A New Dai-Liao-type Conjugate Gradient Method for Unconstrained Optimization Problems

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***Abstract*—Conjugate gradient method is widely acclaimed to be efficient in solving large-scale unconstrained optimization problems. In this study, a new conjugate gradient method constructed based on inexact line search is proposed. Convergence and conjugate descent property of the new method were established based on some assumptions on the objective function and standard Wolfe conditions. Furthermore, numerical experiments established that the method is highly competitive and efficient for the solution of large-scale test functions when subjected to comparison with other methods.**

***Index Terms*—global convergence; step length; strong Wolfe conditions; sufficient descent; unconstrained optimization.**

I. Introduction

THIS paper considers a nonlinear unconstrained optimization problem of the form

$$\min_{u \in R^n} f(u), \tag{1}$$

where $f : R^n \longrightarrow R$ and f is continuously differentiable. The aim of this study is to find an effective conjugate gradient method (CG-method) for the solution of large-scale problems of the form (1).

The CG-methods are veritable tools appropriate in solving equations of the form (1) using the following iterative formula:

$$u_{k+1} = u_k + \alpha_k d_k, \tag{2}$$

where $\alpha_k > 0$ is a step-length found from line search schemes with the direction d_k given by

$$d_k = \begin{cases} -g_k & k = 0, \\ -g_k + \beta_k d_{k-1} & k \geq 1. \end{cases} \tag{3}$$

The formula $\beta_k \in R$ is a scalar called the CG coefficient with g_k representing gradient of $f(x_k)$. Different CG-methods arise from different constructions of the formula β_k.

Notable classical CG-methods from different constructions of the formula β_k include:

$$\begin{cases} \beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \\ \beta_k^{FR} = \frac{||g_k||^2}{||g_{k-1}||^2}, \\ \beta_k^{PRP} = \frac{g_k^T y_{k-1}}{||g_{k-1}||^2}, \\ \beta_k^{LS} = \frac{-g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}}, \\ \beta_k^{DY} = \frac{||g_k||^2}{d_{k-1}^T y_{k-1}}, \\ \beta_k^{CD} = \frac{-||g_k||^2}{d_{k-1}^T g_{k-1}}. \end{cases} \tag{4}$$

where $y_{k-1} = g_k - g_{k-1}$ and $||.||$ is the euclidean norm. For more details on the classical or early CG-methods listed above, consult [6, 12, 13, 19, 20 and 21]. However, several modifications of early methods can be found in the literature [2, 3, 7, 8, 9 16, 22 and 23]. Moreover, a number of hybrid CG-method have also been considered. For details on this, see [1, 10, 9, 17 and 18]. In 2018, Adeleke and Osinuga [3] also introduced hybrid CG-method (hAO method). Global convergence of their method was shown and its numerical performance compared with other methods in the literature.

The Dai-Liao CG-method (DL CG-method) constructed and introduced in [7], where an inexact line search scheme was used to produce a new conjugacy condition which reduces to Hestenes-Stiefel (HS) CG-method motivates the study. The famous Hestenes Stiefel formula is obtained when (3) satisfies the condition $d_k^T y_{k-1} = 0$. Further developments of DL CG-method were made by several authors including Lu $et\ al.$ [16] and Zheng [24]. Lu $et\ al.$ [16] proposed an effective DL- type CG algorithm with a new value of parameter t based on the new conjugacy condition. They further demonstrated its success with image restoration problems while Zheng [24] presented a new family of DL-type CG methods for unconstrained optimization problem where an existing secant equation was modified and considered in DL's conjugacy condition.

Considerable number of authors have worked on the global convergence properties of the aforementioned CG-methods. These include Powell [21], Dai and Yuan [8], Al Baali [4], Zoutendijk [25], Liu, Han and Yin [15] and Hu and Storey [14]. Of interest is the CG-method developed by Dai and Yuan [8] that gives global convergence properties for general functions. Their motivation came from conjugate decent method β_k^{CD}. It is worthy of note that Dai and Yuan [8] showed that conjugate gradient methods defined by β_k^{CD} and β_k^{FR} have global convergence based on certain conditions.

Manuscript received December 19, 2022; revised July 18, 2023.

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In all, the choice of the step-length α_k is a factor for global convergence. Basically, α_k must satisfy the following Wolfe conditions.

$$f(u_k) - f(u_k + \alpha_k d_k) \geq -\delta \alpha_k g_k^T d_k \tag{5}$$

and

$$g(u_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k \tag{6}$$

for $0 < \delta \leq \sigma < 1$.

Further to the work of Dai and Liao [7], this paper gives a new CG-method constructed based on an inexact line search scheme for solving non-linear optimization problems that measure up to the standard of the existing ones. The convergence of our method based on conditions (5)-(6) factored after β_k^{DY} method is discussed. Test problems selected from [5] are used to subject our method to numerical tests and comparison with other classical, proposed and hybrid CG-methods. It is shown that the proposed method compete favorably well with existing methods.
In this paper, section II discusses the construction of the proposed formula β_k and the algorithm for the iteration. Section III looks at the convergence analysis of our method while in section IV, numerical test and comparison in line with Dolan and More [11] performance profile is presented. Conclusion of the work is given in section 5.

II. Construction of new β_k and Algorithm

The formula β_k is such that (2) and (3) become the linear CG-method for the case where a strictly convex quadratic function $f(u)$ is considered with α_k being the exact one-dimensional minimizer in the direction of d_k.
Hence, d_k which is a sequence of search direction is generated in such a way that the CG-method requires that the conjugacy condition holds, i.e.

