

# Dynamical Analysis of a Commensalism System Incorporating Beddington-DeAngelis Functional Response and Allee Effect

Binfeng Xie, Na Zhang, and Cheng Liu

**Abstract**—In this paper, we put forward and study a commensalism system incorporating the Beddington-DeAngelis functional response and the Allee effect in the second species. In the first place, the legitimacy and well-posedness in the biological sense of the system solution, such as positiveness and boundness are proved. Then, by examining the eigenvalues of the Jacobian matrix at the equilibria and applying the comparison principle and Dulac theorem, we may determine all feasible equilibria of the system as well as their local and global stability criteria. In the second place, by examining the Allee effect, we see that: (i) when the Allee level threshold is exceeded, the coexistence equilibrium point exists and is globally stable; otherwise, the second species will perish; (ii) when the positive equilibrium exists, the increase of the Allee effect level will decrease the final density of two species; (iii) the system will take longer to attain a steady state when the Allee effect is increased. Finally, numerical simulations are presented to verify the validity of the main results.

**Index Terms**—Allee effect, Dynamics, Beddington-DeAngelis, commensalism system.

## 1. INTRODUCTION

COMMENSAL symbiosis is a type of biological contact in which one population benefits while the other does not benefit or hurt. The Pacific Blue mullet, for example, relies on whales and dolphins for survival. Many scientists have examined the dynamic behavior of commensal symbiosis systems in recent years, and while linear functional response is commonly used, [1]–[12], nonlinear functional response is clearly preferable. Han and Chen [5] investigated the following non-monotonic functional response commensalism system:

$$\begin{cases} \frac{du}{dt} = u\left(r_1 - k_1u + \frac{cv}{d + v^2}\right), \\ \frac{dv}{dt} = v(r_2 - k_2v). \end{cases} \quad (1.1)$$

In 1959, a Canadian researcher by the name of Holling [13] proposed the matching functional response function for various species to illustrate the predation rate of predator population to prey population based on his experimental

findings, which includes three main types Holling type I (or Linear functional response), II (see [14]–[18]), and III (see [19]). Beddington (see [20], [21]) presented a nonlinear functional response called the Beddington-DeAngelis functional response for the predator-prey system. In reality, it is comparable to the well-known Holling type II functional response, but the denominator includes an additional  $nv$  element to simulate predator interaction. Many researchers hold that predators must find food to share or compete, hence the functional response in the predator-prey model must be predator dependant, implying that the Beddington-DeAngelis type functional response is more appropriate. It is commonly employed in predator-prey systems due to its theoretical and practical significance. See the literatures [22]–[36] and references therein. Furthermore, the Beddington-DeAngelis type functional response admits rich yet biologically meaningful dynamics, attracting researchers to further investigate the Beddington-DeAngelis type systems. Based on the preceding articles, what happens to the dynamic system if we substitute the functional response of system (1.1) with the Beddington-DeAngelis functional response, which is reliant on the second species? As a result, the system is transformed into the following:

$$\begin{cases} \frac{du}{dt} = u\left(r_1 - k_1u + \frac{cv}{1 + mu + nv}\right), \\ \frac{dv}{dt} = v(r_2 - k_2v). \end{cases} \quad (1.2)$$

However, when population density is either high or too sparse, population reproduction is hampered, which is a common occurrence in nature. Allee brings this phenomenon Allee effect in reference [37]. Allee effect occurs for a variety of reasons, including inbreeding [38], mating challenges [39], low density social dysfunction [40], and so forth. Cooperation is frequently useless when the number of organisms is too small, resulting in a population dynamics growth level, since an appropriately high growth rate is required to control certain fatalities caused by environmental changes. Because of the biological significance of the Allee effect, an increasing number of researchers include it into biological mathematical models, such as [16], [41]–[59]. Bazykin [60] presented the following single species model with Allee effect:  $\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)(x - A)$ , where  $r$  is the population's intrinsic per capita growth rate and  $K$  is environmental capacity. When  $0 < A < K$ , the model has a strong Allee effect; when  $A \leq 0$ , the model has a weak Allee effect. They also created a population threshold, which reflects the bare minimum number required to survive. The

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TABLE I  
EXPLANATION OF PARAMETER MEANING IN SYSTEM (1.3)

Parameter	Meaning
$u$	the first population density at time $t$
$v$	the second population density at time $t$
$r_1$	first population's intrinsic growth rate
$r_2$	the maximum intrinsic growth rate of the second population
$a$	the level of Allee of the second species
$\frac{r_1}{k_1}$	the environmental capacity of the first species
$\frac{r_2}{k_2}$	the environmental capacity of the second species
$\frac{cv}{1 + mu + nv}$	Beddington-DeAngelis functional response

population must rise above this limit if the model exhibits a substantial Allee effect. The model with a modest Allee effect, on the other hand, has no threshold. Furthermore, to answer the question "what minimal numbers are required for a species to survive in nature?" Dennis [61] originally described a single species model with additive Allee effect as:  $\frac{dx}{dt} = rx(1 - \frac{x}{K}) - \frac{Ax}{x + M}$ , where  $A$  and  $M$  are constants,  $A$  denotes the magnitude of Allee effect,  $M$  represents the number of single populations, and its fitness is half of its highest value.  $\frac{A}{x + M}$  represents the additive Allee effect, which means that: if  $0 < A < M$ , the system has a weak Allee effect; if  $A > M$ , the system has a strong Allee impact. Dennis discovered that numerical simulation can be used to calculate the critical density, growth, and extinction probabilities in sparse populations suffering from the Allee effect.

