Dynamical Analysis of an Optimal Iterative Scheme and its Real-Life Applications

Navya Kakarlapudi, Mani Sandeep Kumar Mylapalli, *Member, IAENG,* Pravin Singh

***Abstract*— The weight function approach of solving nonlinear equations is presented in this work as a two-step, fourth-order, optimal iterative method. It comes from the idea of the theoretical order of convergence. The performance of the novel method has been assessed on a small number of real-life applications from different domains and contrasted with the fourth-order methods currently in use. Also, we find that this has fewer errors and iterations for the same order than the methods now in use. To examine the dynamical analysis of the proposed method using computer tools, we test some challenging polynomials that explain convergence and other graphical features of the iterative scheme.**

***Index Terms*—Iterative Method, Nonlinear Equation, Convergence Order, Efficiency Index, Weight function, Basins of Attraction.**

I. Introduction

FINDING the solution to a nonlinear equation is common in science and engineering. Analytical methods are exceedingly challenging to use when solving such equations. Their importance has led to the suggestion of several different numerical methods. Newton's method (NM) is the iterative method that is most frequently used to solve the nonlinear equation [6].

Finding the answer to a nonlinear equation

$$h(x)=0 \tag{1.1}$$

and is given by

$$x_{n+1}=x_n-\frac{h(x_n)}{h'(x_n)}, \qquad n=0,1,2,\dots \tag{1.2}$$

which is quadratically convergent for simple zeros. This is the best second-order approach, with an efficiency index (E.I) of $P^{1/n}=1.414$, where P is the convergence order and n is the number of function evaluations.

Several methods have recently been developed that assess additional functions and first-order derivatives to give higher-order convergence or greater effectiveness. One of the current strategies for enhancing the procedures' order is using weight functions.

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Navya Kakarlapudi is a PhD candidate of GITAM (Deemed to be University), Visakhapatnam 530040, INDIA (e-mail: nkakarla@gitam.in).

Mani Sandeep Kumar Mylapalli is an Assistant Professor of Mathematics, GITAM Institute of Science, GITAM (Deemed to be University), Visakhapatnam 530040, INDIA (corresponding author phone: +91 9989865011; e-mail: mmylapal@gitam.edu).

Pravin Singh is an Associate Professor of Mathematics, University of KwaZulu Natal, Durban 4001, SOUTH AFRICA (e-mail: singhprook@gmail.com).

In this research, we proposed some weight function-based fourth-order iterative techniques for nonlinear equations:

Prem B. Chand [1] provided the best two-step fourth-order approach (PJ) as

$$\begin{aligned} y_n &= x_n-\frac{h(x_n)}{h'(x_n)} \\ x_{n+1} &= x_n-\left(1+2\left(\frac{h(y_n)}{h(x_n)}\right)^2\right)\frac{h(x_n)+h(y_n)}{h'(x_n)} \end{aligned} \tag{1.3}$$

King [5] proposed a new fourth-order optimal iterative method (KI):

$$\begin{aligned} y_n &= x_n-\frac{h(x_n)}{h'(x_n)} \\ x_{n+1} &= y_n-\left(\frac{3h(y_n)+h(x_n)}{h(x_n)+h(y_n)}\right)h(y_n)h'(x_n) \end{aligned} \tag{1.4}$$

Francisco I. Chicharro [4] presented the following optimal fourth-order iterative method (FI):

$$\begin{aligned} y_n &= x_n-\frac{h(x_n)}{h'(x_n)} \\ x_{n+1} &= x_n-\frac{h^2(x_n)+h(x_n)h(y_n)+2h^2(y_n)}{h(x_n)h'(x_n)} \end{aligned} \tag{1.5}$$

Potra-P'tak [3] presented the following new three-step optimal fourth-order scheme (PP):

$$\begin{aligned} y_n &= x_n-\frac{h(x_n)}{h'(x_n)} \\ z_n &= x_n-\frac{h(x_n)+h(y_n)}{h'(x_n)} \\ x_{n+1} &= z_n-\frac{(h(y_n))^2}{(h(x_n))^2}\left(2h(x_n)+h(y_n)\right)h'(x_n) \end{aligned} \tag{1.6}$$

Mudassir [14] presented an optimal fourth-order iterative method (TI):

$$\begin{aligned} y_n &= x_n-\frac{h(x_n)}{h'(x_n)} \\ x_{n+1} &= y_n-\frac{h(y_n)}{h'(x_n)}\left(\beta\left(h(y_n)\right)^2+\left(1+\frac{2}{3}\mu\right)^3\right) \end{aligned} \tag{1.7}$$

where $\mu=\dfrac{h(y_n)}{h(x_n)}$ and $\beta\in R$.

The novel two-step optimal iterative approach (PS) of order four that Parimala [12] presented is provided by:

$$y_n = x_n - \frac{h(x_n)}{h'(x_n)}$$
$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}\left(1+\tau+2\tau^2+\frac{2}{3}(\tau)^3\right) \quad (1.8)$$
where $\tau = \frac{h(y_n)}{h(x_n)}$

This study derived an ideal fourth-order iterative scheme using the weight function technique and showed that convergence analysis produces an ideal one. In the following section, we put the suggested strategy into reality with a few real-life applications. Numerical comparisons are provided in Tables I and II to show how our proposed solution is better than the existing methods for the same order. The dynamical comparison of the recommended iterative strategy with comparable methods has also been shown using the PYTHON computer application.

Definition-1: Intermediate value theorem [7]:

A function $h(x)$ is continuous on an interval $[a,b] \subset R$ and $h(a).h(b) < 0$, then, "(1.1)" has at least one real root in the interval (a,b).

Definition-2: Order of convergence [13]:

Let $h(x)$ be a real function with a simple root $x_0 \in R$ and $x_1, x_2, x_3, ...$ be a sequence of real numbers that converges to the root x_0. Then, the order of convergence of the sequence $p \in R^+$ is defined as,

$$\lim_{n\to\infty} \frac{x_{n+1} - x_0}{\left(x_n - x_0\right)^p} \approx C$$

Where $C \neq 0$ is known as the asymptotic error constant.

Definition-3: Error Equation:

The relationship $\varepsilon_{n+1} = C\varepsilon_n^p + O\left(\varepsilon_n^{p+1}\right)$ is referred to as an error equation if $\varepsilon_n = x_n - x_0$ represents the error in the nth iteration.

Definition-4: Efficiency Index [16]:

Let d represent how many times the function was utilized in the specified algorithm and the order of convergence is indicated by p, then the efficiency index is defined as $E.I = p^{1/d}$.

Definition-5: Computational order of convergence (COC) [15]:

Let x be the root of the nonlinear equation $h(x) = 0$, and let x_{n-1}, x_n, x_{n+1} be three successive iterations close to the root. Then the COC can be approximated below

$$\rho = \frac{\log\left(\left|x_{n+1} - x\right| / \left|x_n - x\right|\right)}{\log\left(\left|x_n - x\right| / \left|x_{n-1} - x\right|\right)}$$

It is used to check the convergence order of a given iterative scheme.

Definition-6: Weight function [10]:

The weight function is a real variable sufficiently differentiable function defined on a given interval. These are introduced in an iteration scheme via any arithmetical operation. The main reason for the intrusion of weight function to some specific entity in an iteration process is to enhance the behavior of the iterative method and assist in improving its order of convergence and computational efficiency.

II. OPTIMAL FOURTH-ORDER CONVERGENT ITERATIVE METHOD

Consider x^* is an exact root of the nonlinear equation "(1.1)" where $h(x)$ is continuous and has well-defined first-order derivatives. Let x_n be the root of the n[th] approximation of "(1.1)". Then

$$x^* = x_n + \varepsilon_n \quad (2.1)$$

where ε_n is the error. Thus, we get

$$h\left(x^*\right) = 0 \quad (2.2)$$

writing $h(x^*)$ by Taylor's series about x_n, we have

$$h(x^*) = h(x_n) + (x^* - x_n)h'(x_n) + \frac{(x^* - x_n)^2}{2!}h''(t_n) + ...$$
$$h(x^*) = h(x_n) + \varepsilon_n h'(x_n) + \frac{\varepsilon_n^2}{2!}h''(x_n) + ... \quad (2.3)$$

Here higher powers of ε_n are neglected from ε_n^3 onwards. Using "(2.2)", and "(2.3)", We have

$$\varepsilon_n^2 h''(x_n) + 2\varepsilon_n h'(x_n) + 2h(x_n) = 0$$
$$\varepsilon_n = \left[-2h'(x_n) \pm \sqrt{4h'(x_n) - 8h(x_n)h''(x_n)}\right] \div 2h''(x_n) \quad (2.4)$$

On substituting x^* by x_{n+1} in "(2.1)" and from "(2.4)", we get

$$x_{n+1} = y_n - \frac{2h(y_n)}{h'(y_n)}.\frac{1}{1+\sqrt{1-2\rho_n}}$$

Where, $\rho_n = \frac{h'(x_n) - h'(y_n)}{h'(x_n)}$ and

$h'(y_n) = 2h[y_n, x_n] - h'(x_n)$ has third-order convergence. Now we introduce a weight function $H(\tau) = 1 - \tau$ and $\tau = \frac{h(y_n)}{h(x_n)}$. We get

$$x_{n+1} = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)}.\frac{1}{1+\sqrt{1-2\rho_n}}\right] \quad (2.5)$$

We develop the algorithm by taking "(1.2)" as the first step and "(2.5)" as the second step.

