Positive Solutions for Generalized p-Laplacian Systems with Uncoupled Boundary Conditions

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***Abstract*—We prove that under some assumptions on h and g, it is typical for generalized p-Laplacian systems with uncoupled integral boundary conditions to exhibit intervals for λ and μ such that there are positive functions which satisfied the above systems. Through constructing the appropriate cone, we show that there are positive functions which satisfied the above systems by using a theorem called Guo-Krasnosel'skii theorem.**

***Index Terms*—Generalized p-Laplacian; Positive solutions; Fixed point theorem; Uncoupled integral boundary conditions.**

I. Introduction

In the past few years, mathematical modelling of many nonlinear problems, for example, control theory, image processing, signal, viscoelasticity, physics, biophysics, biophysics, which come from scientific disciplines and various fields of engineering, these practical problems inspire people to consider fractional order differential equations [1-8]. Recently, some authors began to seek positive functions which satisfied the equations involved with p-Laplacian operator.

The following equation contains fractional derivatives with operator called p-Laplace

$$D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) = f(t,u(t)),\ t\in[0,1],$$
$$u(0)=-u(1),\ D_{0+}^{\alpha}u(0)=-D_{0+}^{\alpha}u(1),$$

was studied by the authors in [9].

Some other authors go on to discuss fractional equations with the more generalized operator called p-Laplace.

The following equation contains fractional derivatives and parameter λ with generalized p-Laplacian operator

$$D_{0+}^{\beta}(\phi(D_{0+}^{\alpha}\theta(t))) = \lambda f(\theta(t)),\ t\in(0,1),$$
$$\theta(0)=\theta'(0)=\theta'(1)=0,$$
$$\phi(D_{0+}^{\alpha}\theta(0)) = (\phi(D_{0+}^{\alpha}\theta(1)))'=0,$$

was studied by the authors in [10].

On the other hand, coupled systems with fractional equations have been recently applied in various areas of natural sciences, mathematical biology and engineering due to their widely applications, these topic have been treated by many authors, see the interesting recent paper [11-15,16,17,20].

Om Kalthoum Wanassi and Faten Toumi [11] focused on seeking positive functions which satisfied the following relationship

$$D^{\alpha}x(t)+\mu_1 f(t,x(t),y(t))=0,\ n-1\le\alpha\le n,\ t\in(0,1),$$
$$D^{\beta}y(t)+\mu_2 g(t,x(t),y(t))=0,\ m-1\le\beta\le m,\ t\in(0,1),$$
$$x(1)=\lambda_1\int_0^1 x(s)ds,\ x^{(j)}(0)=0,\ 0\le j\le n-2,$$
$$y(1)=\lambda_2\int_0^1 y(s)ds,\ y^{(j)}(0)=0,\ 0\le j\le m-2.$$

Henderson [12] considered the system contains fractional derivatives

$$D_{0+}^{\alpha}x(t)+f(t,y(t))=0,\ n-1<\alpha\le n,\ t\in(0,1),$$
$$D_{0+}^{\beta}y(t)+g(t,x(t))=0,\ m-1<\beta\le m,\ t\in(0,1),$$

with the uncoupled relationship

$$x(1)=\int_0^1 x(s)dH(s),\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0,$$
$$y(1)=\int_0^1 y(s)dK(s),\ y(0)=y'(0)=\cdots=y^{(m-2)}(0)=0.$$

However, in all of the previously mentioned investigations, there is always either the considered coupled systems were only involved with fractional equations or the considered problems were only involved with generalized p-Laplacian operator. Consequently, our results demonstrate the positive functions which satisfied the coupled fractional systems with generalized p-Laplacian operator. So far, there are no paper considered such coupled systems. Inspired by the above cited literature, the task of this paper is to seek the positive functions which satisfied the for a generalized p-Laplacian system of fractional equations which contains some parameters and is subject to uncoupled boundary conditions

$$\begin{cases} D_{0+}^{\beta}(\phi({}^cD_{0+}^{\alpha}\Phi(t))) = \lambda h(t,\Phi(t),\Psi(t)),\ t\in(0,1),\\ D_{0+}^{\beta}(\phi({}^cD_{0+}^{\alpha}\Psi(t))) = \mu g(t,\Phi(t),v(t)),\ t\in(0,1),\end{cases} \tag{1}$$

