Global Attractor of Networks of n Coupled Reaction-Diffusion Systems of Hindmarsh-Rose Type

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Abstract—**This paper focuses on the networks of n reaction-diffusion systems of the Hindmarsh - Rose type. We prove the existence of the global attractor of these networks in the space $(L^2(\Omega) \times L^2(\Omega))^n$. It is also bounded in $L^\infty(\Omega)$ for any network topology satisfied some necessary conditions of coupling function and also the dynamic of the system.**

Index Terms—**global attractor, coupling function, networks, reaction-diffusion system of Hindmarsh-Rose.**

I. Introduction

THE Hindmarsh - Rose model was introduced as a dimensional reduction of the well-known Hodgkin-Huxley model (see, e.g. [4], [6], [8], [9], [11], [12]). It is more analytically tractable and it maintains a certain biophysical meaning. The model is constituted by two equations in two variables u et v. The first one is the fast variable called excitatory and represents the transmembrane voltage. The second variable is the slow recovery variable and describes the time dependence of several physical quantities, such as the electrical conductance of the ion currents across the membrane. The ordinary differential equations of the Hindmarsh - Rose type are given by:

$$\begin{cases} u_t = f(u) + v + I \\ v_t = 1 - bu^2 - v \end{cases} \tag{1}$$

where t presents the time variable, $f(u) = -u^3 + au^2$, a, b are constants, and I presents the external current.

However, the system (1) is not strong enough to describe the propagation of action potential. To solve this problem, the cable equation is considered. This mathematical system is derived from a circuit model of the membrane and its intracellular and extracellular space to provide a quantitative description of current flow and voltage change both within and between cells, allowing a quantitative and qualitative understanding of how cells function. Hence, the reaction-diffusion equations of the Hindmarsh-Rose type (HR) are considered as follows:

$$\begin{cases} u_t = f(u) + v + I + d\Delta u & \text{on } \Omega \times \mathbb{R}^+ \\ v_t = 1 - bu^2 - v & \text{on } \Omega \times \mathbb{R}^+ \end{cases} \tag{2}$$

where $u = u(x,t), v = v(x,t)$, $d > 0$, $\Omega \subset \mathbb{R}^N$ is a regular bounded open set with Neumann zero flux conditions on the boundary, and N is a positive integer. This system allows the emergence of a rich variety of patterns and relevant phenomena in physiology (see, e.g. [2], [3]). It is a system of two nonlinear partial differential equations of incomplete parabolic type which describes the action potential and the recovery variable in the whole set of neurons. Note that the first equation is similar to the so-called cable equation, which describes the distribution of the potential along the axon of a single neuron (see, e.g. [4], [7]). Neurons connect through synapses, and it leads to two types of connections between cells such as chemical connections and electrical ones. Then, they form neural networks. A neural network describes a population of physically interconnected nerve cells. Communication between cells is mainly due to electrochemical processes. Hereafter, system (2) is considered as a neural model, and a network of n coupled systems (2) is constructed as follows:

$$\begin{cases} u_{it} = f(u_i) + v_i + I + d\Delta u_i + h_i(u,v) \\ v_{it} = 1 - bu_i^2 - v_i \\ \qquad i, j = 1, 2, ..., n, i \neq j \end{cases} \tag{3}$$

where $u = (u_1, u_2, ..., u_n), v = (v_1, v_2, ..., v_n)$, $(u_i, v_i), i = 1, 2, ..., n$ is defined by (2). The function $h_i, i = 1, 2, ..., n$ presents the coupling function describing the type of connections between cells, and also introduces different network topologies (see [15], [16]).

In this work, we prove the existence of the global attractor of the system (3) in the space $(L^2(\Omega) \times L^2(\Omega))^n$. It is also bounded in $L^\infty(\Omega)$ for any network topology satisfied some necessary conditions of coupling function and also the dynamic of the system.

II. Existence of a global attractor of networks of n coupled reaction-diffusion systems of Hindmarsh-Rose type

In this section, we prove the existence of a global attractor for the dynamical system (3). A global attractor is a compact invariant set for the flow that attracts all trajectories (see [1], [10]). The study of the attractor is fundamental for the asymptotic study of the system since the attractor is a set near which the solutions asymptotically evolve.