$$d_n^T H d_m = 0, \qquad for \;\; all \qquad n \neq m \tag{7}$$

H denotes the Hessian for $f(u)$.
Let f be a general nonlinear function, for some $t \in (0, 1)$, gradient g of f and gradient g^2 of g,

$$\alpha_{k-1}^{-1} d_k^T y_{k-1} = d_k^T g^2 (u_{k-1} + t\alpha_{k-1} d_{k-1}) d_{k-1} \tag{8}$$

holds (see [7]).
Following (8), (7) can be replaced with

$$d_k^T y_{k-1} = 0. \tag{9}$$

Going by Dai and Liao's [7] approach that uses the quasi-Newton method, the new approximation matrix, H_k holds for

$$H_k s_{k-1} = y_{k-1} \tag{10}$$

where,

$$s_{k-1} = \alpha_{k-1} d_{k-1}. \tag{11}$$

Hence, d_k is found by putting

$$d_k = -H_k^{-1} g_k. \tag{12}$$

By (10) and (12), then,

$$d_k^T y_{k-1} = d_k^T (H_k s_{k-1}) = -g_k^T s_{k-1}. \tag{13}$$

From (3),

$$d_k = -g_k + \beta_k d_{k-1}. \tag{14}$$

For an inexact line search, multiply (14) by g_{k-1} to give

$$d_k^T g_{k-1} = -g_k^T g_{k-1} + \beta_k g_{k-1}^T d_{k-1} \tag{15}$$

and

$$d_k^T g_{k-1} = d_k^T g_k - d_k^T y_{k-1}. \tag{16}$$

The conjugacy condition (10) can be conveniently replaced by the condition

$$d_k^T y_{k-1} = -t g_k^T s_{k-1} \tag{17}$$

with a scalar $t \geq 0$ (See [7]).
Hence, (16) becomes

$$d_k^T g_{k-1} = d_k^T g_k + t g_k^T s_{k-1}. \tag{18}$$

By using (18) and (15), we have

$$d_k^T g_k + t g_k^T s_{k-1} = -g_k^T g_{k-1} + \beta_k g_{k-1}^T d_{k-1}. \tag{19}$$

Hence,

$$\beta_k = \frac{d_k^T g_k}{g_{k-1}^T d_{k-1}} + \frac{t g_k^T s_{k-1} + g_k^T g_{k-1}}{g_{k-1}^T d_{k-1}}. \tag{20}$$

Given $t \in [0, \infty)$, we define

$$\beta_k = \frac{d_k^T g_k}{g_{k-1}^T d_{k-1}} - \frac{t g_k^T s_{k-1} + g_k^T g_{k-1}}{(-g_{k-1}^T d_{k-1})}. \tag{21}$$

An equivalent representation of

$$\frac{d_k^T g_k}{g_{k-1}^T d_{k-1}} \tag{22}$$

is the formula β_k^{DY}. This is presented as follows.
Multiplying (14) by g_k gives

$$g_k^T d_k = -||g_k||^2 + \beta_k^{DY} g_k^T d_{k-1}. \tag{23}$$

$$g_k^T d_k = -||g_k||^2 + \frac{||g_k||^2}{d_{k-1}^T y_{k-1}} g_k^T d_{k-1}. \tag{24}$$

$$g_k^T d_k = \frac{-||g_k||^2 [d_{k-1}^T (g_k - g_{k-1}) - g_k^T d_{k-1}]}{d_{k-1}^T y_{k-1}}. \tag{25}$$

$$g_k^T d_k = \frac{||g_k||^2 g_{k-1}^T d_{k-1}}{d_{k-1}^T y_{k-1}}. \tag{26}$$

Hence,

$$\frac{d_k^T g_k}{d_{k-1}^T g_{k-1}} = \frac{||g_k||^2}{d_{k-1}^T y_{k-1}}. \tag{27}$$

See [8 (p.180)].
Therefore, β_k becomes

$$\beta_k = \frac{||g_k||^2}{d_{k-1}^T y_{k-1}} + \frac{t g_k^T s_{k-1} + g_k^T g_{k-1}}{d_{k-1}^T g_{k-1}}. \tag{28}$$

From [6 (p. 144)], we shall use the fact that $g_k^T g_{k-1} = 0$.
Hence, the proposed new formula β_k^{AyO} is

$$\beta_k^{AyO} = \frac{||g_k||^2}{d_{k-1}^T y_{k-1}} + \frac{t g_k^T s_{k-1}}{d_{k-1}^T g_{k-1}}. \tag{29}$$

Procedure (2)-(3) with β_k^{AyO} will be called AyO-CG method.

Algorithm (new β_k)
Step 1: With the initial point $u_o \in R^n$ and $\epsilon > 0$, we set $k = 1$ and $d_0 = -g_0$. If $||g_k|| \leq \epsilon$, then stop;
Step 2: Find $\alpha_k > 0$ by conditions (5)-(6);
Step 3: Compute β_k^{AyO} and generate the sequences $\{u_k\}$, $\{g_k\}$ and $\{d_k\}$;
Step 4: Set $k = k + 1$ and proceed to step 2.

III. Analysis on Convergence

The discussion of the convergence analysis for the algorithm stated in section II is carried out here based on the conditions of standard Wolfe line search given in (5) and (6).
In the case when $t = 0$, $\beta_k^{AyO} = \beta_k^{DY}$. Based on this condition, analysis on convergence for our method is addressed in the following ways:
We first consider the descent property of this new method.
Assumption 3.1
The objective function (1) satisfies the conditions highlighted below.
(i) The set

$$U = \{u|f(u) \leq f(v)\} \tag{30}$$

with $v \in R^n$ is bounded.
(ii) Given a neighborhood W of U, f is continuous and differentiable in W and its gradient $g(u)$ with Lipschitz constant L satisfies Lipschitz continuity

$$||g(u) - g(v)|| \leq L||u - v|| \tag{31}$$

for any $u, v \in W$.

Theorem 3.2 addresses descent property of the new method.
Theorem 3.2. *Suppose d_k and g_k are determined by the CG-method algorithm given in section II. Then, d_k satisfies the following conditions.*

$$d_k^T g_k \leq -c||g_k||^2, \tag{32}$$

where c is a constant and for each $k \geq 0$.
Proof: By induction, it is obvious for the case when $k = 0$ that,

$$d_0^T g_0 = -||g_0||^2. \tag{33}$$

Suppose that (32) is true for $k \geq 1$. Pre multiplying (14) by g_k gives

$$d_k^T g_k = -||g_k||^2 + \beta_k^{AyO} d_{k-1}^T g_k. \tag{34}$$

For $s_{k-1} = \alpha_{k-1} d_{k-1}$, we have

$$d_k^T g_k = -||g_k||^2 + (\beta_k^{DY} + \frac{t g_k^T s_{k-1}}{g_{k-1}^T d_{k-1}}) d_{k-1}^T g_k. \tag{35}$$

That is,

$$d_k^T g_k = -||g_k||^2 + \beta_k^{DY} d_{k-1}^T g_k - \frac{t\alpha_{k-1}(g_k^T d_{k-1})^2}{-g_{k-1}^T d_{k-1}}. \tag{36}$$