$$\begin{cases} \Phi'(0)-\delta\Phi(\xi)=0,\ \Phi'(1)+\gamma\Phi(\eta)=0,\ {}^cD_{0+}^{\alpha}\Phi(0)=0,\\ \phi({}^cD_{0+}^{\alpha}\Phi(1))=\sum_{i=1}^{m-2}a_i\phi({}^cD_{0+}^{\alpha}\Phi(\xi_i)),\\ \Psi'(0)-\delta\Psi(\xi)=0,\ \Psi'(1)+\gamma\Psi(\eta)=0,\ {}^cD_{0+}^{\alpha}\Psi(0)=0,\\ \phi({}^cD_{0+}^{\alpha}\Psi(1))=\sum_{i=1}^{m-2}b_i\phi({}^cD_{0+}^{\alpha}\Psi(\eta_i)),\end{cases} \tag{2}$$

where D_{0+}^{β} is the Riemann-Liouville fractional derivative, ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative. $,1<\beta\le 2, 0\le\xi\le\eta\le 1, 1<\alpha\le 2,\ 0\le\delta,\gamma\le 1,,0<a_i,\xi_i<1, 0<b_i,\eta_i<1, \sum_{i=1}^{m-2}a_i\xi_i^{\beta-1}<1, \sum_{i=1}^{m-2}b_i\eta_i^{\beta-1}<1$, λ,μ are two positive parameters. Let $\Delta=\delta(1+\gamma\eta-\gamma\xi)+\gamma$. Furthermore, h,g,ϕ satisfy

(H_1) The odd strictly increasing function $\phi\in C'(R,R)$ and the following relationship holds true such that

$$\varphi_1(x)\phi(y)\le\phi(x,y)\le\varphi_2(x)\phi(y),$$

for two increasing homeomorphisms $\varphi_1,\varphi_2:(0,+\infty)\to(0,+\infty)$;

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(H_2) $h, g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous;
(H_3) $(\alpha-1)(1-\delta\xi) > \Delta$.

A coupled continuous positive functions $(\Phi(t), \Psi(t))$ on the interval [0,1] and $(\Phi, \Psi) \neq (0,0)$ is called the positive solution of the relational expression (1),(2).

II. Some lemmas

Definition 2.1 [8] $\beta > 0$ order integral involved fractional order of function $g : (0,\infty) \to R$ denoted as $I_{0+}^{\beta} g$ is expressed by

$$(I_{0+}^{\beta} g)(s) = \frac{1}{\Gamma(\beta)} \int_0^s (s-\theta)^{\beta-1} g(\theta) d\theta.$$

Definition 2.2 [8] $\beta > 0$ order Riemann-Liouville derivative involved fractional order of function $g : (0,\infty) \to R$ denoted as $D_{0+}^{\beta} g$ is expressed by

$$(D_{0+}^{\beta} g)(s) = \frac{1}{\Gamma(n-\beta)} (\frac{d}{ds})^n \int_0^s \frac{g(\theta)}{(s-\theta)^{\beta-n+1}} d\theta,$$

in the place $n = [\beta] + 1$.

Definition 2.3 [8] $\beta > 0$ order Caputo's derivative involved fractional order about function $g : (0,\infty) \to R$ can be expressed as

$$({}^c D_{0+}^{\beta} g)(s) = \frac{1}{\Gamma(n-\beta)} \int_0^s \frac{g^{(n)}(\theta)}{(s-\theta)^{\beta-n+1}} d\theta,$$

in the place $n = [\beta] + 1$.

Lemma 2.1 [8] Presume $\beta > 0$ and $n = [\beta] + 1$. Assume that $g, {}^c D_{0+}^{\beta} g, D_{0+}^{\beta} g \in L'(0,1)$, one has

$$I^{\beta} \, {}^c D_{0+}^{\beta} g(t) = g(t) - h_1 - h_2 t - \cdots - h_n t^{n-1},$$

$$I^{\beta} D_{0+}^{\beta} g(t) = g(t) - d_1 t^{\beta-1} - d_2 t^{\beta-2} - \cdots - d_n t^{\beta-n},$$

in the place $c_i, d_i,\ i$ can be taken as 1,2 until n are real numbers.