We set $H = ((L^2(\Omega))^2)^n$ and $V = ((H^1(\Omega))^2)^n$. We set also for any function u of H^1,

$$|u|^2 = \int_\Omega u^2 \quad \text{and} \quad \|u\|^2 = \int_\Omega |\nabla u|^2.$$

To get a result of the existence of a global attractor that is strong enough and more general. In this section, we consider the following system:

$$\begin{cases} u_{it} = f(u_i) + v_i + I + d\Delta u_i + h_i(u,v) \\ v_{it} = 1 - bu_i^2 - v_i \\ \qquad i, j = 1, 2, ..., n, i \neq j \end{cases} \tag{4}$$

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where $u = (u_1, u_2, ..., u_n), v = (v_1, v_2, ..., v_n)$, $u_i = u_i(x,t), v_i = v_i(x,t)$, $i = 1,2,...,n$, $x \in \Omega \subset \mathbb{R}^N$ is a regular bounded open set with Neumann zero flux conditions on the boundary, $t \in \mathbb{R}^+$, d is a positive constant, and N is a positive integer. The functions f and h_i are assumed to be twice continuously differentiable in all variables and satisfy:

$$\delta_1|u_i|^p - \delta_3 \leqslant -f(u_i)u_i \leqslant \delta_2|u_i|^p + \delta_3, \quad p > 2, \tag{5}$$

for all $u_i, i = 1,2,...,n$,

$$|h_i(u,v)| \leqslant \delta_4(1 + \sum_{j=1}^{n} |u_j|^{p_1} + |v_i|), \;\; 0 < p_1 < p-1, \tag{6}$$

for all $u_i, v_i, i = 1,2,...,n$, where $\delta_i, i = 1,2,3,4$ are positive constants.

Remark 1. The function f with $f(u) = -u^3 + au^2$ completely satisfies the condition (5), and the coupling function $h_i, i = 1,2,...,n$ that defined as in [15], [16] actually verifies the condition (6). It means what we realize here is more general and even better than working only for Hindmarsh - Rose model. In other words, the reaction-diffusion system of the Hindmarsh-Rose type is just a particular case of what we are considering.

Theorem 1. *The semi-group $S(t)$ associated with the system* (4) *has a globally connected attractor in H.*

Proof: To show the result, we show the existence of an absorbing set in H. We then show the existence of an absorbing set in V, which attracts all bounded sets of H. This implies that the operator $S(t)$ is uniformly compact with the compact injection of H^1 on L^2 and allows us to conclude by applying Theorem 1.1 page 23 in [14] (see also in [1], [10]).

Existence of an absorbing set in H

We multiply the first equation (4) by u_i and the second one by v_i and integrate over Ω. Using the Green formula and (5), (6), we obtain:

$$\begin{aligned}
A(t) &= \sum_{i=1}^{n} \int_\Omega \Big[\frac{d}{2dt}(b|u_i|^2 + |v_i|^2) + bd\|u_i\|^2 + |v_i|^2 \\
&\qquad + b\delta_1|u_i|^p\Big]dx \\
&= \sum_{i=1}^{n} \int_\Omega [bu_i(f(u_i) + v_i + I + d\Delta u_i + h_i(u,v)) \\
&\quad + v_i(1 - bu_i^2 - v_i) + bd(\nabla u_i)^2 + v_i^2 + b\delta_1|u_i|^p\Big] dx \\
&= \sum_{i=1}^{n} \int_\Omega [bu_i f(u_i) + bu_i h_i(u,v) + bu_i v_i + bu_i I \\
&\qquad + v_i - bu_i^2 v_i + b\delta_1|u_i|^p\big]dx \\
&\leqslant \sum_{i=1}^{n} \int_\Omega [bu_i h_i(u,v) - b\delta_1|u_i|^p + b\delta_3 + bu_i v_i + bu_i I \\
&\qquad + v_i - bu_i^2 v_i + b\delta_1|u_i|^p\big] dx \\
&\leqslant \sum_{i=1}^{n} \int_\Omega [bu_i h_i(u,v) + b\delta_3 + bu_i v_i + v_i + bu_i I \\
&\qquad - bu_i^2 v_i\big] dx \\
&\leqslant \sum_{i=1}^{n} \int_\Omega [b\delta_3 + b\delta_4(\sum_{j=1}^{n} |u_j|^{p_1}|u_i| + |u_i|) \\
&\qquad + |v_i|(1 + (b\delta_4 + b + b|u_i|)|u_i|) + bI|u_i|] dx.
\end{aligned}$$

Then we can find the positive constants k_1, k_2 such that:

$$\begin{aligned}
A(t) \leq \sum_{i=1}^{n} \int_\Omega \Bigg[k_1 + k_2 \Bigg((\sum_{j=1}^{n} |u_j|^{p_1}|u_i| + |u_i| \\
+ |v_i|(1 + (1 + |u_i|)|u_i|))\Big]dx.
\end{aligned}$$