Since

$$d_{k-1}^T y_{k-1} = d_{k-1}^T (g_k - g_{k-1}) \tag{37}$$

$$= d_{k-1}^T g_k - d_{k-1}^T g_{k-1} \tag{38}$$

$$\leq |d_{k-1}^T g_k| - d_{k-1}^T g_{k-1}. \tag{39}$$

For (39) and $d_{k-1}^T y_{k-1} > 0$ to be true always, it implies that

$$d_{k-1}^T g_{k-1} < 0 \tag{40}$$

is true.
From (40), $t > 0$ and $\alpha_{k-1} > 0$

$$\frac{\alpha_{k-1} t (g_k^T d_{k-1})^2}{-d_{k-1}^T g_{k-1}} > 0. \tag{41}$$

Therefore,

$$d_k^T g_k \leq -||g_k||^2 + \beta_k^{DY} d_{k-1}^T g_k. \tag{42}$$

By [8], $c = \frac{1}{1+\sigma}$. Hence,

$$d_k^T g_k \leq -(\frac{1}{1+\sigma})||g_k||^2. \tag{43}$$

Therefore, sufficient descent property is satisfied.
We consider next result on convergence for the new method.
Lemma 3.3. *Suppose the conditions in the assumption 3.1 is satisfied and for any equation of the form (3) with descent direction d_k where α_k satisfies the Wolfe conditions in (5) and (6). Then,*

$$\sum_{i=1}^{\infty} \frac{(g_k^T d_k)^2}{||g_k||^2} < \infty. \tag{44}$$

Proof: The comprehensive proof of Lemma 3.3 can be found in [8].
Theorem 3.4. *Suppose the conditions in the assumption 3.1 is satisfied and also that u_k be generated by the algorithm in section II. Then,*

$$\lim_{k\to\infty} \inf ||g_k|| = 0. \tag{45}$$

Proof: The result is proved by contradiction.
Suppose

$$\lim_{k\to\infty} \inf ||g_k|| \neq 0. \tag{46}$$

Given that

$$||g_k|| > 0, \tag{47}$$

we can find a constant $n > 0$, such that,

$$||g_k|| > n \quad \forall k. \tag{48}$$

From (14), we have

$$d_k + g_k = \beta_k d_{k-1}. \tag{49}$$

Take square of both sides of equation (49) with $\beta_k = \beta_k^{AyO}$ to have

$$||d_k||^2 = -||g_k||^2 - 2d_k^T g_k + (\beta_k^{AyO})^2 \, ||d_{k-1}||^2. \tag{50}$$

Divide (50) by $(d_k^T g_k)^2$.

$$\frac{||d_k||^2}{(d_k^T g_k)^2} = -\frac{||g_k||^2}{(d_k^T g_k)^2} - \frac{2d_k^T g_k}{(d_k^T g_k)^2} + \frac{(\beta_k^{AyO})^2 \, ||d_{k-1}||^2}{(d_k^T g_k)^2}. \tag{51}$$

Since $0 \leq \beta_k^{AyO} \leq \beta_k^{DY}$ for $k \geq 1$. The result follows from Theorem 3.3 of [8].

IV. Numerical Test and Discussion of Results

A set of test problems selected from [5] are considered here for the numerical test, evaluations and discussion of results. A comparison of the proposed method with hAO [3], DL [7], NEW$_+$ [24] and PRP [19, 20] is employed in this section. The following abbreviations are adopted in the tables: Q-Quadratic, TD-TRIDIA, QD-Quadratic Diagonal Perturbed, QF-Quadratic QF1, AR-ARGLINB, AP-Almost Perturbed Quadratic, D4-Diagonal 4, EH-Extended Himmalblau, R1-Raydan 1, R2-Raydan 2, ET-Extended Three Exponential Terms, GR- Generalized Rosenbrock, GP-Generalized PSCI, EC-Extended Cliff, ET-Extended Freudenstein and Roth, D9-Diagonal 9, HG-HIMMELBG, MS-MODF SINE, M1-MDF EXPLIN 1, M2-MDF EXPLIN 2, RC-RMODF COSINE, RS-RMODF SINE, EM-Ext MCCORMCK, PO-Power, EB-Extended Booth, CU-Cube and CQ-Chebyquad.

Table I. List of test problems and initial points.