Lemma 2.2 [18] Given $1 < \alpha \leq 2$, for continuous function $k(t)$ on the interval $[0,1]$ and the representation

$$w(t) = \int_0^1 k(s) G(t,s) ds,$$

where

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta(1+\gamma\eta-\gamma t)(\xi-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, \\ +\frac{(1-\delta\xi+\delta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\delta\xi+\delta t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, \xi \geq s, t \geq s; \\ \frac{\delta(1+\gamma\eta-\gamma t)(\xi-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, \\ +\frac{(1-\delta\xi+\delta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\delta\xi+\delta t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, \xi \geq s, s \geq t; \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-\delta\xi+\delta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\delta\xi+\delta t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, \eta \geq s \geq \xi, t \geq s; \\ \frac{(1-\delta\xi+\delta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} \\ +\frac{\gamma(1-\delta\xi+\delta t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \eta \geq s \geq \xi, s \geq t; \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-\delta\xi+\delta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, \\ s \geq \eta, t \geq s; \\ \frac{(1-\delta\xi+\delta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, s \geq \eta, s \geq t, \end{cases} \tag{3}$$

is the unique function satisfied the following equation and condition.

$$\begin{cases} {}^c D_{0+}^{\alpha} w(t) + k(t) = 0, \quad 0 < t < 1, \\ w'(0) - \delta w(\xi) = 0,\ w'(1) + \gamma w(\eta) = 0. \end{cases} \tag{4}$$

Let

$$M = \frac{\delta(1+\gamma\eta) + \gamma(1-\delta\xi+\delta) + (\alpha-1)(1-\delta\xi+\delta)}{\Delta\Gamma(\alpha)},$$

$$\Upsilon = \frac{(1-\delta\xi+\frac{1}{4}\delta)(\alpha-1) - \Delta}{(1+\gamma\eta)\delta + (1-\delta\xi+\delta)\gamma + (\alpha-1)(1-\delta\xi+\delta)}$$

Lemma 2.3 [18] Presume $(H_1), (H_2), (H_3)$ hold. The function $G(t,s)$ expressed by equation (3) conforms to the following relationships:
(i) The expression $G(t,s)$ is positive and continuous;
(ii) The function $G(t,s)$ less than or equal to $M(1-s)^{\alpha-2}$ for s, t belongs to the interval $(0,1)$;
(iii) The following relationship holds

$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t,s)$ greater than or equal to $\Upsilon M(1-s)^{\alpha-2},\ s \in (0,1)$,

for some positive number Υ.

Lemma 2.4 [19] We define

$$p(s) = 1 - \sum_{s \leq \xi_i} a_i (\frac{\xi_i - s}{1-s})^{\beta-1}, \tag{5}$$

then $p(s)$ is nondecreasing and positive on $[0,1]$.

Lemma 2.5 [19] For continuous function $y(t)$ on the interval $[0,1]$, the system

$$D_{0+}^{\beta} \nu(t) + y(t) = 0, \quad 0 < t < 1, \tag{6}$$

$$\nu(0) = 0, \quad \nu(1) = \sum_{i=1}^{m-2} a_i \nu(\xi_i), \tag{7}$$

has a unique expression $\nu(t)$ that satisfies the above problem

$$\nu(t) = \int_0^1 K(t,s) y(s) ds,$$

in this place

$$K(t,s) = \frac{1}{p(0)\Gamma(\beta)} \begin{cases} [(1-s)t]^{\beta-1} p(s) - (t-s)^{\beta-1} p(0), \\ \quad 1 \geq t \geq s \geq 0, \\ p(s)[(1-s)t]^{\beta-1}, \\ \quad 1 \geq s \geq t \geq 0. \end{cases} \tag{8}$$

Lemma 2.6 [19] The Green's function $K(t,s)$ of problem (6), (7) satisfies the following statement:
(a) $K(t,s)$ is positive for s, t belongs to the interval $(0,1)$;
(b) $K(t,s)$ greater than or equal to $mt^{\beta-1}(1-t)(1-s)^{\beta-1}s$ for s, t belongs to the interval $(0,1)$;
(c) $K(t,s)$ less than or equal to $\overline{M}(1-s)^{\beta-1}s$ for s, t belongs to the interval $(0,1)$;
in this place

$$m_1 = \inf_{0<\tau\leq 1} \frac{p(\tau) - p(0)}{\tau},\ m = \frac{p(0) + m_1}{p(0)\Gamma(\beta)},$$

$$M_1 = \sup_{0<\tau\leq 1} \frac{p(\tau) - p(0)}{\tau}, \quad \overline{M} = \frac{p(0)(\beta-1) + M_1}{p(0)\Gamma(\beta)}.$$