By using Young's inequality, we have:

$$\begin{aligned}
k_2 \int_\Omega |v_i|(1 + (1 + |u_i|)|u_i|)dx &\leq \frac{1}{2}\int_\Omega v_i^2 dx \\
&+ \frac{k_2^2}{2}\int_\Omega (1 + (1 + |u_i|)|u_i|)^2 dx,
\end{aligned}$$

and since $p_1 < p-1$, let $\overline{p} = \dfrac{p}{p_1}$, and let $\overline{q}$ be such that $\dfrac{1}{\overline{p}} + \dfrac{1}{\overline{q}} = 1 \Rightarrow \overline{q} = \dfrac{p}{p - p_1} < p$. Then, for some positive constants k_3, k_4, we have:

$$\begin{aligned}
k_2 \sum_{j=1}^{n} |u_j|^{p_1}|u_i| &\leqslant \sum_{j=1}^{n} \left(\frac{b\delta_1}{16n}|u_j|^p + k_3|u_i|^{\overline{q}}\right) \\
&\leqslant \sum_{j=1}^{n} \left(\frac{b\delta_1}{16n}|u_j|^p + \frac{b\delta_1}{16n}|u_i|^p + k_4\right).
\end{aligned}$$

Thus

$$\begin{aligned}
A(t) &\leq \sum_{i=1}^{n} \int_\Omega \Bigg[k_1 + \sum_{j=1}^{n}\left(\frac{b\delta_1}{16n}|u_j|^p + \frac{b\delta_1}{16n}|u_i|^p + k_4\right) \\
&\qquad + k_2|u_i| + \frac{1}{2}v_i^2 + \frac{k_2^2}{2}(1 + (1 + |u_i|)|u_i|)^2\Bigg] dx \\
&\leq \sum_{i=1}^{n} \int_\Omega \Bigg(\frac{b\delta_1}{8}|u_i|^p + k_2|u_i| + \frac{1}{2}v_i^2 \\
&\qquad + \frac{k_2^2}{2}(1 + (1 + |u_i|)|u_i|)^2\Bigg)dx \\
&\qquad + nk_1|\Omega| + n^2k_4|\Omega| \\
&\leq \sum_{i=1}^{n} \int_\Omega \left(\frac{b\delta_1}{4}|u_i|^p + k_5 + \frac{1}{2}v_i^2\right)dx + nk_1|\Omega| \\
&\qquad + n^2k_4|\Omega| \\
&\leq \sum_{i=1}^{n} \int_\Omega \left(\frac{b\delta_1}{4}|u_i|^p + \frac{1}{2}v_i^2\right)dx + nk_1|\Omega| + n^2k_4|\Omega| \\
&\qquad + nk_5|\Omega|,
\end{aligned}$$

for some positive constant k_5.

Combining the above inequalities, we have:

$$\begin{aligned}
\sum_{i=1}^{n} \int_\Omega \Big[\frac{d}{2dt}(b|u_i|^2 + |v_i|^2) + bd\|u_i\|^2 \\
+ \frac{1}{2}|v_i|^2 + \frac{3b\delta_1}{4}|u_i|^p\Big] dx \leq K,
\end{aligned} \tag{7}$$

where $K = nk_1|\Omega| + n^2k_4|\Omega| + nk_5|\Omega|$.

By using again Young's inequality, we have:

$$\frac{1}{2}\int_\Omega u_i^2 dx \leqslant \frac{b\delta_1}{4}\int_\Omega |u_i|^p dx + \overline{k}_5.$$

Hence, (7) implies:

$$\begin{aligned}
\sum_{i=1}^{n} \int_\Omega \Big[\frac{d}{dt}(b|u_i|^2 + |v_i|^2) + 2bd\|u_i\|^2 \\
+ b(|u_i|^2 + |v_i|^2) + b\delta_1|u_i|^p\Big] dx \leq k_6,
\end{aligned} \tag{8}$$

where $k_6 = 2(K + \overline{k}_5)$; (8) gives in particular:

$$\sum_{i=1}^{n}\left[\frac{d}{dt}(b|u_i|^2 + |v_i|^2) + b(|u_i|^2 + |v_i|^2)\right] \leqslant k_6,$$

which yields, using Gronwall's lemma, we have:

$$\sum_{i=1}^{n}(b|u_i|^2 + |v_i|^2) \le \sum_{i=1}^{n}(b|u_i(0)|^2 + |v_i(0)|^2)\exp(-t) + k_6(1 - \exp(-t)). \tag{9}$$