s/n	$Problems$	$Dim.$	$initial\ points$
1	AP	2	$(0.5, 0.5, ..., 0.5)$
2	TD	2	$(1, 1, ..., 1)$
3	QD	2	$(0.5, 0.5, ..., 0.5)$
4	QF	2	$(0, 0, ..., 0)$
5	QF	4	$(0, 0, ..., 0)$
6	AR	2	$(1, 1, ..., 1)$
7	AR	4	$(1, 1, ..., 1)$
8	Q	2	$(0.5, 0.5, ..., 0.5)$
9	$D4$	2	$(1, 1, ..., 1)$
10	$D4$	100	$(1, 1, ..., 1)$
11	$D4$	500	$(1, 1, ..., 1)$
12	$D4$	1000	$(1, 1, ..., 1)$
13	$D4$	10000	$(1, 1, ..., 1)$
14	$D4$	50000	$(1, 1, ..., 1)$
15	$D4$	100000	$(1, 1, ..., 1)$
16	EH	2	$(1, 1, ..., 1)$
17	EH	100	$(1, 1, ..., 1)$
18	EH	500	$(1, 1, ..., 1)$
19	EH	1000	$(1, 1, ..., 1)$
20	EH	10000	$(1, 1, ..., 1)$
21	EH	50000	$(1, 1, ..., 1)$
22	EH	100000	$(1, 1, ..., 1)$
23	$R1$	2	$(1, 1, ..., 1)$
24	$R1$	100	$(1, 1, ..., 1)$
25	$R1$	500	$(1, 1, ..., 1)$
26	$R1$	1000	$(1, 1, ..., 1)$
27	$R1$	10000	$(1, 1, ..., 1)$
28	$R1$	50000	$(1, 1, ..., 1)$
29	$R1$	100000	$(1, 1, ..., 1)$
30	$R2$	2	$(1, 1, ..., 1)$
31	$R2$	100	$(1, 1, ..., 1)$
32	$R2$	500	$(1, 1, ..., 1)$
33	$R2$	1000	$(1, 1, ..., 1)$
34	$R2$	10000	$(1, 1, ..., 1)$
35	$R2$	50000	$(1, 1, ..., 1)$
36	$R2$	100000	$(1, 1, ..., 1)$
37	ET	2	$(0.1, 0.1, ..., 0.1)$
38	ET	100	$(0.1, 0.1, ..., 0.1)$
39	ET	500	$(0.1, 0.1, ..., 0.1)$
40	ET	1000	$(0.1, 0.1, ..., 0.1)$
41	ET	10000	$(0.1, 0.1, ..., 0.1)$
42	GR	2	$(-1.2, 1, ..., -1.2, 1)$
43	GP	2	$(3, 0.1, ..., 3, 0.1)$
44	GP	100	$(3, 0.1, ..., 3, 0.1)$
45	GP	500	$(3, 0.1, ..., 3, 0.1)$
46	GP	1000	$(3, 0.1, ..., 3, 0.1)$
47	EC	2	$(0, -1, ...0, -1)$
48	EC	100	$(0, -1, ...0, -1)$
49	EC	500	$(0, -1, ...0, -1)$
50	EC	1000	$(0, -1, ...0, -1)$
51	EF	2	$(0.5, -2, ..., 0.5, -2)$
52	EF	100	$(0.5, -2, ..., 0.5, -2)$
53	$D9$	2	$(1, 1, ..., 1)$
54	$D9$	100	$(1, 1, ..., 1)$
55	HG	2	$(1.5, 1.5, ..., 1.5)$
56	MS	2	$(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$
57	MS	100	$(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$
58	MS	500	$(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$
59	$M1$	2	$(1, 1, ..., 1)$
60	$M1$	100	$(1, 1, ..., 1)$
61	$M1$	500	$(1, 1, ..., 1)$
62	$M1$	1000	$(1, 1, ..., 1)$
63	$M1$	10000	$(1, 1, ..., 1)$
64	$M1$	50000	$(1, 1, ..., 1)$
65	$M1$	100000	$(1, 1, ..., 1)$
66	$M2$	2	$(1, 1, ..., 1)$
67	RC	2	$(1, 1, ..., 1)$
68	RC	100	$(1, 1, ..., 1)$
69	RC	500	$(1, 1, ..., 1)$
70	RC	1000	$(1, 1, ..., 1)$
71	RS	2	$(1, 1, ..., 1)$
72	RS	100	$(1, 1, ..., 1)$
73	RS	500	$(1, 1, ..., 1)$
74	RS	1000	$(1, 1, ..., 1)$
75	RS	10000	$(1, 1, ..., 1)$
76	RS	50000	$(1, 1, ..., 1)$
77	RS	100000	$(1, 1, ..., 1)$
78	EM	2	$(1, 1, ..., 1)$
79	EM	100	$(1, 1, ..., 1)$
80	EM	500	$(1, 1, ..., 1)$
81	EM	1000	$(1, 1, ..., 1)$
82	EM	10000	$(1, 1, ..., 1)$
83	PO	2	$(1, 1, ..., 1)$
84	EB	2	$(1, 3, ..., 1, 3)$
85	EB	100	$(1, 3, ..., 1, 3)$
86	EB	500	$(1, 3, ..., 1, 3)$
87	EB	1000	$(1, 3, ..., 1, 3)$
88	EB	10000	$(1, 3, ..., 1, 3)$
89	EB	50000	$(1, 3, ..., 1, 3)$
90	EB	100000	$(1, 3, ..., 1, 3)$
91	CU	2	$(1, 1, ..., 1)$
92	CQ	2	$(1, 1, ..., 1)$
93	CQ	100	$(1, 1, ..., 1)$
94	CQ	500	$(1, 1, ..., 1)$
95	CQ	1000	$(1, 1, ..., 1)$
96	CQ	10000	$(1, 1, ..., 1)$

Table II. Numerical result of number of iterations and values of function f.