We take the following fractional system into account

$$D_{0+}^{\beta}(\phi({}^c D_{0+}^{\alpha} \Phi(t))) = \lambda h(t, \Phi(t), \Psi(t)),\ 0 < t < 1, \tag{9}$$

$$\Phi'(0) - \delta\Phi(\xi) = 0,\ \Phi'(1) + \gamma\Phi(\eta) = 0,\ {}^c D_{0+}^{\alpha} \Phi(0) = 0,$$

$$\phi({}^cD_{0^+}^{\alpha}\Phi(1))=\sum_{i=1}^{m-2}a_i\phi({}^cD_{0^+}^{\alpha}\Phi(\xi_i)). \tag{10}$$

Lemma 2.7 Presume $(H_1),(H_2),(H_3)$ hold, the following relationship

$$\Phi(t)=\int_0^1\phi^{-1}(\int_0^1\lambda H(s,\theta)h(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds.$$

satisfies the equation (9), (10).

Proof: Choose $\Psi(t)=\phi({}^cD_{0^+}^{\alpha}\Phi(t))$, thus (9), (10) can be rewritten as

$$D_{0^+}^{\beta}\Psi(t)=\lambda h(t,\Phi(t),\Psi(t)),$$

$$\Psi(0)=0,\quad \Psi(1)=\sum_{i=1}^{m-2}a_i\Psi(\xi_i).$$

From lemma 2.5, one has

$$\Psi(t)=-\int_0^1\lambda K(t,s)h(s,\Phi(s),\Psi(s))ds,$$

which means

$$\phi({}^cD_{0^+}^{\alpha}\Phi(t))=-\int_0^1\lambda K(t,s)h(s,\Phi(s),\Psi(s))ds,$$

thus

$${}^cD_{0^+}^{\alpha}\Phi(t)+\phi^{-1}(\int_0^1\lambda K(t,s)h(s,\Phi(s),\Psi(s))ds)=0$$

and

$$\Phi'(0)-\delta\Phi(\xi)=0,\ \Phi'(1)+\gamma\Phi(\eta)=0.$$

Lemma 2.2 implies that

$$\begin{aligned}\Phi(t)\ &=\textstyle\int_0^1\phi^{-1}(\int_0^1\lambda K(s,\theta)\\ &\quad h(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds.\end{aligned}$$

■

Lemma 2.8 [19] Suppose that (H_1) holds, then

$$\varphi_2^{-1}(x)y\le\phi^{-1}(x\phi(y))\le\varphi_1^{-1}(x)y,\ x,y\ \text{belong to}\ (0,+\infty).$$

Remark 2.1: Similar to Lemma 2.7, one can get similar results for the following fractional expression which contain generalized operator called p-Laplace

$$D_{0^+}^{\beta}(\phi({}^cD_{0^+}^{\alpha}\Psi(t)))=\mu g(t,\Phi(t),\Psi(t)),\ 0<t<1,$$

$$\Psi'(0)-\delta\Psi(\xi)=0,\ \Psi'(1)+\gamma\Psi(\eta)=0,\ {}^cD_{0^+}^{\alpha}\Psi(0)=0,$$

$$\phi({}^cD_{0^+}^{\alpha}\Psi(1))=\sum_{i=1}^{m-2}b_i\phi({}^cD_{0^+}^{\alpha}\Psi(\eta_i)).$$

III. Existence

We introduce

$$h_0^s=\limsup_{\Phi+\Psi\to0^+}\max_{t\in[0,1]}\frac{h(t,\Phi,\Psi)}{\phi(\Phi+\Psi)},$$

$$g_0^s=\limsup_{\Phi+\Psi\to0^+}\max_{t\in[0,1]}\frac{g(t,\Phi,\Psi)}{\phi(\Phi+\Psi)},$$

$$h_\infty^i=\liminf_{\Phi+\Psi\to\infty}\min_{t\in[\frac14,\frac34]}\frac{h(t,\Phi,\Psi)}{\phi(\Phi+\Psi)},$$

$$g_\infty^i=\liminf_{\Phi+\Psi\to\infty}\min_{t\in[\frac14,\frac34]}\frac{g(t,\Phi,\Psi)}{\phi(\Phi+\Psi)}.$$