Algorithm: The iterative scheme is computed by x_{n+1} as

1. $y_n = x_n - \frac{h(x_n)}{h'(x_n)}$

2. $x_{n+1} = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)} \cdot \frac{1}{1+\sqrt{1-2\rho_n}}\right]$ (2.6)

where, $\rho_n = \frac{h'(x_n)-h'(y_n)}{h'(x_n)}$, $h'(y_n) = 2h[y_n, x_n] - h'(x_n)$,

$H(\tau) = 1-\tau$ and $\tau = \frac{h(y_n)}{h(x_n)}$

The above method has fourth-order convergence with three functional evaluations at each step, and the efficiency index is 1.5874 and it is denoted with SS.

III. Convergence Criteria

Theorem [9]: Let $x_0 \in D$ be a single zero of a sufficiently differentiable function h for an open interval D. If x_0 is the neighborhood of x^*. Then the algorithm "(2.6)" has fourth-order convergence.

Proof: Let the single zero of $h(x) = 0$ be x^* and $x^* = x_n + \varepsilon_n$. Thus, $h(x^*) = 0$

Using Taylor's series, writing $h(x^*)$ about x_n, we get

$$h(x_n) = h'(x^*)\left(\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + ...\right) \quad (3.1)$$

$$h'(x_n) = h'(x^*)\left(1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + ...\right) \quad (3.2)$$

Now, we get

$$y_n = x^* + c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n^4 + \dots \quad (3.3)$$

From "(3.3)", we get

$$h(y_n) = h'(x^*)\left(c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_4 - 7c_2c_3 + 5c_2^3\right)\varepsilon_n^4 + \dots\right) \quad (3.4)$$

$$h'(y_n) = h'(x^*)\left(1 + \left(2c_2^2 - c_3\right)\varepsilon_n^2 + \left(6c_2c_3 - 4c_2^3 - 2c_4\right)\varepsilon_n^3 + ..\right) \quad (3.5)$$

and $\frac{h(y_n)}{h'(y_n)} = c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_2^3 - 6c_2c_3 + 3c_4\right)\varepsilon_n^4 + \dots$ (3.6)

From $\rho_n = \frac{h'(x_n)-h'(y_n)}{h'(x_n)}$, we get

$$\rho_n = 2c_2\varepsilon_n + \left(4c_3 - 6c_2^2\right)\varepsilon_n^2 + \left(6c_4 + 16c_2^3 - 20c_2c_3\right)\varepsilon_n^3 + ... \quad (3.7)$$

From "(3.6)",

$$H(\tau) = \begin{pmatrix} 1 - c_2\varepsilon_n - \left(2c_3 - 3c_2^2\right)\varepsilon_n^2 \\ -\left(3c_4 - 10c_2c_3 + 8c_2^3\right)\varepsilon_2^3 + ... \end{pmatrix} \quad (3.8)$$

Putting "(3.3)", "(3.6)", "(3.7)" and "(3.8)" in the second step of "(2.6)", we get

$$\varepsilon_{n+1} = -c_2c_3\varepsilon_n^4 + o\left(\varepsilon_n^5\right)$$

Thus, we derived the convergence of this method which is of fourth order, and its efficiency index is $\sqrt[3]{4} = 1.5874$.

IV. Numerical Examples

This section used a few real-life applications to demonstrate how well the innovative fourth-order iterative technique (SS) performed. We compared the proposed method to various fourth-order methods, such as PJ, KI, FI, PP, TI, and PS. Using the high precision mpmath library and an Intel(R) Core (TM) i5-10210U CPU clocked at 2.11 GHz with a 64-bit operating system, we employ PYTHON for all numerical operations. We take on the stopping criterion $|f(x_n)| < \varepsilon$, with the required precision set to 200 decimal places and the tolerance set to $\varepsilon = 10^{-100}$.

Some real-life applications:

In this section, we present some applications and compare our results to some existing methods in Table II, and the efficiency index is shown in Table I.

Table I
Comparison of Efficiency Index

Methods	p	n	$E.I$
PJ	4.00	3	1.587
KI	4.00	3	1.587
FI	4.00	3	1.587
PP	4.00	3	1.587
TI	4.00	3	1.587
PS	4.00	3	1.587
SS	4.00	3	1.587

where the efficiency-index, the number of functional values each iteration, and the convergence order are $E.I$, n and p.

Application 1. (Depth of Embedment Model, [12])

The following nonlinear equation determines a sheet-pile wall's embedment depth:

$$h_1(x) = \frac{1}{4.62}\left(x^3 + 2.87x^2 - 10.28\right) - x$$

The approximated root is 2.0021187789538272.

Application 2. (Chemical equilibrium problem / Fractional Conversion, [8])

The equation for the fractional conversion of nitrogen, using hydrogen feed that is transformed to ammonia at a "Temperature of 500^0 C" and a "Pressure of 250 atm", is

$$h_2(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674$$

The real root is 0.2777595428417206.

Application 3. (Azeotropic point of a binary solution, [11])

To find out the azeotropic point of a binary solution of the nonlinear equation:

$$h_3(x) = \frac{PQ\left[Q(1-x)^2 - Px^2\right]}{\left[x(P-Q)+Q\right]^2} + 0.14845$$

We took $P = 0.38969$ and $Q = 0.55954$ for this problem. The root of this equation is 0.69147373574714144.

Application 4. (The vertical stress, [12])

Vertical stress is one of the basic stresses experienced by finite underground structures and is given by

$$h_4(x) = \frac{x + CosxSinx}{\pi} - \frac{1}{4}$$

The root of the nonlinear equation $h_4(x)=0$ is 0.416044498810076704З.

Application 5. (Plank's Constant, [17])

The solution to Planck's radiation law problem is

$$h_5(x)=e^{-x}-1+\frac{x}{5}$$

The approximate root of this equation is given by 4.96511423174427630369.

Application 6. (Parachutist's problem [2])

The nonlinear equation in velocity of the parachutist is

$$h_6(x)=\frac{gm}{x}\left(1-e^{-\frac{x}{m}t}\right)-v$$

We took the values of the parameters as "$g=9.8$ m/s^2", "$m=68$ kg", "$t=8$ s", and "$v=41$ m/s". Then the root of the nonlinear equation is 12.533522848184467.

Application 7. (Study of Multifactor effect [17])

The moment of an electron in the space between two parallel plates is

$$\mathrm{x(t)}=x_0+\left(v_0+eE_0(\mathrm{m}\,w)^{-1}\sin(wt_0+\eta)\right)(t-t_0)$$

$$+eE_0\left(\mathrm{m}\,w^2\right)^{-1}\left(\cos(wt_0+\eta)+\sin(wt_0+\eta)\right)$$

Where x_0 is the position of the electron, v_0 is the velocity, e is the charge, m is the mass of the electron at rest, and $\mathrm{E}_0\sin(wt_0+\eta)$ is the RF electric field between plates at a time t_0. For the particular values, it can be reduced in polynomial form as

$$h_7(x)=x-0.5\cos x+\frac{\pi}{4}$$

This function has a simple root at $x^*\approx-0.309466139208214$.

Application 8. (Chemical Engineering)

The equation that Provides the chemical concentration in a mixed reactor is

$$\mathrm{h}_8(x)=1-0.75e^{-0.05x}$$

The root of the nonlinear equation is -5.753641449035618.