s/n	$Prob.$	$Dim.$	AyO	hAO	PRP	NEW_+	DL
			IT/FE	IT/FE	IT/FE	IT/FE	IT/FE
1	AP	2	$50/1.26e-05$	$70/1.26e-05$	$5/1.26e-05$	$12/1.26e-05$	$108/1.26e-05$
2	TD	2	$32/2.90e-15$	$37/5.03e-14$	$29/2.30e-14$	$25/3.86e-15$	$313/1.93e-14$
3	QD	2	$22/3.10e+00$	$33/3.10e+00$	$45/3.10e+00$	F/F	$29/3.10e+00$
4	QF	2	$6/2.00e-01$	$2/2.00e-01$	$53/2.00e-01$	$72/2.00e-01$	$53/2.00e-01$
5	QF	4	$8/3.10e+00$	$2/3.10e+00$	$139/3.10e+00$	$32/6.67e-01$	$102/6.67e-01$
6	AR	2	$6/1.26e-05$	$1488/1.26e-5$	$5/1.26e-05$	$37/F$	$32/1.26e-07$
7	AR	4	$19/5.61e-16$	$14/1.24e-14$	$657/1.21e-16$	$69/F$	$21/4.64e-17$
8	Q	2	$29/9.27e-01$	$15/9.27e-01$	$757/9.27e-01$	$27/9.27e-01$	$40/9.27e-01$
9	$D4$	2	$21/5.97e-17$	$15/1.72e-16$	$799/2.91e-16$	$1103/F$	$70/8.15e-18$
10	$D4$	100	$23/1.60e-16$	$15/6.01e-17$	$799/1.16e-16$	$36/F$	$25/4.47e-18$
11	$D4$	500	$25/1.65e-15$	$15/1.20e-15$	$851/1.17e-16$	$22/3.83e-13$	$104/3.01e-17$
12	$D4$	1000	$25/2.17e-16$	$19/4.17e-17$	$35/5.96e-16$	$408/2.4e-13$	$161/4.05e-15$
13	$D4$	10000	$27/5.67e-15$	$23/3.61e-16$	$64/1.86e-17$	$135/F$	$85/3.63e-15$
14	$D4$	50000	$22/1.01e-14$	$19/2.62e-15$	$887/1.19e-16$	$28/1.05e-14$	$17/1.04e-14$
15	$D4$	100000	$28/2.01e-14$	$19/5.25e-15$	$903/1.18e-16$	$28/2.10e-14$	$30/1.84e-15$
16	EH	2	$34/2.00e-15$	$42/4.17e-15$	$46/4.24e-16$	$26/5.26e-15$	$61/4.76e-15$
17	EH	100	$37/4.32e-16$	$45/1.34e-15$	$48/1.42e-16$	F/F	$72/1.56e-15$
18	EH	500	$39/4.32e-15$	$47/4.96e-17$	$50/4.74e-16$	$77/2.65e-15$	$52/3.62e-15$
19	EH	1000	$34/8.03e-15$	$47/9.92e-17$	$50/9.47e-16$	$75/1.96e-15$	$74/4.62e-15$
20	EH	10000	$40/2.97e-15$	$47/9.92e-16$	$52/6.33e-16$	$56/5.61e-15$	$78/3.98e-15$
21	EH	50000	$41/6.22e-15$	$48/3.24e-15$	$52/3.17e-15$	$152/1.26e-14$	$46/5.23e-15$
22	EH	100000	$42/4.20e-15$	$49/1.04e-15$	$54/4.24e-16$	$155/1.26e-14$	$71/1.47e-14$
23	$R1$	2	$14/5.18E+12$	F/F	F/F	F/F	F/F
24	$R1$	100	$1/1.43e+05$	$1/1.43e+05$	$1/1.43e+05$	$1/1.43e+05$	$1/1.43e+05$
25	$R1$	500	$1/8.73e+07$	$1/8.73e+07$	$1/8.73e+07$	$1/8.73e+07$	$1/8.73e+07$
26	$R1$	1000	$1/1.39e+09$	$1/1.39e+09$	$1/1.39e+09$	$1/1.39e+09$	$1/1.39e+09$
27	$R1$	10000	$1/1.39e+13$	$1/1.39e+13$	$1/1.39e+13$	$1/1.39e+13$	$1/1.39e+13$
28	$R1$	50000	$1/8.68e+15$	$1/8.68e+15$	$1/8.68e+15$	$1/8.68e+15$	$1/8.68e+15$
29	$R1$	100000	$1/1.39e+17$	$1/1.39e+17$	$1/1.39e+17$	$1/1.39e+17$	$1/1.39e+17$
30	$R2$	2	$12/2.00e+00$	$12/2.00e+00$	$11/2.00e+00$	$9/2.00e+00$	$10/2.00e+00$
31	$R2$	100	$13/1.00e+02$	$13/1.00e+02$	$11/1.00e+02$	$14/1.00e+02$	$11/1.00e+02$
32	$R2$	500	$13/5.00e+02$	$14/5.00e+02$	$12/5.00e+02$	$15/5.00e+02$	$11/5.00e+02$
33	$R2$	1000	$13/1.00e+03$	$14/1.00e+03$	$12/1.00e+03$	$49/1.00e+03$	$11/1.00e+03$
34	$R2$	10000	$14/1.00e+04$	$14/1.00e+04$	$12/1.00e+04$	$37/1.00e+04$	$11/1.00e+04$
35	$R2$	50000	$14/5.00e+04$	$15/5.00e+04$	$11/5.00e+04$	$83/5.00e+04$	$11/5.00e+04$
36	$R2$	100000	$14/1.00e+05$	$15/1.00e+05$	$11/1.00e+05$	$61/1.00e+05$	$11/1.00e+05$
37	ET	2	$22/2.95e+00$	$26/2.56e+004$	$43/2.56e+00$	F/F	$31/2.56e+00$
38	ET	100	$25/1.28e+02$	$29/1.28e+02$	$47/1.28e+02$	F/F	$35/1.28e+02$
39	ET	500	$28/6.40e+02$	$30/6.40e+02$	$49/6.40e+02$	F/F	$36/6.40e+02$
40	ET	1000	$135/1.28e+03$	$33/1.28e+03$	$93/1.28e+03$	F/F	$369/1.28e+03$
41	ET	10000	1692/12800	1335/12800	1146/12800	F/F	54700/12800
42	GR	2	$109/7.98e-14$	$120/1.94e-14$	$81447/3.88-14$	$598/F$	F/F
43	GP	2	$19/7.73e-01$	$24/7.73e-01$	$30/7.42e-07$	$22/7.73e-01$	$22/7.73e-01$
44	GP	100	472/98.7	1410/98.7	981/98.7	660/98.7	352/98.7
45	GP	500	526/499	2855/499	812/499	476/499	462/499
46	GP	1000	443/999	3261/999	868/999	770/999	505/999
47	EC	2	47/0.2	276/0.2	120/0.2	39817/48.9	56/48.9
48	EC	100	91/9.99	408/9.99	62773/2450	51773/999	60/999

Table II. contd.