Under the normal maximum norm $\|\cdot\|$, space consisting of continuous function on interval [0,1] denoted by X, which is a Banach space. Mark

$$Y=X\times X,$$

under the norm

$$\|(\Psi,\Phi)\|_Y=\|\Psi\|+\|\Phi\|,$$

Y is a Banach space. We mark P as

$$\begin{aligned}P\ &=\{(\Psi,\Phi)\in Y:\Psi(t)\ge0,\Phi(t)\ge0,\\ &\quad \textstyle\inf_{t\in[\frac14,\frac34]}(\Psi(t)+\Phi(t))\ge\Upsilon\|(\Psi,\Phi)\|_Y\}.\end{aligned}\tag{11}$$

In regard to $\lambda,\mu>0$, operator T_1 maps Y into X, operator T_2 maps Y into X, and operator T maps Y into Y defined as

$$\begin{aligned}T_1(\Phi,\Psi)(t)\ &=\textstyle\int_0^1\phi^{-1}(\int_0^1\lambda\\ &\quad K(s,\theta)h(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds,\end{aligned}\tag{12}$$

$$\begin{aligned}T_2(\Phi,\Psi)(t)\ &=\textstyle\int_0^1\phi^{-1}(\int_0^1\mu\\ &\quad K(s,\theta)g(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds,\end{aligned}\tag{13}$$

and

$$T(\Phi,\Psi)=(T_1(\Phi,\Psi),T_2(\Phi,\Psi)),\quad (\Phi,\Psi)\in Y. \tag{14}$$

Then if (Φ,Ψ) satisfies $T(\Phi,\Psi)=(\Phi,\Psi)$, the (Φ,Ψ) is a coupled solutions for (1), (2).

Lemma 3.1 Let $(H_1),(H_2),(H_3)$ are true, then T denoted by (14) mapping P to P is compact, moreover, T is continuous.

Proof: Take any element (Φ,Ψ) in P. Operator $T_1(\Phi,\Psi)(t)$ is nonnegative because of the nonnegativity of $K(t,s)$ and function $G(t,s)$ and h. Operator $T_2(\Phi,\Psi)(t)$ is nonnegative because of the nonnegativity of $K(t,s)$ and function $G(t,s)$ and g.

Lemma 2.3 implies that

$$\begin{aligned}\|T_1(\Phi,\Psi)(t)\|\ &\le\textstyle\int_0^1M(1-s)^{\alpha-2}\phi^{-1}(\int_0^1\lambda\\ &\quad K(s,\theta)h(\theta,\Phi(\theta),\Psi(\theta))d\theta)ds,\end{aligned}$$

$$\begin{aligned}\textstyle\inf_{t\in[\frac14,\frac34]}T_1(\Phi,\Psi)(t)\ &\ge\textstyle\int_0^1\Upsilon M(1-s)^{\alpha-2}\phi^{-1}(\int_0^1\lambda\\ &\quad K(s,\theta)h(\theta,\Phi(\theta),\Psi(\theta))d\theta)ds\\ &\quad \ge\Upsilon\|T_1(\Phi,\Psi)\|,\end{aligned}$$

$$\begin{aligned}\|T_2(\Phi,\Psi)(t)\|\ &\le\textstyle\int_0^1M(1-s)^{\alpha-2}\phi^{-1}(\int_0^1\mu\\ &\quad K(s,\theta)g(\theta,\Phi(\theta),\Psi(\theta))d\theta)ds,\end{aligned}$$

$$\begin{aligned}\textstyle\inf_{t\in[\frac14,\frac34]}T_2(\Phi,\Psi)(t)\ &\ge\textstyle\int_0^1\Upsilon M(1-s)^{\alpha-2}\phi^{-1}(\int_0^1\mu\\ &\quad K(s,\theta)g(\theta,\Phi(\theta),\Psi(\theta))d\theta)ds\\ &\quad \ge\Upsilon\|T_2(\Phi,\Psi)\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\textstyle\inf_{t\in[\frac14,\frac34]}(T_1(\Phi,\Psi)(t)+T_2(\Phi,\Psi)(t))\\ &\ge\textstyle\inf_{t\in[\frac14,\frac34]}T_1(\Phi,\Psi)(t)+\inf_{t\in[\frac14,\frac34]}T_2(\Phi,\Psi)(t)\\ &\ge\Upsilon\|T_1(\Phi,\Psi)\|+\Upsilon\|T_2(\Phi,\Psi)\|=\Upsilon\|T(\Phi,\Psi)\|_Y.\end{aligned}$$