This inequality (9) gives us the existence of an absorbing set in H. There exists $\beta > 0$ such that for all bounded B of H, there exists a time $T(B)$ such that $S(t)B \subset B_0$ for all $t > T$, where B_0 is the ball of radius β on H. Let $r > 0$, and K denotes a constant that depends only on $\Omega, a, b, c, f, \epsilon, r$. By integrating (8) between t and $t + r$, we obtain:

$$\begin{aligned}\sum_{i=1}^{n}\left(2bd\int_t^{t+r}\|u_i\|^2 + b\delta_1\int_t^{t+r}\int_\Omega |u_i|^p\right) &\leqslant rk_6 + \sum_{i=1}^{n}(b|u_i|^2 + |v_i|^2) \\ &\leqslant rk_6 + K.\end{aligned} \tag{10}$$

Again integrating (8) in t, we also get for $(u_i(0,x), v_i(0,x)) = (u_i(0), v_i(0)) = (u_{i0}, v_{i0}) \in B \subset B_0$,

$$\int_0^t \|u_i\|^2 \leqslant \frac{k_6 t + K}{2bd}, \text{ for all } t \ge 0.$$

The solution v_i of (4) can be written as follows:

$$v_i(t) = v_{i1}(t) + v_{i2}(t),$$

with

$$\begin{cases} v_{i1}(t) = \displaystyle\int_0^t (bu_i + 1)\exp(-(t-s))ds \\ v_{i2}(t) = v_i(0)\exp(-t) \end{cases} \tag{11}$$

and we define the families S_{i1}, S_{i2} of operators from H to H by setting:

$$\begin{cases} S_{i1} : (u_i(0), v_i(0)) \to (u_i(t), v_{i1}(t)) \\ S_{i2} : (u_i(0), v_i(0)) \to (0, v_{i2}(t)) \end{cases} \tag{12}$$

It is straightforward that S_{i2} satisfies, for all bounded set $B \subset H$,

$$r_B(t) \le \exp(-t)\sup_{\varphi\in B}|\varphi|.$$

Uniform compactness of operator $S(t)$

Our aim now is to check the uniform compactness of the operators $S_{i1}(t)$ by using uniform in time a priori on $u_i(t)$ and $v_{i1}(t)$ (see also in [10]).

We multiply the first equation in (4) by $-\Delta u_i$ and integrate over Ω. Thanks to the Green formula, we have:

$$\begin{aligned}\overline{A}(t) &= \sum_{i=1}^{n}\int_\Omega\left[\frac{d}{2dt}(b\|u_i\|^2) + bd(\Delta u_i)^2\right]dx \\ &= \sum_{i=1}^{n}\int_\Omega [b\nabla u_i \nabla(f(u_i) + v_i + I + h_i(u,v))]dx \\ &= \sum_{i=1}^{n}\int_\Omega [-bf(u_i)\Delta u_i - ah_i(u,v)\Delta u_i - bv_i\Delta u_i \\ &\qquad -bI\Delta u_i]dx.\end{aligned}$$

Due to (5), there exists a constant $k_7 > 0$ such that

$$|f(u_i)| \leqslant k_7(1 + |u_i|^{p-1}), \text{ for all } u_i \in \mathbb{R}, i = 1, 2, ..., n,$$

for all $x \in \Omega$. Hence, by using also (6), we get:

$$\begin{aligned}\overline{A}(t) &\leqslant \sum_{i=1}^{n}\int_\Omega [(b\delta_4(1 + \sum_{j=1}^{n}|u_j|^{p_1} + |v_i|) + ak_7(1 + |u_i|^{p-1}) \\ &\qquad + b|v_i| + I)\,|\Delta u_i|]dx \\ &\leqslant \sum_{i=1}^{n}\int_\Omega \Big[\Big(a\delta_4(1 + n|u_i|^{p_1} + |v_i|) + ak_7(1 + |u_i|^{p-1}) \\ &\qquad + b|v_i| + I)\,|\Delta u_i|]dx.\end{aligned}$$