Table II

M	Itr1	Itr2	Itr3	Itr4	Itr5	CPU
	$h_1(x)$	$\mathbf{X_0}$	1.6			
PJ	0.4134	0.0113	4.56e-09	1.23e-34	6.63e-137	0.004343
KI	0.5138	0.1116	7.70e-05	2.47e-17	2.59e-67	0.004747
FI	0.4533	0.0512	2.95e-06	3.74e-23	9.60e-91	0.004487
PP	0.4334	0.0312	3.45e-07	5.49e-27	3.53e-106	0.003829
TI	0.3924	0.0097	3.18e-09	3.62e-35	1.16e-120	0.009011
PS	0.4400	0.0379	7.05e-07	1.69e-25	3.49e-100	0.004678
SS	0.3999	0.0021	5.33e-13	1.99e-51	5.55e-200	0.003645
	$h_1(x)$	$\mathbf{X_0}$	3.8			
PJ	1.5799	0.2175	0.0004	9.34e-15	2.18e-57	0.006128
KI	1.5194	0.0766	0.0019	8.93e-12	4.45e-45	0.006769
FI	1.5463	0.2505	0.0011	5.81e-13	5.62e-50	0.006694
PP	1.5631	0.2341	6.80e-04	8.73e-14	2.27e-53	0.005989
TI	5.2579	0.0838	4.34e-06	4.33e-23	6.91e-85	0.006351
PS	1.5575	0.2395	7.95e-04	1.70e-13	3.51e-52	0.006972
SS	1.8608	0.0629	4.64e-07	1.15e-27	4.32e-110	0.005527
	h2(x)	$\mathbf{X_0}$	0.1			
PJ	0.1727	0.0059	4.05e-09	9.32e-34	2.60e-132	0.003996
KI	0.3712	0.1918	0.0016	5.01e-11	4.49e-41	0.005432
FI	0.2459	0.0681	5.90e-05	6.43e-17	9.02e-65	0.006339
PP	0.2148	0.0371	5.84e-06	5.08e-21	2.90e-81	0.002875
TI	0.1578	0.0199	7.76e-07	1.47e-24	3.35e-88	0.002872
PS	0.2252	0.0474	1.51e-05	2.43e-19	1.64e-74	0.003075
SS	0.1778	0.0001	8.82e-21	4.37e-81	8.16e-202	0.002786
	h2(x)	$\mathbf{X_0}$	0.5			
PJ	0.2206	0.0016	2.46e-11	1.26e-42	8.77e-168	0.003979
KI	0.2199	0.0023	1.92e-10	9.64e-39	6.13e-152	0.002425
FI	0.2202	0.0019	7.98e-11	2.14e-40	1.10e-158	0.002471
PP	0.4404	0.0018	4.61e-11	1.97e-41	6.62e-163	0.002605
TI	0.2214	0.0007	1.62e-12	2.78e-47	2.24e-156	0.002483
PS	0.2203	0.0019	5.58e-11	4.52e-41	1.95e-161	0.002615
SS	0.2214	0.0008	3.52e-13	1.10e-50	1.83e-200	0.002303
	h3(x)	$\mathbf{X_0}$	0.4			
PJ	0.2942	0.0027	2.27e-11	1.07e-43	5.23e-173	0.003374
KI	0.3086	0.0171	9.74e-08	1.05e-28	1.42e-112	0.003518
FI	0.3005	0.0091	5.30e-09	6.22e-34	1.18e-133	0.003494
PP	0.2973	0.0059	7.29e-10	1.63e-37	4.25e-148	0.003604
TI	0.2963	0.0048	2.84e-10	3.48e-39	1.55e-132	0.003643
PS	0.2984	0.0069	1.55e-09	3.76e-36	1.31e-142	0.003721
SS	0.2902	0.0012	3.80e-13	3.56e-51	3.26e-201	0.002953
	h3(x)	$\mathbf{X_0}$	0.9			
PJ	0.2078	0.0006	8.97e-14	2.62e-53	1.63e-201	0.003445
KI	0.2070	0.0015	5.73e-12	1.26e-37	2.95e-180	0.003397
FI	0.2074	0.0011	1.18e-12	1.54e-48	4.39e-192	0.003578
PP	0.2076	0.0008	3.85e-13	1.31e-50	2.53e-200	0.003565
TI	0.2077	0.0008	2.24e-13	1.35e-51	8.98e-180	0.003531
PS	0.2075	0.0009	5.76e-13	7.27e-50	1.84e-197	0.003696
SS	0.2088	0.0003	2.44e-15	6.07e-60	3.26e-201	0.003320
	h4(x)	$\mathbf{X_0}$	1			
PJ	4.2896	4.7622	0.1114	9.41e-05	2.95e-17	0.008302
KI	10.443	247.15	1210.1	932.45	17.8354	0.041744
FI	5.2232	38.063	33.054	0.4793	0.1105	0.010576
PP	0.4668	0.1172	0.0001	3.09e-16	4.22e-63	0.008784
TI	0.9309	0.3458	0.0013	7.03e-13	1.06e-49	0.008879
PS	2.0323	7.7807	5.6426	0.9184	0.2482	0.011472
SS	0.5341	0.0497	7.45e-07	3.64e-26	2.08e-103	0.008160
	h4(x)	$\mathbf{X_0}$	-0.5			
PJ	0.9724	0.0563	4.88e-06	2.13e-22	7.80e-88	0.008397
KI	0.9966	0.0806	5.13e-05	5.04e-18	4.64e-70	0.007385
FI	0.9833	0.0673	1.65e-05	4.14e-20	1.61e-78	0.008348
PP	0.9778	0.0618	9.33e-06	3.52e-21	7.08e-83	0.007697
TI	0.9747	0.0586	6.90e-06	9.85e-22	2.07e-80	0.009156
PS	0.9747	0.0636	1.14e-05	8.26e-21	2.29e-81	0.008323
SS	0.9642	0.0482	6.58e-07	2.22e-26	2.86e-104	0.006970
	h5(x)	$\mathbf{X_0}$	10			
PJ	5.0341	0.0007	3.41e-17	1.72e-70	3.91e-200	0.004657
KI	5.0341	0.0007	4.17e-17	4.55e-70	7.83e-200	0.011100
FI	5.0341	0.0007	3.79e-17	2.85e-70	2.61e-201	0.004041
PP	5.0341	0.0007	3.60e-17	2.22e-70	3.26e-200	0.004398
TI	5.0341	0.0006	2.92e-17	9.94e-70	3.91e-200	0.005328
PS	5.0341	0.0007	3.66e-17	2.42e-70	3.26e-200	0.004198
SS	5.0342	0.0007	2.87e-17	7.37e-71	6.53e-201	0.003755
	h5(x)	$\mathbf{X_0}$	3			
PJ	2.0112	0.0461	5.38e-10	1.06e-41	1.59e-168	0.004876
KI	2.0992	0.1342	4.01e-08	3.89e-34	3.45e-138	0.007342
FI	2.0503	0.0852	6.47e-09	2.42e-37	4.75e-151	0.008089
PP	2.0307	0.0656	2.25e-09	3.39e-39	1.74e-158	0.004057
TI	1.9885	0.0398	3.18e-10	2.11e-44	3.92e-173	0.005817
PS	2.0372	0.0721	3.30e-09	1.59e-38	8.66e-156	0.007978
SS	1.9861	0.0210	2.06e-11	1.98e-47	1.67e-181	0.003522
	h6(x)	$\mathbf{X_0}$	15			
PJ	2.4701	0.0034	1.17e-14	1.65e-60	1.35e-198	0.005478
KI	2.4797	0.0130	7.05e-12	6.03e-49	3.25e-197	0.012790
FI	2.4746	0.0079	6.73e-13	3.39e-53	9.14e-199	0.005292
PP	2.4723	0.0057	1.33e-13	3.94e-56	1.13e-198	0.008539
TI	2.4316	0.0349	1.70e-10	9.35e-44	1.04e-148	0.007365
PS	2.4731	0.0065	2.43e-13	4.82e-55	6.40e-199	0.006078
SS	2.4654	0.0012	6.51e-17	5.62e-70	2.22e-199	0.004944
	h6(x)	$\mathbf{X_0}$	3			