Because of the small population, it is difficult for the population to find a partner and proliferate, hence the Allee effect impacts the innate rate of increase of the species. The second population's intrinsic growth rate is represented as  $A(a, v) = \frac{r_2 v}{a + v}$ , where  $r_2$  denotes the maximum intrinsic growth rate of the second population,  $v$  denotes the prey population density at time  $t$ , and  $a > 0$  denotes the extent of Allee, which can be used to determine the extent of Allee impact on the second species.  $A(a, v)$  satisfy  $\lim_{a \rightarrow +\infty} A(a, v) = 0$ ,  $\lim_{v \rightarrow 0} A(a, v) = 0$ ,  $\lim_{a \rightarrow 0} A(a, v) = r_2$ ,  $\lim_{v \rightarrow +\infty} A(a, v) = r_2$ ,  $\frac{\partial A(a, v)}{\partial a} < 0$ . The Allee effect is introduced into the second population using system (1.2) to create a commensalism system with the Beddington-DeAngelis functional reaction and Allee phenomenon as follows:

$$\begin{cases} \frac{du}{dt} = u \left( r_1 - k_1 u + \frac{cv}{1 + mu + nv} \right), \\ \frac{dv}{dt} = v \left( \frac{r_2 v}{a + v} - k_2 v \right). \end{cases} \quad (1.3)$$

All parameters in the system are positive, In Table I, we explain what each parameter of system (1.3) means.

The rest of the essay is organized as follows: Section 2 establishes the legality and well-posedness of the system solution in biological terms, such as boundedness and positivity. The presence and stability of the system's equilibria are discussed. The influence of the Allee effect on the system

is examined in Section 3. The numerical simulation and discussion in Section 4 serve to illustrate the validity of the theoretical conclusions. Section 5 concludes this study with a brief conclusion.

## 2. QUALITATIVE AND STABILITY ANALYSIS OF THE SYSTEM

### A. Positivity and boundedness of solutions

In order to conduct a qualitative analysis of the system (1.3), we introduce the following two lemmas:

**Lemma 2.1.** [62] *If  $p, q > 0$  and  $\frac{dx}{dt} \leq (\geq) x(t)(p - qx)$  with  $x(0) > 0$ , then*

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{p}{q} (\liminf_{t \rightarrow +\infty} x(t) \geq \frac{p}{q}).$$

The following lemma can also be used to replace the aforementioned lemma:

**Lemma 2.2.** [62] *If  $p, q > 0$  and  $\frac{dx}{dt} \leq (\geq) x(t)(p - qx)$  with  $x(0) > 0$ , then for all  $t \geq 0$*

$$x(t) \leq \frac{p}{q - Ce^{-pt}}, \quad \text{with } C = q - \frac{p}{x(0)}.$$

*In particular  $x(t) \leq \max\{x(0), \frac{p}{q}\}$  for all  $t \geq 0$ .*

The positivity and boundedness of solutions in system (1.3) are provided by the following proposition.

**Proposition 2.1.** (i) *All solutions  $(u(t), v(t))$  of system (1.3) with the initial conditions  $u(t) > 0, v(t) > 0$  are positive for all  $t \geq 0$ .*

(ii) *For any  $t \geq 0$ , all solutions  $(u(t), v(t))$  to system (1.3) with initial conditions  $u(t) > 0$  and  $v(t) > 0$  are bounded.*

*Proof:* (i) Since  $u = 0$  and  $v = 0$  are invariant sets in system (1.3), any trajectory originating in  $R_+^2$  cannot cross the coordinate axes if  $u(0) > 0$  and  $v(0) > 0$ . Therefore,  $u(t) > 0$  and  $v(t) > 0$  for all  $t \geq 0$ . As a result, for any  $t \geq 0$ , all solutions  $(u(t), v(t))$  of system (1.3) with the initial conditions  $u(t) > 0, v(t) > 0$  are positive.

(ii) By applying the positivity of variables  $u, v$ , we have

$$\frac{dv}{dt} \leq v(r_2 - k_2 v).$$

From Lemma 2.2, we get

$$v(t) \leq \max\{v(0), \frac{r_2}{k_2}\} \equiv M_1 \quad \text{for all } t \geq 0.$$

Then, from the first equation of the system (1.3), we obtain

$$\frac{du}{dt} \leq u(r_1 + cM_1 - k_1 u).$$

From Lemma 2.2, we have

$$u(t) \leq \max\{u(0), \frac{r_1 + cM_1}{k_1}\} \equiv M_2 \quad \text{for all } t \geq 0.$$

Therefore all solutions  $(u(t), v(t))$  to system (1.3) under the initial conditions  $u(t) > 0$  and  $v(t) > 0$  are bounded for any  $t \geq 0$ . ■

**B. Equilibria and the conditions for their existence**

The following equations' nonnegative solutions characterize biologic equilibria in the system (1.3):

$$\begin{cases} u\left(r_1 - k_1u + \frac{cv}{1 + mu + nv}\right) = 0, \\ v\left(\frac{r_2v}{a + v} - k_2v\right) = 0. \end{cases} \quad (2.1)$$

From the second equation of (2.1), we obtain two nonnegative solutions  $v = 0$  or  $v = \frac{r_2 - ak_2}{k_2}$ , when  $0 < a < a_1$  holds, where  $a_1 = \frac{r_2}{k_2}$ . Consequently, the following are the biologic equilibrium points in the system (1.3):

- (i) The boundary equilibria  $B_0(0, 0)$  and  $B_1(u_1, 0)$  always exist without any restrictions, where  $u_1 = \frac{r_1}{k_1}$ .
- (ii) If  $0 < a < a_1$  holds, the system (1.3) also has another boundary equilibrium  $B_2(0, v^*)$  and positive equilibrium  $B_3(u^*, v^*)$ , where  $v^* = \frac{r_2 - ak_2}{k_2}$ , and  $u^*$  is the positive solution of the following single-variable quadratic expression:

$$A_1u^2 + A_2u + A_3 = 0,$$

where  $A_1 = mk_1$ ,  $A_2 = k_1 + nk_1v^* - mr_1$ ,  $A_3 = -(r_1 + cv^* + nr_1v^*)$ . It is easy to find that the equation has a unique positive solution:

$$u^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}.$$

Here, we express the existence conditions of the equilibria of system (1.3) as the following proposition:

**Proposition 2.2.** *The system (1.3) always exists an trivial equilibrium  $B_0(0, 0)$  and a border equilibrium  $B_1(u_1, 0)$ ; If  $0 < a < a_1$  holds, the system (1.3) has another border equilibrium  $B_2(0, v^*)$  and positive equilibrium  $B_3(u^*, v^*)$ .*

**C. Local and global stability of the equilibria**

In order to obtain the stability of the equilibria of system (1.3), we first calculate the Jacobian matrix of the system at any point  $(u, v)$  as follows:

$$J(u, v) = \begin{pmatrix} r_1 - 2k_1u + \frac{cv(nv + 1)}{(1 + mu + nv)^2} & \frac{cu(mv + 1)}{(1 + mu + nv)^2} \\ 0 & \frac{r_2v^2 + 2ar_2v}{(a + v)^2} - 2k_2v \end{pmatrix},$$

The following theorems will now be used to describe the dynamics of the model system at each stable state in turn:

**Theorem 2.1.**  $B_0(0, 0)$  is a saddle point, if  $a = \frac{r_2}{k_2}$  holds; while  $B_0(0, 0)$  is a saddle node, if  $a \neq \frac{r_2}{k_2}$  holds.

*Proof:* The Jacobian matrix of system (1.3) at  $B_0$  is

$$J(0, 0) = \begin{pmatrix} r_1 & 0 \\ 0 & 0 \end{pmatrix},$$

its eigenvalues are  $\lambda_1 = r_1 > 0$  and  $\lambda_2 = 0$ , so the extinction equilibrium  $B_0(0, 0)$  is non-hyperbolic. By making time

transformation  $\tau = tr_1$  and extending the system (1.3) into a third-order power series centered on point  $B_0(0, 0)$ , we get

$$\begin{cases} \frac{du}{d\tau} = u - \frac{k_1}{r_1}u^2 + \frac{c}{r_1}uv - \frac{cm}{r_1}u^2v - \frac{cn}{r_1}uv^2 \\ \quad + O(|u, v|^4), \\ \frac{dv}{d\tau} = \frac{1}{r_1}\left(\frac{r_2}{a} - k_2\right)v^2 - \frac{r_2}{r_1a^2} + O(|u, v|^4). \end{cases} \quad (2.2)$$

Therefore, according to Theorem 7.1 in [63],  $B_0(0, 0)$  is a saddle point, if  $a = \frac{r_2}{k_2}$  holds; while  $B_0(0, 0)$  is a saddle node, if  $a \neq \frac{r_2}{k_2}$  holds. ■

**Theorem 2.2.**  $B_1(u_1, 0)$  is globally attractive if  $a \geq a_1$  holds; and  $B_1(u_1, 0)$  is unstable if  $0 < a < a_1$  holds.

*Proof:* From the second equation of system (1.3), we denote

$$F(v) = \frac{r_2v}{a + v} - k_2v,$$

then

$$F(v) = \frac{-k_2v^2 + (r_2 - ak_2)v}{a + v}.$$

(i) When  $0 < a < a_1$ , i.e.,  $r_2 - ak_2 > 0$  then  $F(v) = 0$  has a unique positive  $v = v^*$ , which satisfies  $vF(v) > 0$  for  $v \in (0, v^*)$  and  $vF(v) < 0$  for  $v > v^*$ , hence

$$\lim_{t \rightarrow +\infty} v(t) = v^*, \quad (2.3)$$

that is,  $v^*$  is globally attractive if  $0 < a < a_1$  holds, hence,  $B_1(u_1, 0)$  is unstable if  $0 < a < a_1$  holds.

(ii) When  $a \geq a_1$ , i.e.,  $r_2 - ak_2 \leq 0$  then  $F(v) = 0$  doesn't have positive solution,  $v = 0$  is the unique equilibrium, and  $vF(v) < 0$  for  $v > 0$ , then

$$\lim_{t \rightarrow +\infty} v(t) = 0. \quad (2.4)$$

From (2.4), for arbitrary  $\varepsilon > 0$ , there exists  $T > 0$ , when  $t > T$ , we derive

$$v(t) < \varepsilon. \quad (2.5)$$

From the first equation of system (1.3) and (2.5), we get

$$u(r_1 - k_1u) \leq \frac{du}{dt} \leq u(r_1 - k_1u + c\varepsilon).$$

From the comparison principle of differential equation, we get

$$\frac{r_1}{k_1} \leq u(t) \leq \frac{r_1 + c\varepsilon}{k_1}, \quad \text{for } t > T,$$

let  $\varepsilon \rightarrow 0$ , then

$$\lim_{t \rightarrow +\infty} u(t) = u_1. \quad (2.6)$$

From (2.4) and (2.6), we obtain that  $B_1(u_1, 0)$  is globally attractive if  $a \geq a_1$  holds. ■

**Theorem 2.3.**  $B_2(0, v^*)$  is saddle, if  $a < a_1$  holds.

*Proof:* System (1.3)'s Jacobian matrix around  $B_2$  is

$$J(0, v^*) = \begin{pmatrix} \frac{cr_2 + k_2r_1 + nr_1r_2}{k_2 + nr_2} & 0 \\ 0 & \frac{-r_2v^*}{(a + v^*)^2} \end{pmatrix},$$

TABLE II  
STEADY STATE AND ITS STABILITY IN THE SYSTEM (1.3)

Equilibrium	Existence	Type
$B_0(0, 0)$	Always exists	$a = a_1$ , Saddle. $a \neq a_1$ , Saddle node.
$B_1(u_1, 0)$	Always exists	$a \geq a_1$ , Globally attractive; $0 < a < a_1$ , Unstable.
$B_2(0, v^*)$	$0 < a < a_1$	$0 < a < a_1$ , Saddle.
$B^*(u^*, v^*)$	$0 < a < a_1$	$0 < a < a_1$ , Stable node; $0 < a < a_1$ , Globally asymptotically stable node.