Setting $k_8 = \max(a\delta_4, ak_7, a\delta_4 n, b, I)$, we have:

$$\begin{aligned}\overline{A}(t) &\leqslant \sum_{i=1}^{n} k_8\int_\Omega \Big[\Big(3 + |u_i|^{p_1} + |u_i|^{p-1} + 2|v_i|\Big)|\Delta u_i|\Big]dx \\ &\leqslant \sum_{i=1}^{n}\Big[\frac{k_8^2}{2bd}\int_\Omega\Big(3 + |u_i|^{p_1} + |u_i|^{p-1} + 2|v_i|\Big)^2 \\ &\qquad + \frac{bd}{2}\int_\Omega (\Delta u_i)^2\Big]dx.\end{aligned}$$

Since $p_1 < p - 1$, thus

$$\begin{aligned}&\sum_{i=1}^{n}\int_\Omega\left[\frac{d}{dt}(b\|u_i\|^2)\right]dx \\ &\leqslant \sum_{i=1}^{n}\left[\frac{7k_8^2}{2bd}\int_\Omega\Big(9 + |u_i|^{2p_1} + |u_i|^{2(p-1)} + |v_i|^2\Big)\right]dx \\ &\leqslant \sum_{i=1}^{n} k_9\int_\Omega\Big(1 + |u_i|^{2(p-1)} + |v_i|^2\Big)dx.\end{aligned} \tag{13}$$

for some positive constant k_9.

Thanks to Lemma 1 below for $k = 1$ and (10), we can check the existence of a constant k_{10} such that, for all $t \ge T + r$,

$$\int_t^{t+r}\int_\Omega |u_i|^{2(p-1)} \leqslant k_{10}.$$

This implies by applying the uniform Gronwall lemma (see [5] page 822, [1]):

$$\sum_{i=1}^{n} b\,\|u_i\|^2 \leqslant k_{11}, \quad \forall t \geqslant T + 2r. \tag{14}$$

We now derive a time-uniform estimate of $v_{i1}(t)$ in $H^1(\Omega)$. First, it is easy to deduce from (9) and (11) that there exists a constant $k_{12} > 0$ such that, for all $t \ge 0$,

$$|v_{i1}|^2 \leqslant k_{12}. \tag{15}$$

We then set:

$$w_j = \frac{\partial v_{i1}}{\partial x_j}, \ j = 1, 2, ..., N,$$

and w_j satisfies:

$$\begin{cases} \dfrac{\partial w_j}{\partial t} = b\dfrac{\partial u_i}{\partial x_j} - w_j \\ w_j(0) = 0 \end{cases} \tag{16}$$

By multiplying (16) by w_j and integrating over Ω, we get:

$$\begin{aligned}\frac{d}{2dt}|w_j|^2+|w_j|^2&=\int_\Omega\left(w_j(b\frac{\partial u_i}{\partial x_j}-w_j)+bw_j^2\right)\\&\leqslant\int_\Omega bw_j\frac{\partial u_i}{\partial x_j}\\&\leqslant\frac{1}{2}|w_j|^2+k_{13}\int_\Omega\left(\frac{\partial u_i}{\partial x_j}\right)^2.\end{aligned}$$

Then this implies that:

$$\frac{d}{dt}|w_j|^2+|w_j|^2\leqslant 2k_{13}\int_\Omega\left(\frac{\partial u_i}{\partial x_j}\right)^2 \tag{17}$$

Summing (17) from $j=1$ to $j=N$, we finally obtain,

$$\frac{d}{dt}\|v_{i1}\|^2+b\|v_{i1}\|^2\leqslant 2k_{13}\|u_i\|^2.$$

We then integrate this inequality; this gives, since $v_{i1}(0)=0$,

$$\begin{aligned}\|v_{i1}\|^2&\leqslant 2k_{13}\int_0^t\|u_i(s)\|^2\exp((s-t))ds\\&\leqslant 2k_{13}\int_0^{T+2r}\|u_i(s)\|^2\exp((s-t))ds\\&\qquad+2k_{13}\int_{T+2r}^t\|u_i(s)\|^2\exp((s-t))ds\\&\leqslant 2k_{13}\frac{k_6(T+2r)+K}{2bd}+2k_{13}k_{11}.\end{aligned} \tag{18}$$

The estimates (9), (14), (15) and (18) provide the uniform compactness of the operators S_{i1}. Indeed, if $(u_i(0),v_i(0))$ belongs to a bounded subset and for all $t\geq T+2r$, then $S_{i1}(u_i(0),v_i(0))$ belongs to a bounded set in $(H^1(\Omega))^2$ independently of t and relatively compact in H. We have proved the existence of a bounded absorbing set. Hence, this gives Theorem 1. ∎

Remark 2. The result of Theorem 1 shows the existence of the global attractor of the system (3) in the space $(L^2(\Omega)\times L^2(\Omega))^n$. Next, we prove that this attractor is also bounded in $L^q(\Omega)$ for all $q\in[1,+\infty)$ (Lemma 1), and better in $L^\infty(\Omega)$ (Theorem 2).