PJ	9.1118	0.4215	2.69e-06	4.57e-27	3.78e-110	0.012908
KI	8.8592	0.6740	4.59e-05	1.09e-21	3.39e-88	0014644
FI	8.9732	0.5601	1.54e-05	9.20e-24	1.18e-96	0.012069
PP	9.0425	0.4908	7.03e-06	3.08e-25	1.13e-102	0.005793
TI	DIV	DIV	DIV	DIV	DIV	DIV
PS	9.0194	0.5139	9.27e-06	1.03e-24	1.54e-100	0.007359
SS	9.8436	0.3103	2.90e-07	2.22e-31	7.56e-128	0.005267
	h7(x)	X_0	-0.2			
PJ	0.1094	7.6e-6	1.99e-22	9.08e-89	2.04e-200	0.002129
KI	0.1094	1.7e-5	1.38e-20	5.35e-81	1.34e-200	0.002545
FI	0.1094	1.2e-5	2.63e-21	4.87e-84	1.55e-200	0.002381
PP	0.1095	1.0e-5	8.52e-22	4.23e-86	7.34e-201	0.002494
TI	0.1094	9.2e-6	5.39e-22	1.37e-81	3.67e-201	0.002495
PS	0.1094	1.1e-5	1.28e-21	2.32e-85	6.94e-201	0.002444
SS	0.1094	8.6e-7	4.78e-27	4.3e-108	0	0.002040
	h7(x)	X_0	-1			
PJ	0.7008	0.0102	6.43e-10	9.93e-39	5.63e-154	0.002383
KI	0.7748	0.0843	6.44e-06	2.53e-22	5.97e-88	0.002867
FI	0.7321	0.0416	2.91e-07	7.83e-28	2.96e-110	0.002738
PP	0.7164	0.0259	3.54e-08	1.26e-31	2.00e-125	0.002748
TI	0.7088	0.0182	7.99e-09	2.97e-34	2.28e-118	0.002845
PS	0.7217	0.0311	7.98e-08	3.55e-30	1.39e-119	0.002937
SS	0.6855	0.0049	5.27e-12	6.49e-48	1.49e-191	0.002252
	h8(x)	X_0	-15			
PJ	9.0842	0.6124	2.50e-08	1.43e-35	1.50e-144	0.002375
KI	8.9600	0.2840	6.23e-07	1.49e-29	4.85e-120	0.002344
FI	9.0186	0.0077	1.78e-07	6.83e-32	1.47e-129	0.002364
PP	9.0514	0.1949	7.41e-08	1.57e-33	3.14e-136	0.002440
TI	9.0752	0.1711	3.97e-08	1.16e-34	1.08e-121	0.002447
PS	9.0405	0.2058	1.01e-07	5.99e-33	7.38e-134	0.002416
SS	9.3118	0.0655	1.92e-10	1.42e-44	4.28e-181	0.002202
	h8(x)	X_0	2			
PJ	7.8917	0.1381	1.32e-08	1.10e-36	5.33e-149	0.002344
KI	DIV	DIV	DIV	DIV	DIV	DIV
FI	8.3032	0.5496	5.87e-06	8.05e-26	2.84e-105	0.002313
PP	8.0975	0.3438	7.09e-07	1.32e-29	1.57e-120	0.002371
TI	8.0247	0.2710	2.48e-07	1.78e-31	3.89e-112	0.002655
PS	8.1661	0.4124	1.60e-06	3.78e-28	1.16e-114	0.002505
SS	7.7035	0.0501	6.55e-11	1.92e-46	1.41e-188	0.002229

Where M represents methods, x_0 represents the starting approximation and DIV represents divergent.

The graph of residual fall for nonlinear equations using simultaneous methods PJ, KI, FI, PP, TI, PS, and SS in the aforementioned practical applications.