its eigenvalues are  $\lambda_1 = \frac{cr_2 + k_2r_1 + nr_1r_2}{k_2 + nr_2} > 0$  and  $\lambda_2 = \frac{-r_2v^*}{(a + v^*)^2} < 0$ . Hence  $B_2(0, v^*)$  is saddle, if  $a < a_1$  holds. ■

**Theorem 2.4.** *If  $a < a_1$  holds, then  $B^*(u^*, v^*)$  is locally asymptotically stable; furthermore,  $B^*(u^*, v^*)$  exhibits asymptotic global stability.*

*Proof:* System (1.3)'s Jacobian matrix around  $B^*$  is

$$J(u^*, v^*) = \begin{pmatrix} -k_1u^* - \frac{cmu^*v^*}{(1 + mu^* + nv^*)^2} & \frac{cu^*(1 + mu^*)}{(1 + mu^* + nv^*)^2} \\ 0 & \frac{-r_2v^*}{(a + v^*)^2} \end{pmatrix},$$

its eigenvalues are  $\lambda_1 = -k_1u^* - \frac{cmu^*v^*}{(1 + mu^* + nv^*)^2} < 0$  and  $\lambda_2 = \frac{-r_2v^*}{(a + v^*)^2} < 0$ . Hence  $B_2(u^*, v^*)$  is locally asymptotically stable, if  $a < a_1$  holds. In order to prove that  $B^*$  is globally asymptotically stable, we first construct a Dulac function as follows:

$$D(u, v) = \frac{1}{uv^2}.$$

Let

$$P(u, v) = u \left( r_1 - k_1u + \frac{cv}{1 + mu + nv} \right),$$

$$Q(u, v) = v \left( \frac{r_2v}{a + v} - k_2v \right).$$

Then

$$\frac{\partial(PD)}{\partial u} + \frac{\partial(QD)}{\partial v} = -\frac{1}{v^2} \left( k_1 + \frac{cmv}{(1 + mu + n)^2} \right) - \frac{r_2}{u(a + v)^2} < 0.$$

The Dulac Theorem in reference [64] states that the first quadrant is devoid of limit cycles.  $B^*(u^*, v^*)$  is hence global asymptotic stable. ■

As illustrated in Table II, we may summarize the stability and existence of equilibrium points.

### 3. THE INFLUENCE OF ALLEE EFFECT ON THE TWO POPULATIONS

This section discusses the Allee effect's impact on each population when a positive equilibrium is present.

#### A. Without Allee effect

If the second species does not exhibit the Allee effect, i.e.,  $a = 0$ , system (1.3) transforms into system (1.2). We find that  $B^*(u^*, v^*)$  is always globally asymptotically stable in the absence of any constraints.

#### B. Incorporate Allee effect

We know that  $u^* = \frac{-A_2 + \sqrt{A_2^2 - A_1A_3}}{2A_1}$  and  $v^* = \frac{r_2 - ak_2}{k_2}$  are continuous functions of  $a$ , and  $u^*$  satisfies

$$r_1 - k_1u^* + \frac{cv^*}{1 + mu^* + nv^*} = 0$$

Take the derivative of parameter  $a$  on both sides of the aforementioned equation, we can arrive at the following conclusion.

$$\frac{du^*}{da} = \frac{-c(1 + mu^*)}{k_1(1 + mu^* + nv^*)^2 + cmv^*} < 0,$$

and

$$\frac{dv^*}{da} = -1 < 0.$$

As a result, the final densities of the two population  $u^*$  and  $v^*$  decrease as Allee level rises. The first population density  $u^*$  and the second population density  $v^*$  reach their maximum values when there is no Allee effect present in the system. Fig. 1 (a) depicts the relationship between  $u^*$  and  $a$ , while Fig. 1 (b) depicts the link between  $v^*$  and  $a$ .

We also find that:

$$\lim_{a \rightarrow a_1} v^* = 0, \quad \lim_{a \rightarrow a_1} u^* = u_1.$$

That is, as the Allee extent increases and approaches  $a_1$ , the second species will go extinct.

### 4. COMPUTATIONAL SIMULATION

We show a number of numerical simulations in this part to support the theoretical analysis and clarify the dynamics of the system (1.3). We looked into the system (1.3)'s dynamic behavior in order to modify the Allee effect's strength.

**Example 4.1** Set  $a = 4$ ,  $r_1 = 1.1$ ,  $r_2 = 0.9$ ,  $k_1 = 1$ ,  $k_2 = 0.3$ ,  $c = 0.7$ ,  $m = 0.8$ ,  $n = 0.6$ , then  $a_1 = 3$ , and  $a \geq a_1$ . According to theorems 2.1 and 2.2, the system (1.3) has an trivial equilibrium  $B_0 = (0, 0)$ , which is saddle, and a border equilibrium  $B_1(1.1, 0)$ , which is globally asymptotically stable implying that the second species will eventually die out regardless of the initial value of population density. Fig. 2 depicts the simulation findings.