Lemma 1. *The attractor called A defined in Theorem 1 is bounded in $L^q(\Omega)$ for all $q\in[1,+\infty)$.*

Proof: Let $\alpha(k)=k(p-2)+2$. Since $\alpha(k)\to+\infty$ as $k\to+\infty$, it suffices to prove that A is bounded in $L^{\alpha(k)}(\Omega)$ for all $k\in\mathbb{N}$. Let $r>0$ be fixed; we shall prove by induction on k that:

$$A \text{ is bounded in } L^{\alpha(k)}(\Omega), \tag{19}$$

$$\sup_{(u_{i0},v_{i0})\in A}\int_t^{t+r}\int_\Omega|u_i|^{\alpha(k+1)}\leqslant K,\quad\text{for all } t\geqslant 0, \tag{20}$$

where $(u_i,v_i), i=1,2,...,n$ is the solution of (4)-(6). Hereafter, we denote by K any constant that depends on the data and k.

- When $k=0\Rightarrow\alpha(0)=2$ and $\alpha(1)=p$. We have already proved this in Theorem 1. Let $(u_{i0},v_{i0})\in A, i=1,2,...,n$, we infer from (10) that,

$$\sum_{i=1}^n\left(b\delta_1\int_t^{t+r}\int_\Omega|u_i|^p\right)\leqslant rk_6+\sum_{i=1}^n(b|u_i|^2+|v_i|^2)\leqslant K,$$

thanks to the case $k=0$. Hence, we proved for $k=0$.

- We assume that (19) and (20) are hold for $k-1$ $(k\geq 1)$. In particular, there exists a constant $K>0$ such that

$$\int_\Omega|v_i|^{\alpha(k-1)}\leqslant K,\quad\text{for all } (u_i,v_i)\in A. \tag{21}$$

Let $(u_{i0},v_{i0})\in A, i=1,2,...,n$. By multiplying the first equation in (4) by $u_i|u_i|^{\alpha(k)-2}$ and integrating over Ω, we obtain:

$$\begin{aligned}&\sum_{i=1}^n\frac{1}{\alpha(k)}\frac{d}{dt}\int_\Omega|u_i|^{\alpha(k)}\\&=\sum_{i=1}^n\int_\Omega[f(u_i)u_i|u_i|^{\alpha(k)-2}+du_i|u_i|^{\alpha(k)-2}\Delta u_i\\&\qquad+u_i|u_i|^{\alpha(k)-2}(v_i+I+h(u_i,v_i))]\\&=\sum_{i=1}^n\int_\Omega[f(u_i)u_i|u_i|^{\alpha(k)-2}-(\alpha(k)-1)d|u_i|^{\alpha(k)-2}(\nabla u_i)^2\\&\qquad+u_i|u_i|^{\alpha(k)-2}(v_i+I+h(u_i,v_i))]\\&\leqslant\sum_{i=1}^n\int_\Omega\Big[f(u_i)u_i|u_i|^{\alpha(k)-2}\\&\qquad+u_i|u_i|^{\alpha(k)-2}(v_i+I+h(u_i,v_i))\Big]\\&\leqslant\sum_{i=1}^n\int_\Omega\Big[(v_i+I+h(u_i,v_i))u_i|u_i|^{\alpha(k)-2}\\&\qquad-|u_i|^{\alpha(k)-2}\delta_1|u_j|^p+|u_i|^{\alpha(k)-2}\delta_3\Big].\end{aligned}$$

Then, this implies:

$$\begin{aligned}&\sum_{i=1}^n\frac{1}{\alpha(k)}\frac{d}{dt}\int_\Omega|u_i|^{\alpha(k)}+\delta_1\int_\Omega|u_i|^{\alpha(k+1)}\\&\leqslant\sum_{i=1}^n\int_\Omega\Big[(v_i+I+h(u_i,v_i))u_i|u_i|^{\alpha(k)-2}+\delta_3|u_i|^{\alpha(k)-2}\Big]\\&\leqslant\sum_{i=1}^n\int_\Omega[\delta_3|u_i|^{\alpha(k)-2}+\delta_4|u_i|^{\alpha(k)-1}+\sum_{j=1}^n\delta_4|u_j|^{p_1}|u_i|^{\alpha(k)-1}\\&\qquad+(|v_i|+I+\delta_4|v_i|)|u_i|^{\alpha(k)-1}]\\&\leqslant\sum_{i=1}^n\int_\Omega(\frac{\delta_1}{4}|u_i|^{\alpha(k+1)}+K+(\delta_4|v_i|+|v_i|+I)|u_i|^{\alpha(k)-1}\\&\qquad+\sum_{j=1}^n\delta_4|u_j|^{p_1}|u_i|^{\alpha(k)-1}),\end{aligned}$$