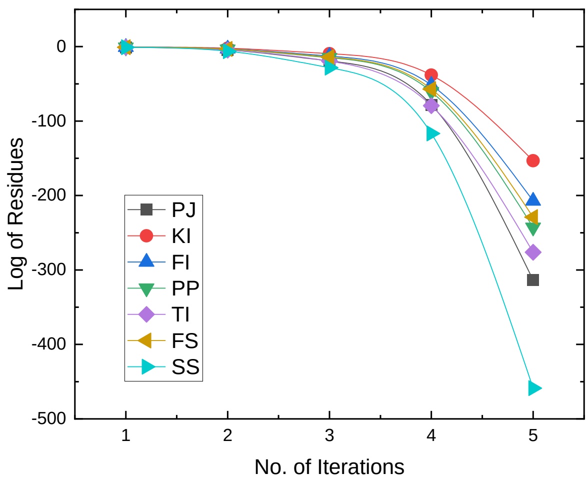


Fig. 1. $h_1(x)$ at x_0=1.6

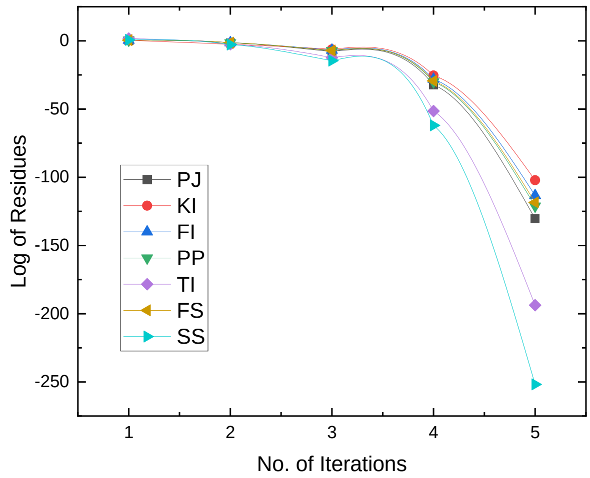


Fig. 2. $h_1(x)$ at x_0=3.8

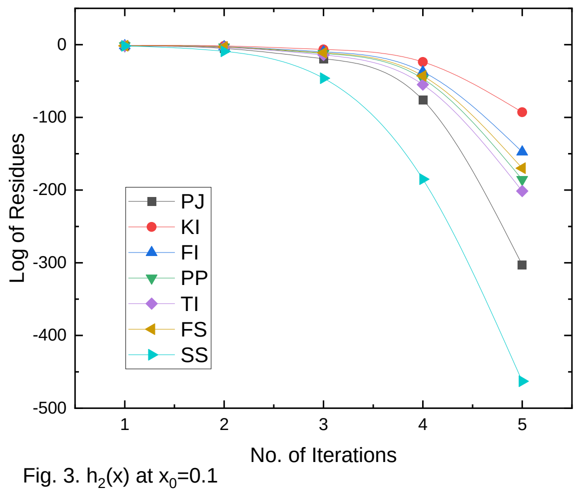


Fig. 3. $h_2(x)$ at x_0=0.1

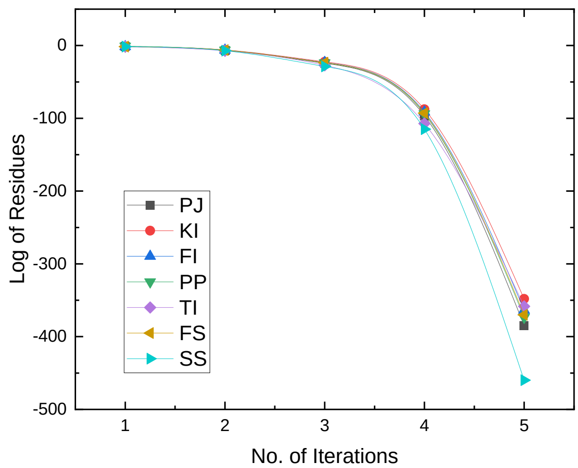


Fig. 4. $h_2(x)$ at x_0=0.5

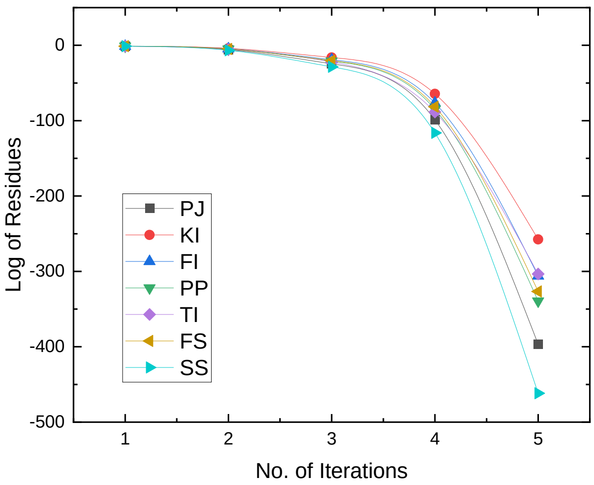


Fig. 5. $h_3(x)$ at x_0=0.4

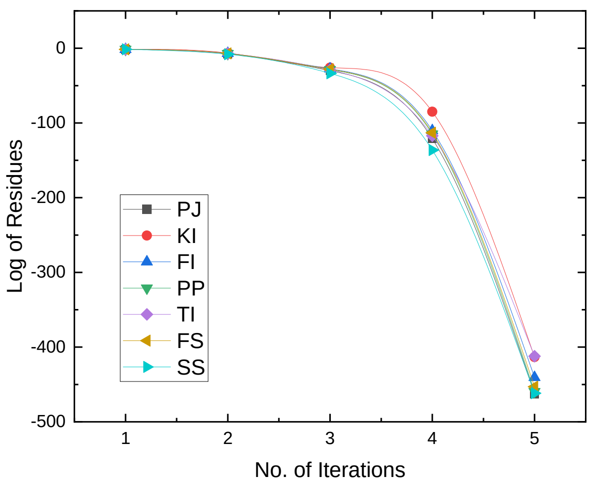


Fig. 6. $h_3(x)$ at x_0=0.9

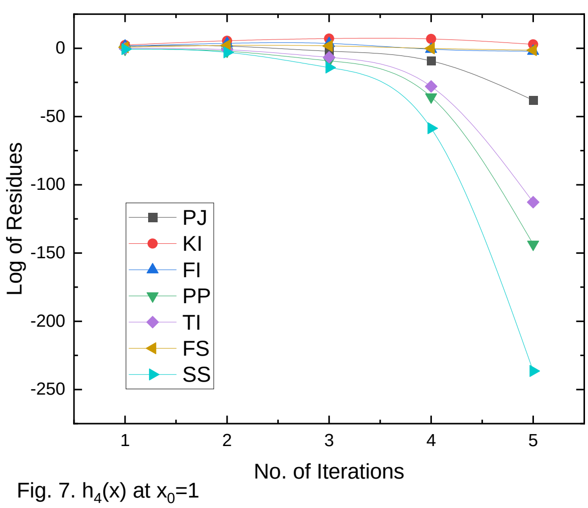


Fig. 7. $h_4(x)$ at x_0=1

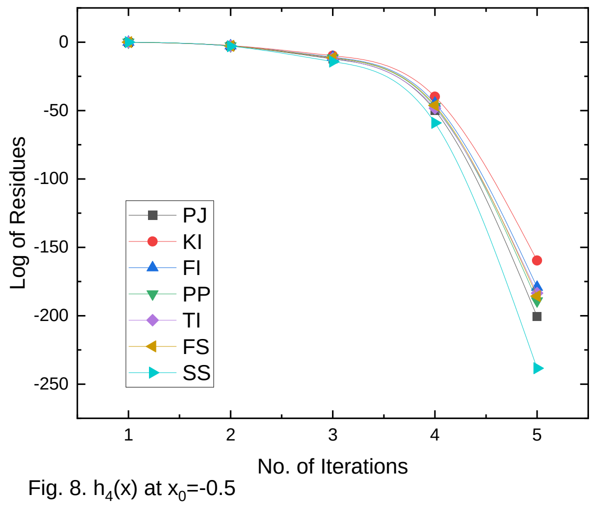


Fig. 8. $h_4(x)$ at x_0=-0.5

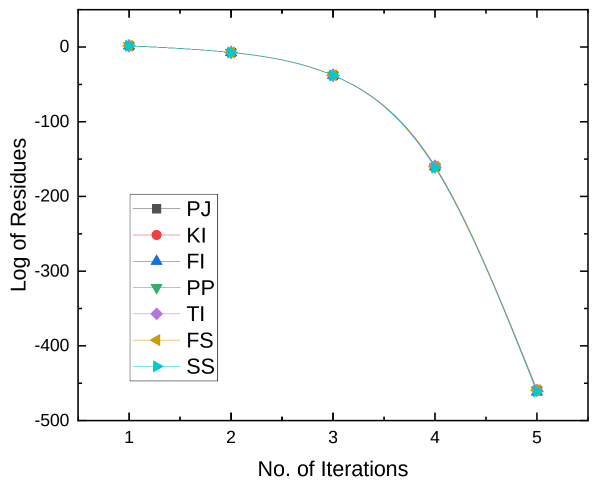


Fig. 9. $h_5(x)$ at x_0=10

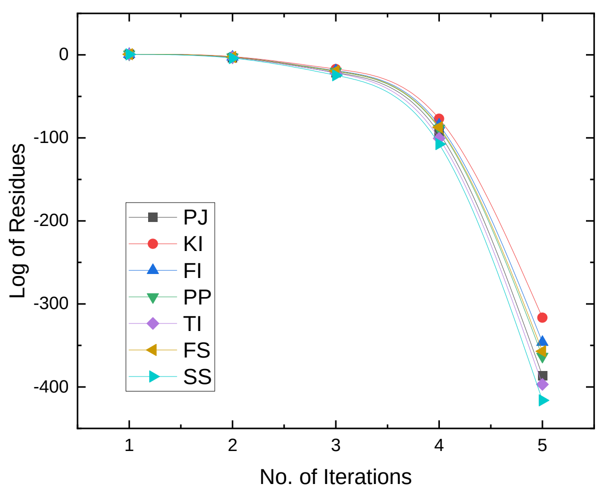


Fig. 10. $h_5(x)$ at x_0=3

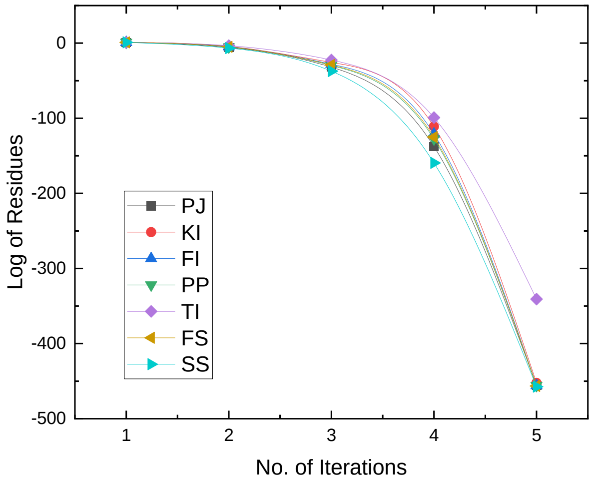


Fig. 11. $h_6(x)$ at x_0=1.5

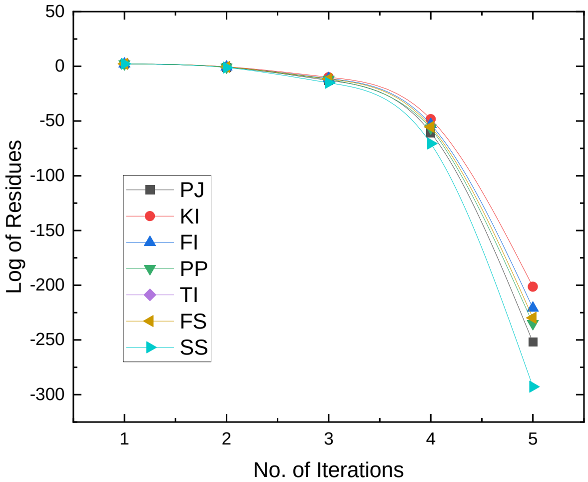


Fig. 12. $h_6(x)$ at x_0=3

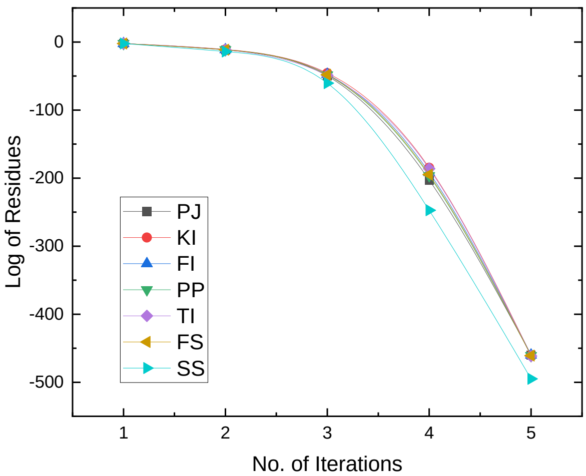


Fig. 13. $h_7(x)$ at x_0=-0.2

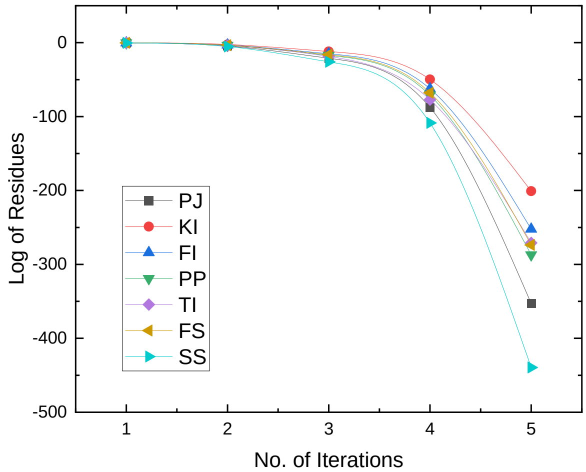


Fig. 14. $h_7(x)$ at x_0=-1

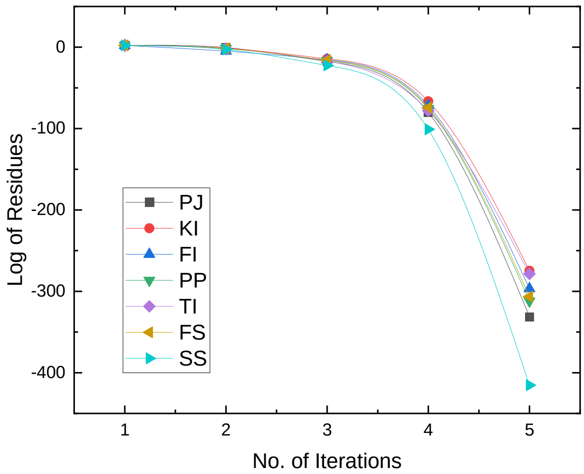


Fig. 15. $h_8(x)$ at x_0=-15

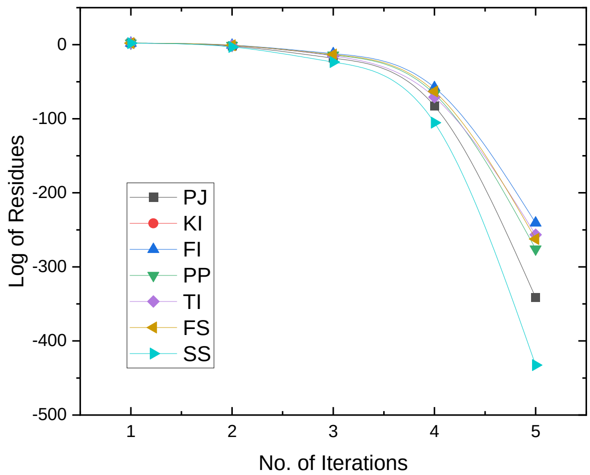


Fig. 16. $h_8(x)$ at x_0=2

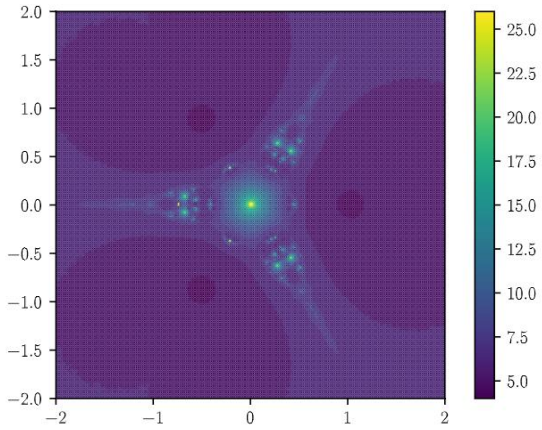