s/n	$Prob.$	$Dim.$	AyO	hAO	PRP	NEW_+	DL
			IT/FE	IT/FE	IT/FE	IT/FE	IT/FE
49	EC	500	$111/4.99e+1$	$223/4.99e+1$	$3796/4.99e+1$	F/F	$89/4.99e+1$
50	EC	1000	$117/9.99e+1$	$178/9.99e+1$	$347/1.61e-14$	F/F	$63/9.99e+1$
51	EF	2	$92/4.90e+1$	$106/4.90e+1$	$3796/4.90e+1$	$167/4.90e+1$	F/F
52	EF	100	$518/2.45e+3$	$142/2.45e+3$	$347/2.45e+3$	F/F	$124/2.45e+3$
53	$D9$	2	$127/1.00e+0$	$202/1.00e+0$	$109/1.00e+0$	$29/1.00e+0$	F/F
54	$D9$	100	538/99	273/99	445/99	$41395/F$	806/99
55	HG	2	$377/7.68e-7$	F/F	F/F	$117/7.53e-7$	$77/7.20e-7$
56	MS	2	$52/-0.125$	$36/-1.06$	F/F	$367/F$	F/F
57	MS	100	$88/-53.1$	$38/-53.1$	F/F	$115/F$	F/F
58	MS	500	$105/-266$	$38/-266E$	F/F	$106/F$	F/F
59	$M1$	2	13/2	32/2	66/2	46/2	114/2
60	$M1$	100	16/100	37/100	79/100	55/100	131/100
61	$M1$	500	16/500	39/500	82/500	59/500	137/500
62	$M1$	1000	16/1000	40/1000	84/1000	61/1000	141/1000
63	$M1$	10000	18/10000	42/10000	18/10000	61/10000	151/10000
64	$M1$	50000	29/50000	44/50000	95/50000	18/50000	159/50000
65	$M1$	100000	29/100000	45/100000	96/100000	19/100000	162/100000
66	$M2$	2	14/5.04	11/5.04	247/5.04	F/F	139/5.04
67	RC	2	$6/-1.00$	$11/-1.00$	$21/-1.00$	$18/-1.00$	$24/-1.00$
68	RC	100	$6/-50$	$27/-50$	$23/-50$	$33/F$	$27/-50$
69	RC	500	$6/-250$	$104/-250$	$23/-250$	$72/F$	$30/-250$
70	RC	1000	$6/-500$	$75/-500$	$23/-500$	$12/F$	$33/-500$
71	RS	2	$18/-1.00$	$28/-1.00$	$57-1.00$	$24/-1.00$	$99/-1.00$
72	RS	100	$20/-50$	$31/-50$	$65/-50$	F/F	$7/-50$
73	RS	500	$21/-250$	$33/-250$	$68/-250$	F/F	$121/-250$
74	RS	1000	$22/-500$	$34/-500$	$70/-5002$	F/F	$123/-500$
75	RS	10000	$23/-5000$	$36/-5000$	$75/-5000$	F/F	$132/-5000$
76	RS	50000	$24/-25000$	$38/-25000$	$78/-25000$	$18217/-5.1e+7$	$139/-25000$
77	RS	100000	$24/-50000$	$38/-50000$	$80/-50000$	$2514/1.00E+06$	$141/-50000$
78	EM	2	$18/-1.91$	$22/-1.91$	$28/-1.91$	$39/-1.91$	$24/-11.3$
79	EM	100	$21/-95.7$	$24/-95.7$	$35/-95.7$	$64/-95.7$	$60/-1.1E+6$
80	EM	500	$22/-478$	$26/-478$	$41/-478$	$132/-2050$	$38/-19300$
81	EM	1000	$22/-957$	$26/-957$	$43/-957$	$57/-957$	$46/-4.69e+5$
82	EM	10000	$34/-9570$	$33/-9570$	$62/-9570$	$554/-727e+6$	$39/-5.12e+5$
83	PO	2	$22/7.74e-1$	$11/7.74e-1$	F/F	F/F	$2/7.74e-1$
84	EB	2	$27/1.23e-14$	$30/7.81e-15$	$26/3.15e-14$	$17/1.25e-14$	$28/1.36e-13$
85	EB	100	$30/3.13e-14$	$32/4.21e-14$	$29/5.89e-15$	$18/1.72e-14$	$33/3.12e-15$
86	EB	500	$32/8.62-15$	$35/1.54-14$	$29/2.94-14$	$24/3.40-14$	$18/F$
87	EB	1000	$32/1.72e-14$	$35/3.08e-14$	$29/5.89e-14$	$37/2.67e-15$	$18/F$
88	EB	10000	$34/1.00-14$	$38/1.37e-13$	$31/8.06e-15$	$29/1.93e-14$	$18/F$
89	EB	50000	$40/2.19e-14$	$42/2.80e-14$	$31/4.03e-14$	$20/6.45e-15$	$38/6.14e-14$
90	EB	100000	$41/6.02e-15$	$42/5.60e-14$	$31/8.06e-14$	$53/7.48e-14$	$36/1.49e-13$
91	CU	2	$559/9.2e-14$	$141/6.80e-13$	$1998/2.6e-16$	$352/5.34e-14$	$142/2.11e-16$
92	CQ	2	$1/1.50e+0$	$1/1.50e+0$	$1/1.50e+0$	$1/1.50e+0$	$1/1.50e+0$
93	CQ	100	$27/1.50e+0$	$87/1.50e+0$	F/F	F/F	$2/1.50e+0$
94	CQ	500	$48/1.50e+0$	F/F	F/F	F/F	$2/1.50e+0$
95	CQ	1000	$58/1.50e+0$	F/F	F/F	$461/1.50e+0$	$2/1.50e+0$
96	CQ	10000	$69/1.50e+0$	F/F	F/F	F/F	$2/1.50e+0$