**Example 4.2** Set  $a = 0.1$ ,  $r_1 = 1.1$ ,  $r_2 = 0.9$ ,  $k_1 = 1$ ,  $k_2 = 0.3$ ,  $c = 0.7$ ,  $m = 0.8$ ,  $n = 0.6$ , then  $a_1 = 3$ , and  $a < a_1$ . According to theorems 2.1, 2.2, 2.3, and 2.4, the system (1.3) has an trivial equilibrium  $B_0 = (0, 0)$ , which is saddle; and two border equilibria  $B_1(1.1, 0)$  and  $B_2(0, 2.9)$ , which are unstable; and a positive  $B^*(1.6045, 2.9)$  which is global asymptotic stable, implying that the two species will

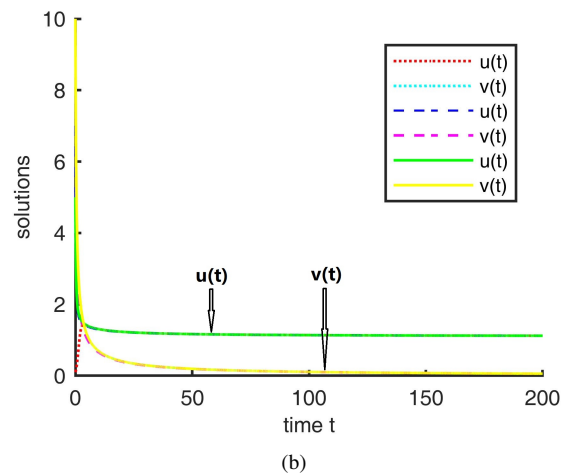
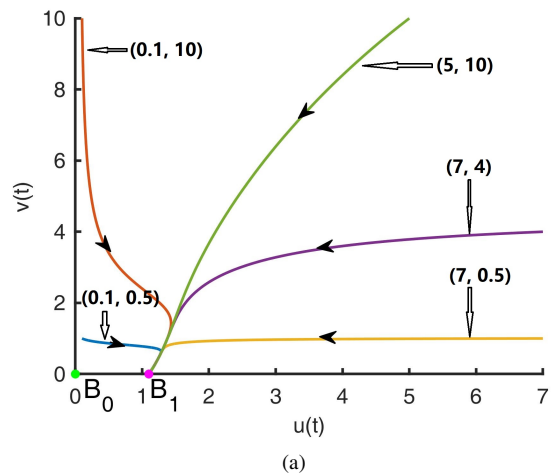
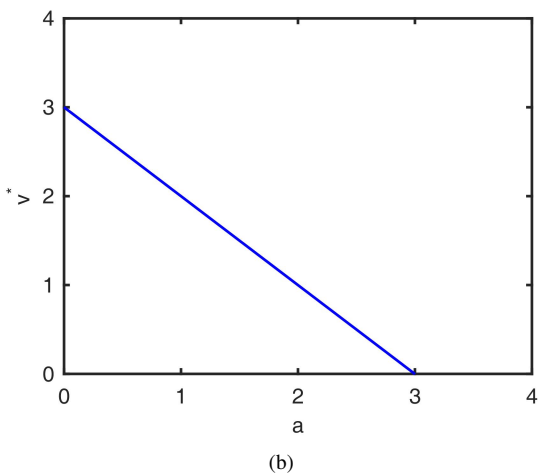
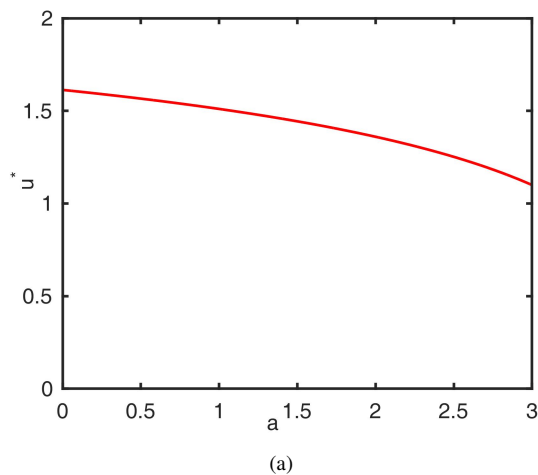


Fig. 1. (a) the function image of  $u^*$  about  $a$ ; (b) the function image of  $v^*$  about  $a$ .

Fig. 2. (a) The phase diagram of system (1.3) for  $a = 4$ , the initial values are  $(0.1, 10)$ ,  $(0.1, 0.5)$ ,  $(7, 4)$ ,  $(7, 0.5)$ ,  $(5, 10)$ ; (b) Solution curves for  $a = 4$ , the initial values are  $(0.1, 10)$ ,  $(7, 4)$ ,  $(5, 10)$ .

coexist regardless of the initial value of population density. Fig. 3 depicts the simulation findings.

**Example 4.3** Set  $r_1 = 1.1$ ,  $r_2 = 0.9$ ,  $k_1 = 1$ ,  $k_2 = 0.3$ ,  $c = 0.7$ ,  $m = 0.8$ ,  $n = 0.6$ , then  $a_1 = 3$ . We use  $a = 0, 1, 2$ , which satisfy  $a < a_1$ , to examine the Allee effect's impact on the system. the system (1.3) has an unique positive equilibrium  $B^* = (1.6134, 3)$ ,  $(1.5107, 2)$ ,  $(1.3604, 1)$ , which is globally asymptotically stable, implying that the two species will coexist; Two population density  $u^*$  and  $v^*$  decrease with the increase of the extent of Allee  $a$ . The two population density  $u^*$  and  $v^*$  achieve the maximum values 1.6134 and 3 respectively as  $a = 0$ . The bigger the Allee effect, that is, the higher the amount of Allee  $a$ , the longer it takes for the system to stabilize. Fig. 4 depicts the simulation findings.

### 5. CONCLUSION

In this paper, we propose and investigate a commensalism system with Beddington-DeAngelis functional response and Allee effect in the second species. We find that: (i) when the Allee level is weak enough, i.e.,  $a$  is less than threshold value  $a_1$ , the coexistence equilibrium point exists and is globally asymptotically stable, otherwise, the second species will perish due to strong Allee effect; (ii) The final densities of two species can be decreased

when the degree of Allee is increased and the positive equilibrium is present. The first population density  $u^*$  and the second population density  $v^*$  achieve the maximum values, when there is no Allee effect in the system; (iii) increasing the Allee level will increase the time for the system to reach steady state. The conclusion of this paper will enrich the dynamic results of profit biased system.

