and T. Nazir, "A novel generalization of Bézier-like curves and surfaces with shape parameters", *Mathematics*, vol. 10, 2022.

[10] X. Kong and J. Chen, "Two extensions of the quadratic nonuniform B-spline curve with local shape parameter series", *Mathematical Problems in Engineering*, 2021.

[11] X. Liu, W. Xu, Y. Guan and Y Shang, "Hyperbolic polynomial uniform B-spline curves and surfaces with shape parameter", *Graphical Models*, vol. 72, pp. 1-6, 2010.

[12] A. Majeed and F. Qayyum, "New rational cubic trigonometric B-spline curves with two shape parameters", *Computational and Applied Mathematics*, vol. 39, 2020.

[13] X. B. Qin and Q. S. Xu, "C^1 positivity-preserving interpolation schemes with local free parameters", *IAENG International Journal of Computer Science*, vol. 43, no. 2, pp. 219-227, 2016.

[14] X. B. Qin and Q. S. Xu, "C^1 rational cubic/linear trigonometric interpolation spline with positivity-preserving property", *Engineering Letters*, vol. 25, no. 2, pp. 152-159, 2017.

[15] J. C. Li and C. Z. Liu, "A class of polynomial spline curve with free parameters that naturally interpolates the data points", *IAENG International Journal of Applied Mathematics*, vol. 50, no. 3, pp. 643-647, 2020.

[16] J. C. Li, C. Z. Liu and S. J. Liu, "The quartic Catmull-Rom spline with local adjustability and its shape optimization", *Advances in Continuous and Discrete Models*, 2022.