The graphical behavior is reflected in "Fig. 1" to "Fig. 16", using Origin Pro software for graphical comparisons. Compared to other existing methods, PJ, KI, FI, PP, TI, and PS, the error of the proposed approach SS goes to zero in fewer iterations, as can be seen from the residual fall graph. The error graph thus shows the potency and quick convergence of the suggested approach SS.

V. BASINS OF ATTRACTION

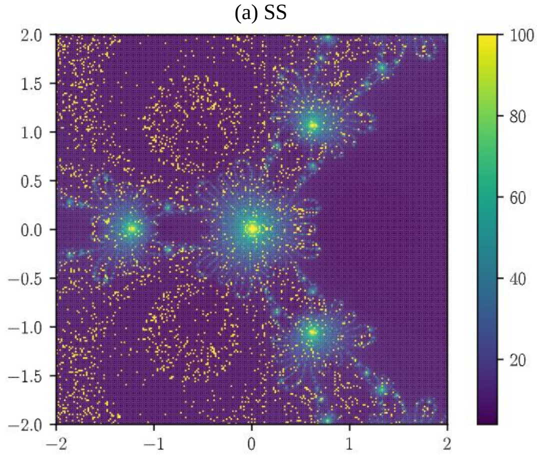
Use the suggested method (SS) beginning at each initial point $z^{(0)}$ in the square region to ascertain the basins of attraction of the root in terms of fractal graphs. Assume a square region $R\times R=[-2,2]\times[-2,2]$ with 250X250 grid points that contains all the roots of the pertinent complex polynomial $\left(z_i^*=1,2,3,...\right)$. If the sequence generated by the iterative technique converges to a polynomial root z_i^* with a tolerance of $\left|f\left(z^{(j)}\right)\right|<10^{-16}$ and a maximum of 100 iterations, we can say that $z^{(0)}$ is in the root's basins of attraction. If $\left|z^{(N)}-z_i^*\right|<10^{-16}$ and the iterative procedure starting at $z^{(0)}$ hits a root in N iterations, the point $z^{(0)}$ is assigned a dark blue hue. The initial point has diverged, and a yellow color has been assigned to it, if $N>100$.

To find the complex roots of the polynomials $f_1(z)=1-z^3$, $f_2(z)=z^{11}-1$ and $f_3(z)=z^2+\frac{1}{z}$, the proposed approach (SS) and other fourth-order methods already in use (PJ, KI, FI, PP, TI, and PS) have the following basins of attraction.