### REFERENCES

- [1] Z. Miao, X. Xie, and L. Pu, "Dynamic behaviors of a periodic lotka-voltterra commensal symbiosis model with impulsive," *Commun. Math. Biol. Neurosci.*, vol. 2015, pp. Article-ID3, 2015.
- [2] R. Han and F. Chen, "Global stability of a commensal symbiosis model with feedback controls," *Commun. Math. Biol. Neurosci.*, vol. 2015, pp. Article-ID15, 2015.
- [3] R. Wu and L. Li, "Dynamic behaviors of a commensal symbiosis model with ratio-dependent functional response and one party can not survive independently," *J. Math. Comput. Sci.*, vol. 16, no. 3, pp. 495–506, 2016.
- [4] T. Li, Q. Lin, and J. Chen, "Positive periodic solution of a discrete commensal symbiosis model with Holling II functional response," *Commun. Math. Biol. Neurosci.*, vol. 2016, pp. Article-ID22, 2016.
- [5] J. Chen and R. Wu, "A commensal symbiosis model with non-monotonic functional response," *Commun. Math. Biol. Neurosci.*, vol. 2017, pp. Article-ID5, 2017.
- [6] Q. Lin, "Dynamic behaviors of a commensal symbiosis model with non-monotonic functional response and non-selective harvesting in a partial closure," *Commun. Math. Biol. Neurosci.*, vol. 2018, pp. Article-ID4, 2018.

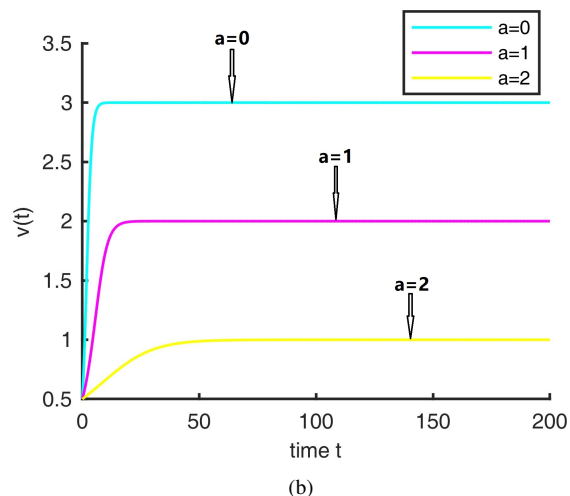
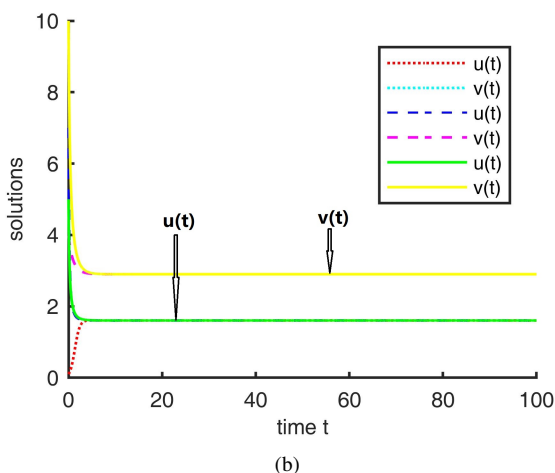
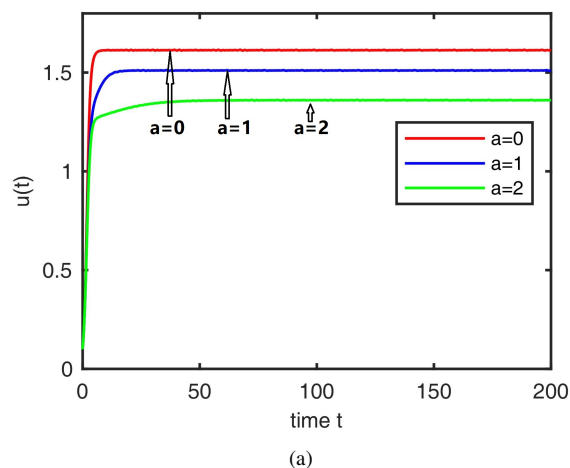
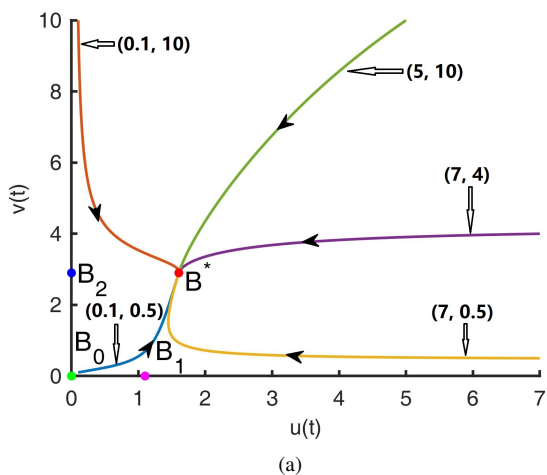


Fig. 3. (a) The phase diagram of system (1.3) for  $a = 0.1$ , the initial values are  $(0.1, 10)$ ,  $(0.1, 0.5)$ ,  $(7, 4)$ ,  $(7, 0.5)$ ,  $(5, 10)$ ; (b) Solution curves for  $a = 0.1$ , the initial values are  $(0.1, 10)$ ,  $(7, 4)$ ,  $(5, 10)$ .

Fig. 4. (a) Solution curves of the first species for  $a = 0$  or  $a = 1$  or  $a = 2$  (b) Solution curves of the second species for  $a = 0$  or  $a = 1$  or  $a = 2$ . With the initial value  $(0.1, 0.5)$ .

[7] F. Chen, Y. Xue, Q. Lin, and X. Xie, "Dynamic behaviors of a lotka-volterra commensal symbiosis model with density dependent birth rate," *Advances in Difference Equations*, vol. 2018, no. 1, pp. 1–14, 2018.

[8] B. Chen, "Dynamic behaviors of a commensal symbiosis model involving allee effect and one party can not survive independently," *Advances in Difference Equations*, vol. 2018, no. 1, pp. 1–12, 2018.