Table III. Numerical result of CPU time and gradient norm

s/n	$Prob.$	$Dim.$	AyO	hAO	PRP	NEW_+	DL
			CPU/GN	CPU/GN	CPU/GN	CPU/GN	CPU/GN
1	AP	2	$0.8/6.59e-7$	$1.152/5.91e-7$	$0.75/2.13e-16$	$0.208/2.68e-7$	$2.164/4.00e-7$
2	TD	2	$0.695/1.97e-7$	$0.82/5.47e-7$	$0.18/7.36e-7$	$0.441/9.47e-7$	$5.997/9.77e-7$
3	QD	2	$0.187/3.49e-7$	$0.756/6.50e-7$	$0.752/5.77e-7$	F/F	$0.508/8.20e-7$
4	QF	2	$0.117/9.03e-8$	$0.05/2.98e-15$	$0.889/8.01e-7$	$1.182/9.75e-7$	$0.99/7.20e-7$
5	QF	4	$0.15/4.76e-10$	$0.04/1.95e-15$	$2.398/9.79e-7$	$0.864/5.24e-7$	$1.885/9.58e-7$
6	AR	2	$0.119/7.26e-7$	$23.86/9.77e-7$	$0.1/2.13e-16$	$0.625/F$	$0.775/7.01e-7$
7	AR	4	$0.317/3.08e-7$	$0.245/6.91e-7$	$12.459.87e-7$	$1.115/F$	$0.434/6.52e-7$
8	Q	2	$0.494/5.29e-8$	$0.375/2.15e-8$	$15.01/9.58e-7$	$0.457/2.74e-7$	$0.91/7.91e-7$
9	$D4$	2	$0.202/2.97e-7$	$0.15/1.20e-17$	$0.25/1.15e-16$	$18.312/F$	$1.468/6.84e-7$
10	$D4$	100	$0.289/1.67e-7$	$0.283/4.82e-8$	$18.62/9.62e-7$	$0.825/F$	$0.538/2.02e-7$
11	$D4$	500	$0.597/5.29e-7$	$0.55/2.15e-7$	$66.90/9.66e-7$	$0.396/9.89e-7$	$4.46/6.45e-7$
12	$D4$	1000	$0.409/5.66e-7$	$0.292/2.41e-7$	$0.859/7.62e-7$	$6.818/9.72e-7$	$3.05/8.90e-7$
13	$D4$	10000	$0.809/8.73e-7$	$0.178/2.34e-7$	$0.559/4.94e-7$	$1.538/F$	$1.652/8.74e-7$
14	$D4$	50000	$2.237/5.96e-7$	$1.512/2.70e-7$	$197.0/9.77e-7$	$2.223/5.95e-7$	$3.088/7.19e-8$
15	$D4$	100000	$4.223/8.43e-7$	$341.3/3.82e-7$	$4.002/9.71e-7$	$1.538/8.41e-7$	$13.48/3.65e-8$
16	EH	2	$0.6/4.13e-7$	$0.774/9.83e-7$	$0.944/2.61e-7$	$0.498/8.64e-7$	$1.276/8.89e-7$
17	EH	100	$0.672/7.81e-7$	$0.881/5.97e-7$	$1.071/4.78e-7$	F/F	$1.574/5.58e-7$
18	EH	500	$0.733/6.17e-7$	$0.929/1.06e-7$	$1.226/2.76e-7$	$1.683/6.41e-7$	$1.412/8.30e-7$
19	EH	1000	$0.744/8.73e-7$	$1.088/1.50e-7$	$1.484/3.91e-7$	$1.613/5.68e-7$	$1.778/9.58e-7$
20	EH	10000	$1.678/4.78e-7$	$2.611/4.74e-7$	$4.965/3.19e-7$	$4.206/8.63e-7$	$4.374/8.87e-7$
21	EH	50000	$5.033/9.27e-7$	$7.48/9.28e-7$	$18.69/7.14e-7$	$22.75/8.70e-7$	$6.856/8.79e-7$
22	EH	100000	$10.12/6.19e-7$	$14.88/4.23e-7$	$34.34/2.61e-7$	$43.77/8.51e-7$	$19.70/8.98e-7$
23	$R1$	2	$0.231/0.00e+0$	F/F	F/F	F/F	F/F
24	$R1$	100	$0.009/0.00e+0$	$0.012/0.00e+0$	$0.011/0.00e+0$	$0.018/0.00e+0$	$0.021/0.00e+0$
25	$R1$	500	$0.012/0.00e+0$	$0.017/0.00e+0$	$0.018/0.00e+0$	$0.018/0.00e+0$	$0.024/0.00e+0$
26	$R1$	1000	$0.018/0.00e+0$	$0.017/0.00e+0$	$0.017/0.00e+0$	$0.019/0.00e+0$	$0.017/0.00e+0$
27	$R1$	10000	$0.026/0.0e+0$	$0.029/0.0e+0$	$0.033/0.0e+0$	$0.041/0.0e+0$	$0.027/0.00e+0$
28	$R1$	50000	$0.06/0.00e+0$	$0.06/0.00e+0$	$0.059/0.00e+0$	$0.058/0.00e+0$	$0.059/0.00e+0$
29	$R1$	100000	$0.079/0.00e+0$	$0.089/0.00e+0$	$0.087/0.00e+0$	$0.083/0.0e+0$	$0.095/0.0e+0$
30	$R2$	2	$0.227/2.06e-7$	$0.277/3.86e-7$	$0.168/1.40e-7$	$0.16/1.77e-11$	$0.193/1.90e-7$
31	$R2$	100	$0.22/2.30e-7$	$0.267/5.32e-7$	$0.169/9.87e-7$	$0.229/3.29e-9$	$0.217/2.38e-9$
32	$R2$	500	$0.215/5.15e-7$	$0.263/2.23e-7$	$0.19/5.55e-10$	$0.26/4.80e-13$	$0.233/5.31e-9$
33	$R2$	1000	$0.208/7.28e-7$	$0.235/3.16e-7$	$0.22/7.85e-10$	$0.96/3.23e-14$	$0.229/7.51e-9$
34	$R2$	10000	$0.347/1.05e-7$	$0.36/1.00e-6$	$0.292/5.96e-7$	$1.41/5.29e-12$	$0.311/2.38e-8$
35	$R2$	50000	$0.622/2.35e-7$	$0.665/4.06e-7$	$0.492/5.55e-9$	$7.934/1.51e-9$	$0.525/9.29e-9$
36	$R2$	100000	$0.994/3.33e-7$	$0.993/5.74e-7$	$0.735/7.85e-9$	$10.15/2.27e-7$	$0.696/1.31e-8$
37	ET	2	$0.378/7.67-7$	$0.467/7.64e-7$	$0.738/5.96e-7$	F/F	$0.628/7.23e-7$
38	ET	100	$0.475/6.27e-7$	$0.516/7.24e-7$	$0.628/8.49e-7$	F/F	$0.662/9.88e-7$
39	ET	500	$0.56/9.61e-7$	$0.556/9.11e-7$	$0.964/9.66e-7$	F/F	$0.731/9.65e-7$
40	ET	1000	$9.672/7.16e-7$	$0.644/9.63e-7$	$2.071/4.01e-7$	F/F	$8.029/9.17e-7$
41	ET	10000	$60.25/3.20e-7$	$63.34/9.91e-7$	$157.5/7.63e-7$	F/F	$1605/6.79e-7$
42	GR	2	$1.898/9.35e-7$	$2.6/2.32e-7$	$26.48/9.80e-7$	$9.849/F$	F/F
43	GP	2	$0.402/3.22e-7$	$0.389/4.85e-7$	$0.589/9.97e-7$	$0.404/8.51e-7$	$0.426/1.78e-7$
44	GP	100	$8.014/7.48e-7$	$23.98/8.54e-7$	$17.21/9.82e-7$	$11.39/8.25e-7$	$6.843/9.15e-7$
45	GP	500	$9.714/7.16e-7$	$50.08/9.60e-7$	$15.07/5.96e-7$	$8.362/9.86e-7$	$9.323/9.46e-7$
46	GP	1000	$8.007/8.24e-7$	$59.69/9.00e-7$	$17.33/6.37e-7$	$14.60/9.91e-7$	$10.53/7.42e-7$
47	EC	2	$0.828/6.00e-7$	$4.777/8.52e-7$	$2.412/7.90e-7$	$640.6/6.60e-7$	$1.086/6.51e-7$
48	EC	100	$1.592/6.34e-7$	$2.822/7.89e-7$	$7.234/5.31e-7$	$1095/1.00e-6$	$1.265/7.09e-7$