Furthermore, the continuity of operator $T:P\to P$ can be obtained from the continuity of G,K and h,g. By using the classical proof method, we can prove T_1 and T_2 mapping P to P are compact, and then the operator T is completely continuous. ■

Denote

$$B=\int_0^1 M(1-s)^{\alpha-2}\varphi_1^{-1}(\int_0^1 \overline{M}(1-\theta)^{\beta-1}\theta d\theta)ds,$$

$$\begin{aligned} C \quad &= \int_{\frac{1}{4}}^{\frac{3}{4}} \Upsilon M(1-s)^{\alpha-2}\varphi_2^{-1}(s^{\beta-1}(1-s)) \\ &\varphi_2^{-1}(\int_{\frac{1}{4}}^{\frac{3}{4}}(1-\theta)^{\beta-1}\theta m d\theta)ds. \end{aligned}$$

Theorem 3.1 Let $(H_1),(H_2),(H_3)$ are true and $h_0^s, g_0^s, h_\infty^i, g_\infty^i \in (0,\infty)$, then for each $\lambda \in (\varphi_2(\frac{1}{2C\Upsilon})\frac{1}{h_\infty^i}, \varphi_1(\frac{1}{2B})\frac{1}{h_0^s})$, and $\mu \in (\varphi_2(\frac{1}{2C\Upsilon})\frac{1}{g_\infty^i}, \varphi_1(\frac{1}{2B})\frac{1}{g_0^s})$, there exists a coupled positive function $(\Phi(t),\Psi(t))$ satisfies the system (1), (2).

Proof: From (H_2) and the definitions of h_0^s, g_0^s, there exist $R_1>0$ satisfying

$$h(t,\Phi,\Psi)\le (h_0^s+\varepsilon)\phi(\Phi+\Psi),$$

$$g(t,\Phi,\Psi)\le (g_0^s+\varepsilon)\phi(\Phi+\Psi),$$

for all $0<t<1,\ \Phi,\Psi\ge 0$ and $0\le\Phi+\Psi\le R_1$.

Select Ω_1 as $\{(\Phi,\Psi)\in Y : \|(\Phi,\Psi)\|_Y < R_1\}$. For any (Φ,Ψ) belongs to $P\cap\partial\Omega_1$, we get

$$\begin{aligned} T_1(\Phi,\Psi)(t) \quad &\le \int_0^1 M(1-s)^{\alpha-2}\phi^{-1}(\lambda\int_0^1 \overline{M}(1-\theta)^{\beta-1} \\ &\theta(h_0^s+\varepsilon)\phi(\Phi(\theta)+\Psi(\theta))d\theta)ds \\ &\le \varphi_1^{-1}(\lambda(h_0^s+\varepsilon))B(\|\Phi\|+\|\Psi\|) \\ &\le \tfrac{1}{2}\|(\Phi,\Psi)\|_Y, \end{aligned}$$

thus,

$$\|T_1(\Phi,\Psi)\|\le\frac{1}{2}\|(\Phi,\Psi)\|_Y.$$

Similarly, one has

$$\begin{aligned} T_2(\Phi,\Psi)(t) \quad &\le \int_0^1 M(1-s)^{\alpha-2}\phi^{-1}(\mu\int_0^1 \overline{M}(1-\theta)^{\beta-1} \\ &\theta(g_0^s+\varepsilon)\phi(\Phi(\theta)+\Psi(\theta))d\theta)ds \\ &\le \varphi_1^{-1}(\mu(g_0^s+\varepsilon))B(\|\Phi\|+\|\Psi\|) \\ &\le \tfrac{1}{2}\|(\Phi,\Psi)\|_Y, \end{aligned}$$

thus,

$$\|T_2(\Phi,\Psi)\|\le\frac{1}{2}\|(\Phi,\Psi)\|_Y.$$

Therefore,

$$\begin{aligned} \|T(\Phi,\Psi)\|_Y \quad &= \|T_1(\Phi,\Psi)\|+\|T_2(\Phi,\Psi)\| \\ &\le \tfrac{1}{2}\|(\Phi,\Psi)\|_Y+\tfrac{1}{2}\|(\Phi,\Psi)\|_Y \\ &= \|(\Phi,\Psi)\|_Y, \quad \text{for } (\Phi,\Psi)\in P\cap\partial\Omega_1. \end{aligned} \tag{15}$$