[9] Q. Lin, "Allee effect increasing the final density of the species subject to the allee effect in a lotka-volterra commensal symbiosis model," *Advances in Difference Equations*, vol. 2018, no. 1, pp. 1–9, 2018.

[10] R. Wu, L. Li, and Q. Lin, "A holling type commensal symbiosis model involving allee effect," *Commun. Math. Biol. Neurosci.*, vol. 2018, pp. Article-ID6, 2018.

[11] B. Chen, "The influence of commensalism on a lotka-volterra commensal symbiosis model with michaelis-menten type harvesting," *Advances in Difference Equations*, vol. 2019, no. 1, pp. 1–14, 2019.

[12] C. Lei, "Dynamic behaviors of a holling type commensal symbiosis model with the first species subject to allee effect," *Commun. Math. Biol. Neurosci.*, vol. 2019, pp. Article-ID3, 2019.

[13] Holling and S. C., "Some characteristics of simple types of predation and parasitism," *Canadian Entomologist*, vol. 91, no. 7, pp. 385–398, 1959.

[14] B. Xie, "Impact of the fear and Allee effect on a Holling type II prey-predator model," *Advances in Difference Equations*, vol. 2021, no. 1, p. 464, 2021.

[15] B. Xie, Z. Zhang, and N. Zhang, "Influence of the fear effect on a Holling Type II prey-predator system with a Michaelis-Menten type harvesting," *International Journal of Bifurcation and Chaos*, vol. 31, no. 14, p. 2150216, 2021.

[16] B. Xie and Z. Zhang, "Impact of allee and fear effects in a fractional order prey-predator system incorporating prey refuge," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 33, no. 1, p. 013131, 2023.

[17] N. Zhang and Y. Kao, "A fractional-order food chain system incorporating holling-ii type functional response and prey refuge," *International Journal of Bifurcation and Chaos*, vol. 32, no. 10, p. 2250143, 2022.

[18] N. Zhang, Y. Kao, F. Chen, B. Xie, and S. Li, "On a predator-prey system interaction under fluctuating water level with nonselective harvesting," *Open Mathematics*, vol. 18, no. 1, pp. 458–475, 2020.

[19] B. Xie and N. Zhang, "Influence of fear effect on a holling type iii prey-predator system with the prey refuge," *AIMS Mathematics*, vol. 7, no. 2, pp. 1811–1830, 2022.

[20] J. R. Beddington, "Mutual interference between parasites or predators and its effect on searching efficiency," *The Journal of Animal Ecology*, pp. 331–340, 1975.

[21] D. L. DeAngelis, R. Goldstein, and R. V. O'Neill, "A model for tropic interaction," *Ecology*, vol. 56, no. 4, pp. 881–892, 1975.

[22] R. S. Cantrell and C. Cosner, "On the dynamics of predator-prey models with the beddington-deangelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 1, pp. 206–222, 2001.

[23] T. W. Hwang, "Global analysis of the predator-prey system with Beddington-DeAngelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 1, pp. 395–401, 2003.

[24] M. Fan and Y. Kuang, "Dynamics of a nonautonomous predator-prey system with the beddington-deangelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 15–39, 2004.

[25] Z. Liu and R. Yuan, "Stability and bifurcation in a delayed predator-prey system with beddington-deangelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 2, pp. 521–537, 2004.

[26] Y. Takeuchi *et al.*, "Permanence, extinction and periodic solution of predator-prey system with beddington-deangelis functional response,"