where K is a positive constant. Thus we have,

$$\begin{aligned}&\sum_{i=1}^n\frac{1}{\alpha(k)}\frac{d}{dt}\int_\Omega|u_i|^{\alpha(k)}+\frac{3\delta_1}{4}\int_\Omega|u_i|^{\alpha(k+1)}\\&\qquad\leqslant K+\sum_{i=1}^n\int_\Omega((\delta_4|v_i|+v_i+I)|u_i|^{\alpha(k)-1}\\&\qquad\qquad+\sum_{j=1}^n\delta_4|u_j|^{p_1}|u_i|^{\alpha(k)-1}).\end{aligned}$$

Let $\overline{p}=\dfrac{\alpha(k+1)}{\alpha(k)-1}$ and let $\overline{q}$ be such that $\dfrac{1}{\overline{p}}+\dfrac{1}{\overline{q}}=1$, we can check that $p_1\overline{q}<\alpha(k+1)$. With the Young inequality, we have:

$$\delta_4 \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega |u_j|^{p_1}|u_i|^{\alpha(k)-1}$$
$$\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega m_1|u_j|^{p_1\overline{q}} + \frac{\delta_1}{8n}|u_i|^{\alpha(k+1)}$$
$$\leqslant \sum_{i=1}^{n}\int_\Omega nm_1|u_i|^{p_1\overline{q}} + \frac{\delta_1}{8}|u_i|^{\alpha(k+1)}$$
$$\leqslant \sum_{i=1}^{n}\int_\Omega \frac{\delta_1}{4}|u_i|^{\alpha(k+1)} + K,$$

and

$$\int_\Omega (\delta_4\,|v_i| + |v_i| + I)|u_i|^{\alpha(k)-1} \leq \frac{\delta_1}{4}\int_\Omega |u_i|^{\alpha(k+1)} + m_2\int_\Omega |v_i|^{\overline{q}} + K,$$

where K is a suitable positive constant.
Since $\overline{q} \leq \alpha(k-1)$, using the Holder inequality and thanks to (21), we also have:

$$\int_\Omega |v_i|^{\overline{q}} \leqslant |\Omega|^{1-\frac{\overline{q}}{\alpha(k-1)}}\left(\int_\Omega |v_i|^{\alpha(k-1)}\right)^{\frac{\overline{q}}{\alpha(k-1)}} \leqslant K.$$

Combining the above inequalities, we finally obtain:

$$\sum_{i=1}^{n}\frac{1}{\alpha(k)}\frac{d}{dt}\int_\Omega |u_i|^{\alpha(k)} + \frac{\delta_1}{4}\int_\Omega |u_i|^{\alpha(k+1)} \leqslant K. \quad (22)$$

Thanks to the induction assumption (20) for $k-1$, we can apply the uniform Gronwall lemma to (22) and we conclude that:

$$\int_\Omega |u_i|^{\alpha(k)} \leqslant K, \quad \forall t \geqslant r.$$

Since $S(r)A = A$, this implies:

$$\int_\Omega |u_i|^{\alpha(k)} \leqslant K, \quad \text{for all } (u_i, v_i) \in A, i = 1, 2, ..., n. \quad (23)$$

By integrating (22) between t and $t+r$, and using (23), we find,

$$\sup_{(u_{i0},v_{i0})\in A}\int_t^{t+r}\int_\Omega |u_i|^{\alpha(k+1)} \leqslant K, \quad \text{for all } t \geqslant 0.$$

It remains to check that,

$$\int_\Omega |v_i|^{\alpha(k)} \leqslant K, \quad \text{for all } (u_i, v_i) \in A, i = 1, 2, ..., n, \quad (24)$$

for all $t \geqslant 0$.