According to the denotation of h_∞^i, g_∞^i, one has

$$h(t,\Phi,\Psi)\ge (h_\infty^i-\varepsilon)\phi(\Phi+\Psi),$$

$$g(t,\Phi,\Psi)\ge (g_\infty^i-\varepsilon)\phi(\Phi+\Psi),$$

for all $\frac{1}{4}\le t\le\frac{3}{4},\ \Phi,\Psi\ge 0$ and $\Phi+\Psi\ge\overline{R_2}$, where $\overline{R_2}$ is some positive constant. Let $R_2=\max\{2R_1,\frac{\overline{R_2}}{\Upsilon}\}$

Select Ω_2 as $\{(\Phi,\Psi)\in Y : \|(\Phi,\Psi)\|_Y < R_2\}$. For any (Φ,Ψ) belongs to $P\cap\partial\Omega_2$, we obtain

$$\begin{aligned} \Phi(t)+\Psi(t) \quad &\ge \inf_{t\in[\frac{1}{4},\frac{3}{4}]}(\Phi(t)+\Psi(t))\ge\Upsilon\|(\Phi,\Psi)\|_Y \\ &= \Upsilon R_2\ge\overline{R_2}. \end{aligned}$$

So, we deduce

$$\begin{aligned} \|T_1(\Phi,\Psi)(t)\| \quad &\ge \int_0^1 \phi^{-1}(\lambda\int_0^1 K(s,\theta) \\ &h(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} \phi^{-1}(\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} K(s,\theta) \\ &h(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} \phi^{-1}(\lambda \\ &\int_{\frac{1}{4}}^{\frac{3}{4}} ms^{\beta-1}(1-s)(1-\theta)^{\beta-1} \\ &\theta(h_\infty^i-\varepsilon)\phi(\Phi(\theta)+\Psi(\theta))d\theta)\Upsilon M(1-s)^{\alpha-2}ds \\ &\ge \varphi_2^{-1}(\lambda(h_\infty^i-\varepsilon))C\Upsilon\|(\Phi,\Psi)\|_Y \\ &\ge \tfrac{1}{2}\|(\Phi,\Psi)\|_Y. \end{aligned}$$

Thus,

$$\|T_1(\Phi,\Psi)\|\ge\frac{1}{2}\|(\Phi,\Psi)\|_Y.$$

Similarly, one get

$$\begin{aligned} \|T_2(\Phi,\Psi)(t)\| \quad &\ge \int_0^1 \phi^{-1}(\mu\int_0^1 K(s,\theta) \\ &g(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} \phi^{-1}(\mu\int_{\frac{1}{4}}^{\frac{3}{4}} K(s,\theta) \\ &g(\theta,\Phi(\theta),\Psi(\theta))d\theta)G(t,s)ds \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} \Upsilon M(1-s)^{\alpha-2}\phi^{-1}(\mu \\ &\int_{\frac{1}{4}}^{\frac{3}{4}} ms^{\beta-1}(1-s)(1-\theta)^{\beta-1} \\ &\theta(g_\infty^i-\varepsilon)\phi(\Phi(\theta)+\Psi(\theta))d\theta)ds \\ &\ge \varphi_2^{-1}(\mu(g_\infty^i-\varepsilon))C\Upsilon\|(\Phi,\Psi)\|_Y \\ &\ge \tfrac{1}{2}\|(\Phi,\Psi)\|_Y. \end{aligned}$$

Thus,

$$\|T_2(\Phi,\Psi)\|\ge\frac{1}{2}\|(\Phi,\Psi)\|_Y.$$

Therefore,

$$\begin{aligned} \|T(\Phi,\Psi)\|_Y \quad &= \|T_1(\Phi,\Psi)\|+\|T_2(\Phi,\Psi)\| \\ &\ge \tfrac{1}{2}\|(\Phi,\Psi)\|_Y+\tfrac{1}{2}\|(\Phi,\Psi)\|_Y \\ &= \|(\Phi,\Psi)\|_Y, \quad \text{for } (\Phi,\Psi)\in P\cap\partial\Omega_2. \end{aligned} \tag{16}$$

Therefore, in view of (15), (16) and theorem of Guo-Krasnosel'skii type for finding fixed point to the problem, we can find a coupled function (Φ,Ψ), which belongs to $P\cap(\overline{\Omega_2}\backslash\Omega_1)$ such that

$$R_1\le\|\Phi\|+\|\Psi\|\le R_2.$$

Obviously, (Φ,Ψ) is a coupled positive function satisfied the problem (1), (2). ■

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