- Journal of Mathematical Analysis and Applications*, vol. 317, no. 2, pp. 464–474, 2006.
- [27] G. Huang, W. Ma, and Y. Takeuchi, “Global properties for virus dynamics model with beddington–deangelis functional response,” *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1690–1693, 2009.
- [28] J. P. Tripathi, S. Abbas, and M. Thakur, “Dynamical analysis of a prey–predator model with beddington–deangelis type function response incorporating a prey refuge,” *Nonlinear Dynamics*, vol. 80, no. 1, pp. 177–196, 2015.
- [29] X. Guan and F. Chen, “Dynamical analysis of a two species a-mensalism model with beddington–deangelis functional response and allee effect on the second species,” *Nonlinear Analysis: Real World Applications*, vol. 48, pp. 71–93, 2019.
- [30] S. Khajanchi and S. Banerjee, “Role of constant prey refuge on stage structure predator–prey model with ratio dependent functional response,” *Applied Mathematics and Computation*, vol. 314, pp. 193–198, 2017.
- [31] K. Sarkar and S. Khajanchi, “Impact of fear effect on the growth of prey in a predator–prey interaction model,” *Ecological Complexity*, vol. 42, no. March 2020, p. 100826, 2020.
- [32] S. Khajanchi, “Modeling the dynamics of stage-structure predator–prey system with Monod–Haldane type response function,” *Applied Mathematics and Computation*, vol. 302, pp. 122–143, 2017.
- [33] A. K. Misra, R. K. Singh, P. K. Tiwari, S. Khajanchi, and Y. Kang, “Dynamics of algae blooming: effects of budget allocation and time delay,” *Nonlinear Dynamics*, vol. 100, no. 2, pp. 1779–1807, 2020.
- [34] K. Sarkar, S. Khajanchi, P. Chandra Mali, and J. J. Nieto, “Rich dynamics of a predator–prey system with different kinds of functional responses,” *Complexity*, vol. 2020, 2020.
- [35] S. Saha and G. Samanta, “Analysis of a predator–prey model with herd behavior and disease in prey incorporating prey refuge,” *International Journal of Biomathematics*, vol. 12, no. 01, p. 1950007, 2019.
- [36] N. Zhang, Y. Kao, and B. Xie, “Impact of fear effect and prey refuge on a fractional order prey–predator system with beddington–deangelis functional response,” *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 32, no. 4, p. 043125, 2022.
- [37] W. Allee, *Animal aggregations: a study in general sociology*. Chicago: University of Chicago Press, 1931.
- [38] P. A. Stephens and W. J. Sutherland, “Consequences of the Allee effect for behaviour, ecology and conservation,” *Trends in Ecology & Evolution*, vol. 14, no. 10, pp. 401–405, 1999.
- [39] F. Courchamp, L. Berec, and J. Gascoigne, *Allee effects in ecology and conservation*. Oxford University Press, 2008.
- [40] G. M. Luque, T. Giraud, and F. Courchamp, “Allee effects in ants,” *Journal of Animal Ecology*, vol. 82, no. 5, pp. 956–965, 2013.
- [41] A. Morozov, S. Petrovskii, and B.-L. Li, “Bifurcations and chaos in a predator–prey system with the Allee effect,” *Proceedings of the Royal Society of London. Series B: Biological Sciences*, vol. 271, no. 1546, pp. 1407–1414, 2004.
- [42] C. Celik and O. Duman, “Allee effect in a discrete-time predator–prey system,” *Chaos, Solitons & Fractals*, vol. 40, no. 4, pp. 1956–1962, 2009.
- [43] G.-Q. Sun, Z. Jin, L. Li, and Q.-X. Liu, “The role of noise in a predator–prey model with Allee effect,” *Journal of Biological Physics*, vol. 35, no. 2, pp. 185–196, 2009.
- [44] J. Zu and M. Mimura, “The impact of allee effect on a predator–prey system with Holling type II functional response,” *Applied Mathematics and Computation*, vol. 217, no. 7, pp. 3542–3556, 2010.
- [45] W.-X. Wang, Y.-B. Zhang, and C.-z. Liu, “Analysis of a discrete-time predator–prey system with Allee effect,” *Ecological Complexity*, vol. 8, no. 1, pp. 81–85, 2011.
- [46] J. Wang, J. Shi, and J. Wei, “Predator–prey system with strong Allee effect in prey,” *Journal of Mathematical Biology*, vol. 62, no. 3, pp. 291–331, 2011.
- [47] M. Sen, M. Banerjee, and A. Morozov, “Bifurcation analysis of a ratio-dependent prey–predator model with the Allee effect,” *Ecological Complexity*, vol. 11, pp. 12–27, 2012.
- [48] S. K. Sasmal, “Population dynamics with multiple Allee effects induced by fear factors—a mathematical study on prey–predator interactions,” *Applied Mathematical Modelling*, vol. 64, pp. 1–14, 2018.
- [49] Y. Ye, H. Liu, Y.-m. Wei, M. Ma, and K. Zhang, “Dynamic study of a predator–prey model with weak Allee effect and delay,” *Advances in Mathematical Physics*, vol. 2019, 2019.
- [50] L. Lai, Z. Zhu, and F. Chen, “Stability and bifurcation in a predator–prey model with the additive Allee effect and the fear effect,” *Mathematics*, vol. 8, no. 8, p. 1280, 2020.
- [51] S. K. Sasmal, “Population dynamics with multiple allee effects induced by fear factors - a mathematical study on prey–predator interactions,” *Applied Mathematical Modelling*, vol. 64, pp. 1–14, 2018.
- [52] M. Sardar and S. Khajanchi, “Is the allee effect relevant to stochastic cancer model?” *Journal of Applied Mathematics and Computing*, pp. 1–23, 2021.
- [53] J. D. Flores and E. González-Olivares, “Dynamics of a predator–prey model with Allee effect on prey and ratio-dependent functional response,” *Ecological Complexity*, vol. 18, pp. 59–66, 2014.
- [54] P. Aguirre, E. González-Olivares, and S. Torres, “Stochastic predator–prey model with Allee effect on prey,” *Nonlinear Analysis: Real World Applications*, vol. 14, no. 1, pp. 768–779, 2013.
- [55] X. Guo and Z. Guo, “A Markov-switching predator–prey model with Allee effect for preys,” *International Journal of Biomathematics*, vol. 13, no. 03, p. 2050018, 2020.
- [56] D. Manna, A. Maiti, and G. Samanta, “Deterministic and stochastic analysis of a predator–prey model with Allee effect and herd behaviour,” *Simulation*, vol. 95, no. 4, pp. 339–349, 2019.
- [57] A. Maiti, P. Sen, D. Manna, and G. Samanta, “A predator–prey system with herd behaviour and strong Allee effect,” *Nonlinear Dyn. Syst. Theory*, vol. 16, no. 1, pp. 86–101, 2016.
- [58] S. Saha, A. Maiti, and G. Samanta, “A Michaelis–Menten predator–prey model with strong Allee effect and disease in prey incorporating prey refuge,” *International Journal of Bifurcation and Chaos*, vol. 28, no. 06, p. 1850073, 2018.
- [59] S. Saha and G. Samanta, “Influence of dispersal and strong Allee effect on a two-patch predator–prey model,” *International Journal of Dynamics and Control*, vol. 7, no. 4, pp. 1321–1349, 2019.
- [60] A. D. Bazykin, *Nonlinear dynamics of interacting populations*. World Scientific, 1998.
- [61] B. Dennis, “Allee effects: population growth, critical density, and the chance of extinction,” *Natural Resource Modeling*, vol. 3, no. 4, pp. 481–538, 1989.
- [62] F. Chen, “On a nonlinear nonautonomous predator–prey model with diffusion and distributed delay,” *Journal of Computational and Applied Mathematics*, vol. 180, no. 1, pp. 33–49, 2005.
- [63] Z. Zhang, T. Ding, W. Huang, and Z. Dong, *Qualitative Theory of Differential Equation*. Science Press, Beijing, 1992.
- [64] J. Chen and H. Zhang, “The qualitative analysis of two species predator–prey model with Holling’s type III functional response,” *Applied Mathematics and Mechanics*, vol. 71, pp. 73–80, 1986.