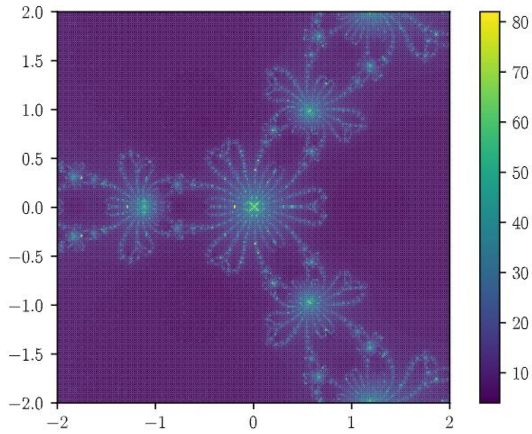
Example 1. $f_1(z)=1-z^3$



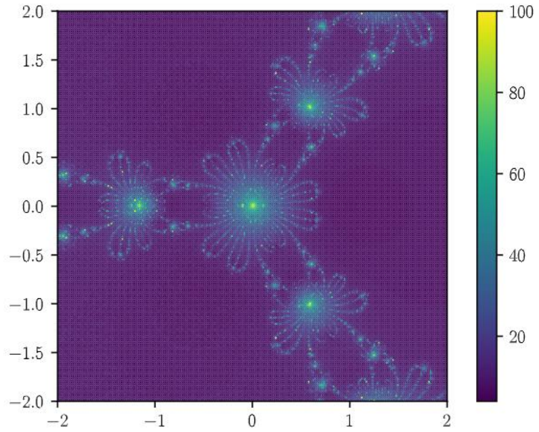
(a) SS



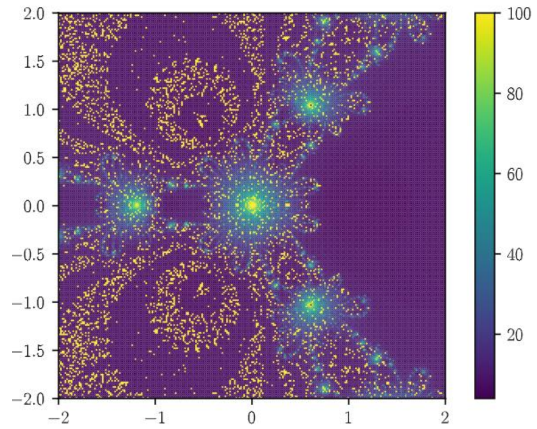
(b) PJ



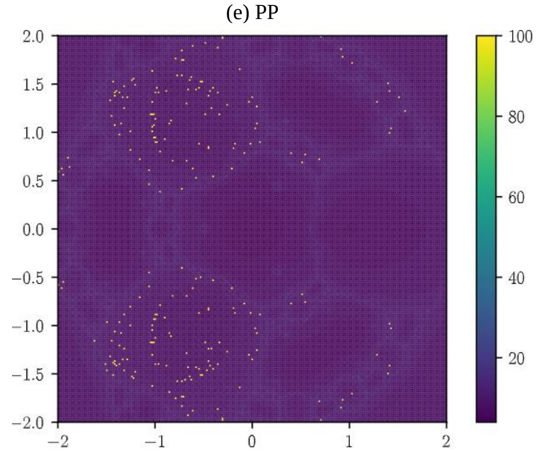
(c) KI



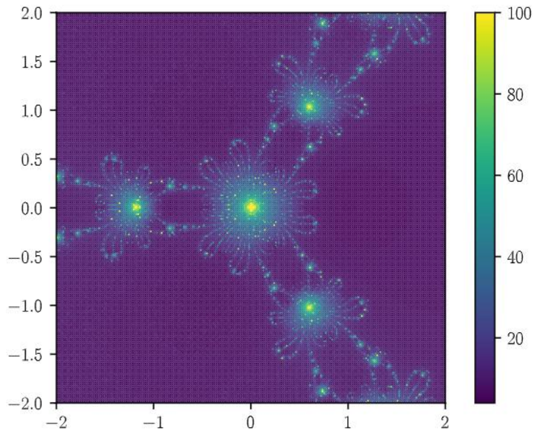
(d) FI



(e) PP



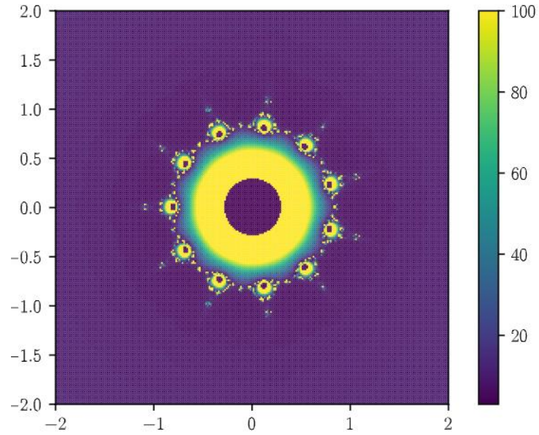
(f) TI



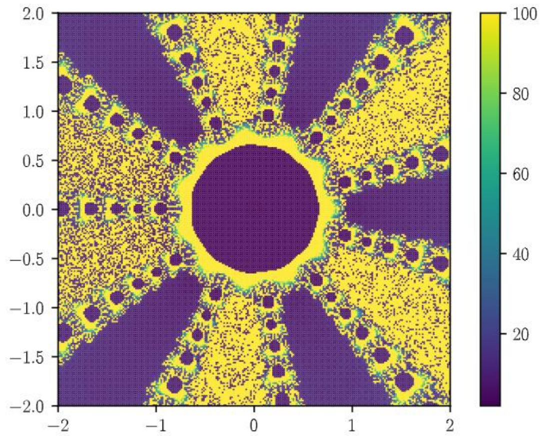
(g) PS

Fig. 1. The polynomiographs obtained by the suggested methods,

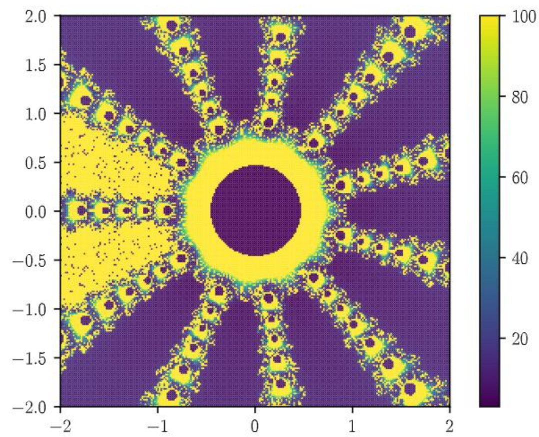
Example 2. $f_2(z) = z^{11} - 1$



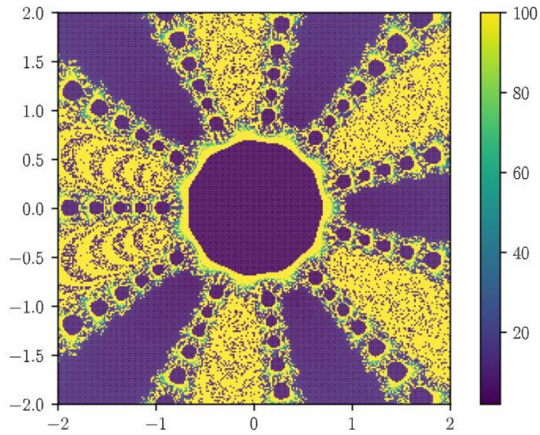
(a) SS



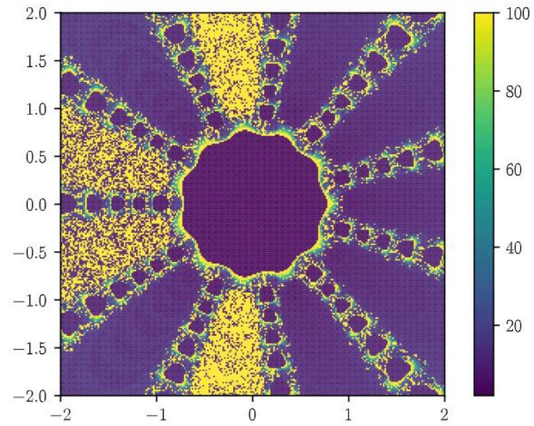
(b) PJ



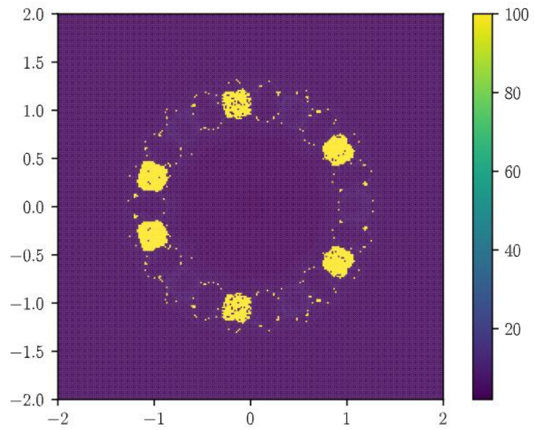
(c) KI



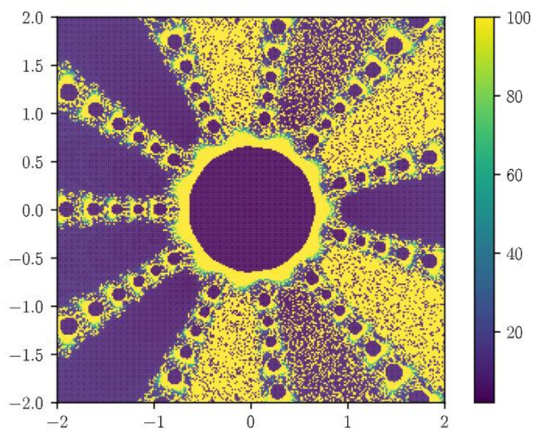
(d) FI



(e) PP



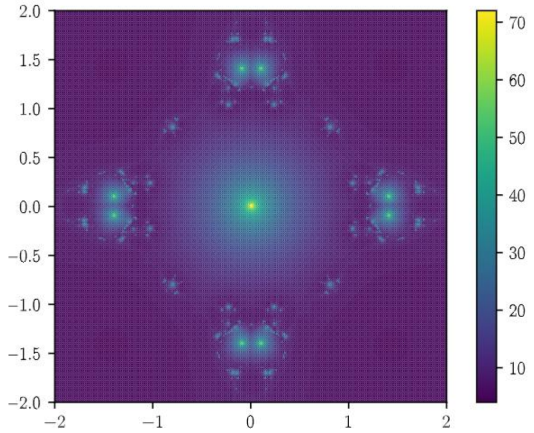
(e) TI



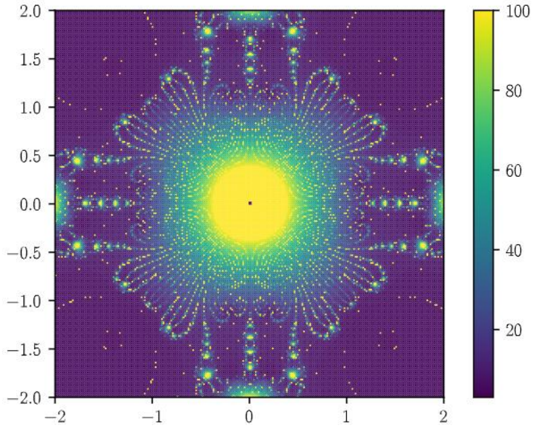
(f) PS

Fig. 2. The polynomiographs obtained by the suggested methods.