Let $(u_i, v_i) \in A, i = 1, 2, ..., n$. We claim that there exists a sequence $(u_{im}, v_{im}) \in A, i = 1, 2, ..., n$ and a sequence $t_m \to +\infty$ such that,

$$S_{i1}(t_m).(u_{im}, v_{im}) \to (u_i, v_i), \text{ as } m \to +\infty$$

where $S_{i1}(t)$ is given by (12). Indeed, this follows from the functional invariance of A and the property for $S_{i2}(t) = S(t) - S_{i1}(t)$. Then, by (11) and (23), we have, for all $t \geq 0$,

$$\|v_{i1m}(t)\|_{L^{\alpha(k)}(\Omega)} \leqslant \int_0^t \|bu_{im} + 1\|_{L^{\alpha(k)}(\Omega)} \exp(-(t-s))ds$$
$$\leqslant K\int_0^t \||u_{im}| + 1\|_{L^{\alpha(k)}(\Omega)} \exp(-(t-s))ds$$
$$\leqslant K.$$

which implies (24), since there exists a subsequence m_l such that,

$$\|v_i(t)\|_{L^{\alpha(k)}(\Omega)} \leqslant \liminf_{l\to+\infty} \|v_{i1m_l}(t_{m_l})\|_{L^{\alpha(k)}(\Omega)}.$$

We complete the lemma. ■

Theorem 2. *Under assumptions* (5) *and* (6)*, the global attractor A defined in Theorem 1 is bounded in $L^\infty(\Omega)$.*

Proof: Let $(\overline{u}_i, \overline{v}_i) \in A$ and let $\overline{t} > 0$. Since A is a functional invariant set, there exists a solution (u_i, v_i) of (4), (5) and (6) satisfying $(u_{i0}, v_{i0}) \in A$ and $(u_i(\overline{t}), v_i(\overline{t})) = (\overline{u}_i, \overline{v}_i)$. Introducing the semigroup $\Sigma(t)$ associated with the linear operator $\frac{\partial}{\partial t} - d\Delta - I$ and with the boundary condition of Neumann type, it is classical that u_i can be written as:

$$u_i(t) = \Sigma(t)u_{i0} + \int_0^t \Sigma(t-s)\left\{f(u_i(s)) + h_i(u(s), v(s)) + v_i(s)\right\}ds, \quad (25)$$

for all $t \geq 0$.

The semigroup $\Sigma(t)$ satisfies the regularity property (see Rothe [13]),

$$\|\Sigma(t)\varphi\|_{L^\infty(\Omega)} \leqslant km(t)^{-\frac{1}{2}}e^{-\lambda t}\|\varphi\|_{L^n(\Omega)},$$

where $m(t) = \min(1, t)$, λ is the smallest eigenvalue of the operator $-d\Delta - I$ associated with the boundary condition de Neumann type, and k is a positive constant. Also, by Lemma 1, there exists a constant $K > 0$ such that:

$$\begin{cases} \|u_i\|_{L^n(\Omega)} \leqslant K \\ \|f(u_i) + h_i(u, v) + v_i + u_i\|_{L^n(\Omega)} \leqslant K \end{cases}$$

Hence, we deduce from (25) that:

$$\|u_i(t)\|_{L^\infty(\Omega)}$$
$$\leqslant kK\left\{m(t)^{-\frac{1}{2}}e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}m(t-s)^{-\frac{1}{2}}ds\right\},$$
(for all $t \geqslant 0$)
$$\leqslant kK\left\{m(\frac{\overline{t}}{2})^{-\frac{1}{2}} + 2 + \frac{1}{\lambda}\right\}, \text{ for all } t \geqslant \frac{\overline{t}}{2}.$$

In particular, $\overline{u}_i = u_i(\overline{t})$ satisfies:

$$\|\overline{u}_i\|_{L^\infty(\Omega)} \leqslant kK\left\{m(\frac{\overline{t}}{2})^{-\frac{1}{2}} + 2 + \frac{1}{\lambda}\right\},$$

for all $(\overline{u}_i, \overline{v}_i) \in A$.

Finally, the bound on $\|\overline{v}_i\|_{L^\infty(\Omega)}$ follows from the one on $\|\overline{u}_i\|_{L^\infty(\Omega)}$ as in Lemma 1 above. This concludes the proof of Theorem 2. ■

III. Conclusion

In this paper, we proved the existence of a global attractor of networks of n coupled reaction-diffusion systems according to different network topologies that satisfied some necessary conditions of the coupling function and the function f. The global attractor exists in the space $(L^2(\Omega) \times L^2(\Omega))^n$, and also bounded in $L^\infty(\Omega)$. Our results are more general and even better than said the paper title. In other words, the Hindmarsh-Rose model is just a particular case satisfying all assumptions we made.

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