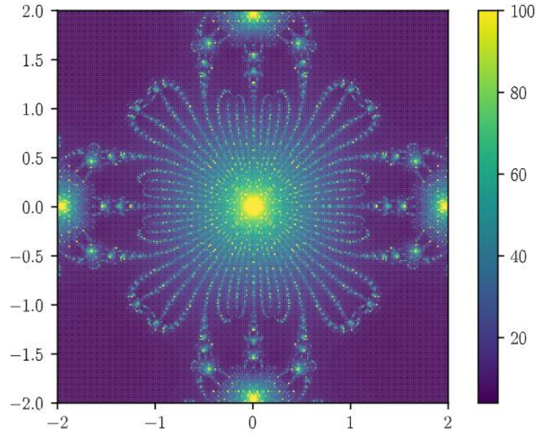
Example 3: $f_3(z) = z^2 + \frac{1}{z}$



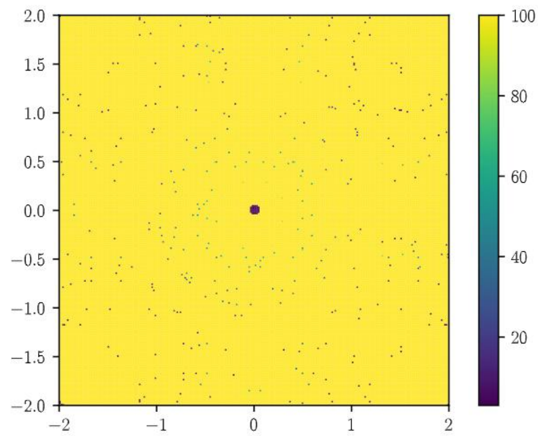
(a) SS



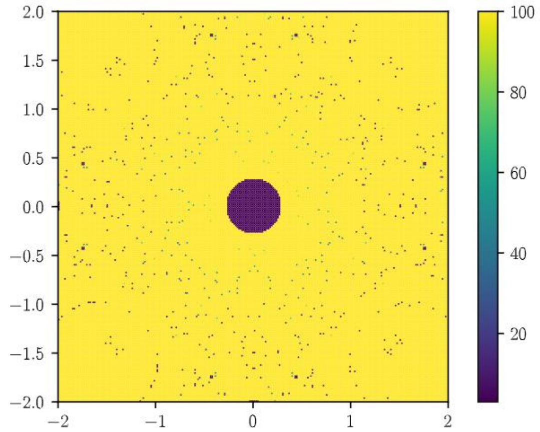
(b) PJ



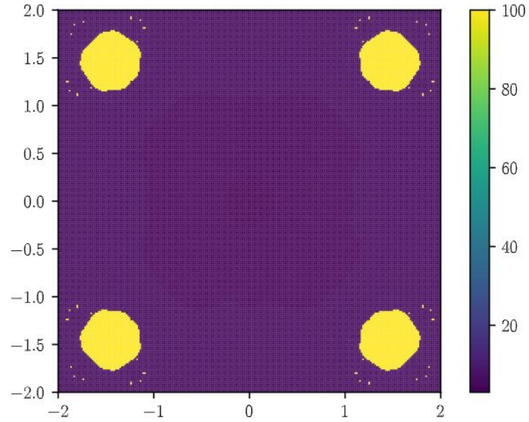
(c) KI



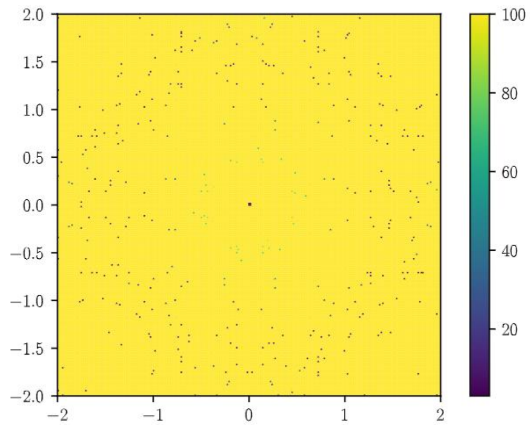
(d) FI



e) PP



(f) TI



(g) PS

Fig. 3. The polynomiographs obtained by the suggested method SS, PJ, KI, FI, PP, TI, PS for $f_3(z)$.

The fractal graphs of the polynomial $f_1(z)$ for the proposed SS and other comparative approaches are shown in “Fig. 1”, “Fig. 2”, and “Fig. 3”. As can be seen from the fractal graphs, the approach SS works quite well because there is no chaotic behavior. Near the boundary points, the methods PJ, KI, FI, PP, TI, and PS exhibit some erratic behavior. The proposed technique SS is the best for all three polynomials in terms of the number of iterations per convergent point, according to the facts shown above.

VI. Conclusions

This work proposes a novel two-step optimal fourth-order iterative approach for solving nonlinear equations with the weight function. This strategy is also particularly appropriate for applications. Tables 1 and 2 demonstrate that the suggested method performs better than other well-known algorithms in terms of iterations, errors, and functional evaluations. Theoretical and COC are proven in the problems under consideration. For solving real-world issues, our proposed strategy (SS) performs better than other existing techniques, such as the PJ, KI, FI, PP, TI, and PS, in terms of iterations and errors. Further detailed plane research has been done for these methods to reveal their fractal graphs and basins of attraction for solving complex polynomials.

Acknowledgments

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Navya Kakarlapudi. The author was born in Visakhapatnam, and the date of birth is 10th December 1990. The author has completed a master's degree in Applied Mathematics at Andhra University, Visakhapatnam, Andhra Pradesh, in the year 2013 and pursuing a doctoral degree in Numerical Analysis at GITAM (Deemed to be University), Visakhapatnam, Andhra Pradesh since 2020. The author's major field of study is numerical analysis.

She has been working as a guest lecturer of mathematics at the Visakha Institute of Engineering and Technology, Visakhapatnam, Andhra Pradesh, since 2021. Previously she worked as an Assistant Professor of Mathematics at the Coastal Institute of Technology and Management, Visakhapatnam, Andhra Pradesh. Some of her previous publications are: Higher order iterative method to solve nonlinear equations, ECS Transactions, vol. 107 (1), 2022, pp. 9141-9148; An Efficient twelfth order convergent scheme to solve nonlinear equations, ECS Transactions, vol. 107 (1), 2022, pp. 14423-14431. The current research interest includes developing application-oriented numerical methods.

Mani Sandeep Kumar Mylapalli. This author became a Member (M) of IAENG in 2017. The author was born in Visakhapatnam, and his date of birth is 26th August 1982. The author completed a master's degree in Applied Mathematics at Andhra University, Visakhapatnam, Andhra Pradesh, in 2004 and earned a doctoral degree in Numerical Analysis at Andhra University, Visakhapatnam, Andhra Pradesh, in the year 2010. The author's major field of study is numerical analysis.

He has worked as an Assistant Mathematics Professor at GITAM (Deemed to be University), Visakhapatnam, Andhra Pradesh, since 2010. Previously he worked as an Assistant Professor of Mathematics at MVGR college of engineering for two years. Some of his previous publications are: An optimal three-step method for solving nonlinear equations, Journal of Critical Reviews, vol. 7 (6), 2020, pp. 100-103; A new two-step sixth-order iterative method with high efficiency-index, Advances in Mathematics: Scientific Journal, 9(7), 2020, pp. 5265-5272 and An iterative method with twelfth order convergence for solving nonlinear equations, Advances and Applications in Mathematical Sciences, 2021, 20, pp. 1633-1643. The goal of the current research is to create novel iterative techniques for solving linear and nonlinear systems of equations.

Dr. Mylapalli will be a Life Member (LM) of IMS in 2021 and an Outreach Member (OM) of SIAM in 2021.

Pravin Singh. The author was born in Durban, South Africa, and the date of birth is 8th June 1965. The author completed a master's degree in physics, The University of Durban Westville in 1989 and earned a doctoral degree in Approximation Theory at the University of Natal, Durban, in 1994. The author's major field of study is Approximation Theory.

He is currently working as a Senior Lecturer of Mathematics at the University of Kwazulu-Natal, Durban, for more than 30 years. Some previous publications are: Approximation of Linear Functionals, P. Singh and J.R. Mika, Quaestiones Mathematicae, vol. 20 (1), 1997, pp. 17-27; The duality property of the Discrete Fourier transform based on Simpson's rule, P. Singh and V. Singh, Math. Meth. Appl. Sci., vol. 35, 2012, pp. 776-781; A Generalization of Partial b-Metric and fixed-point theorems, Aust. J. Math. Anal. Appl., vol. 19 (1), 2022. His current research interests are the Collocation Approximation of PDEs and Fixed-Point Theory.