Table III. contd.

s/n	$Prob.$	$Dim.$	AyO	hAO	PRP	NEW_+	DL
			CPU/GN	CPU/GN	CPU/GN	CPU/GN	CPU/GN
49	EC	500	$2.037/9.98e-7$	$4.148/9.43e-7$	$21.7/9.43e-7$	F/F	$2/4.92e-9$
50	EC	1000	$2.079/4.78e-7$	$3.686/7.53e-7$	$25.27/8.94e-7$	F/F	$1.313/9.63e-7$
51	EF	2	$1.652/6.71e-7$	$2.02/4.36e-7$	$93.78/8.95e-7$	$3.145/9.16e-7$	F/F
52	EF	100	$10.581/8.70e-7$	$2.852/7.89e-7$	$12.22/5.31e-7$	F/F	$2.87/5.48e-7$
53	$D9$	2	$2.291/5.86e-7$	$4.172/2.75e-7$	$2.361/5.47e-7$	$0.491/7.57e-7$	F/F
54	$D9$	100	$4.383/9.56e-7$	$5.751/5.17e-7$	$11.42/7.94e-7$	$801.8/F$	$18.47/6.79e-7$
55	HG	2	$7.427/9.97e-7$	F/F	F/F	$2.581/9.78e-7$	$1.503/9.20e-7$
56	MS	2	$1.138/7.73e-7$	$0.865/3.37e-7$	F/F	$6.823/F$	F/F
57	MS	100	$1.494/3.86e-7$	$0.63/4.11e-7$	F/F	$3.044/F$	F/F
58	MS	500	$1.891/5.65e-7$	$0.699/9.18e-7$	F/F	$5.156/F$	F/F
59	$M1$	2	$0.265/8.36e-7$	$0.825/7.89e-7$	$1.398/8.46e-7$	$0.76/8.93e-7$	$2.826/9.07e-7$
60	$M1$	100	$0.387/2.89e-7$	$0.579/7.35e-7$	$1.247/9.19e-7$	$0.87/8.02e-7$	$2.505/9.63e-7$
61	$M1$	500	$0.264/6.47e-7$	$0.638/7.30e-7$	$1.371/9.72e-7$	$0.985/8.50e-7$	$2.66/9.41e-7$
62	$M1$	1000	$0.287/9.14e-7$	$0.673/6.88e-7$	$1.416/9.45e-7$	$1.084/9.06e-7$	$2.726/8.52e-7$
63	$M1$	10000	$0.469/3.97e-7$	$1.076/9.68e-7$	$2.064/3.80e-7$	$1.642/8.71e-7$	$4.198/8.83e-7$
64	$M1$	50000	$1.334/6.44e-7$	$1.807/9.62e-7$	$3.86/8.51e-7$	$3.536/8.50e-7$	$6.879/9.47e-7$
65	$M1$	100000	$1.806/9.10e-7$	$2.74/9.7e-7$	$5.669/9.98e-7$	$6.223/9.64e-7$	$10.84/9.84e-7$
66	$M2$	2	$0.24/6.83e-7$	$0.21/1.81e-7$	$4.31/9.59e-7$	$2.654/F$	$2.755/9.61e-7$
67	RC	2	$0.099/9.31e-9$	$0.195/1.93e-7$	$0.374/2.46e-7$	$0.371/3.67e-7$	$0.531/3.82e-7$
68	RC	100	$0.106/6.58e-8$	$0.52/7.67e-7$	$0.482/3.81e-7$	$0.739/F$	$0.556/1.33e-7$
69	RC	500	$0.112/1.47e-7$	$2.282/8.75e-7$	$0.54/4.79e-7$	$1.057F$	$0.642/3.65e-7$
70	RC	1000	$0.16/2.08e-7$	$1.647/4.99e-7$	$0.603/6.77e-7$	$0.567/F$	$0.747/5.46e-7$
71	RS	2	$0.3/5.63e-7$	$0.45/6.34e-7$	$0.886/8.11e-7$	$0.51/7.69e-7$	$0.143/9.1e-10$
72	RS	100	$0.354/7.84e-7$	$0.649/9.68e-7$	$1.038/8.95e-7$	F/F	$0.135/6.42e-9$
73	RS	500	$0.372/7.81e-7$	$0.586/7.79e-7$	$1.305/9.98e-7$	F/F	$2.468/8.97e-7$
74	RS	1000	$0.414/4.88e-7$	$0.682/6.61e-7$	$1.283/8.87e-7$	F/F	$2.484/9.80e-7$
75	RS	10000	$0.614/6.87e-7$	$0.989/7.53e-7$	$2.181/8.79e-7$	F/F	$3.83/9.71e-7$
76	RS	50000	$1.247/6.80e-7$	$1.768/6.06e-7$	$3.581/9.80e-7$	$1152/9.31e-7$	$0.337/8.80e-7$
77	RS	100000	$1.801/9.62e-7$	$2.782/8.57e-7$	$5.971/8.71e-7$	$412.5/6.58e-7$	$3.83/9.61e-7$
78	EM	2	$0.314/8.47e-7$	$0.374/3.9e-7$	$0.561/3.84e-7$	$0.685/6.27e-7$	$0.472/2.26e-7$
79	EM	100	$0.377/6.71e-7$	$0.521/5.93e-7$	$0.632/8.73e-7$	$1.111/9.97e-7$	$1.449/4.60e-7$
80	EM	500	$0.389/5.31e-7$	$0.49/6.70e-7$	$0.882/1.33e-7$	$2.558/8.64e-7$	$0.777/6.27e-8$
81	EM	1000	$0.391/7.51e-7$	$0.553/9.47e-7$	$1.043/7.97e-7$	$1.149/7.15e-7$	$0.986/6.38e-7$
82	EM	10000	$1.124/4.15e-7$	$1.098/8.52e-7$	$3.519/1.54e-8$	$20.85/6.80e-7$	$1.503/4.79e-7$
83	PO	2	$0.363/9.29e-9$	$0.187/8.12e-7$	F/F	F/F	$0.035/6.3e-16$
84	EB	2	$0.495/5.57e-7$	$0.509/4.84e-7$	$0.495/4.06e-7$	$0.319/2.67e-7$	$0.552/7.72e-7$
85	EB	100	$0.306/9.73e-7$	$0.534/8.17e-7$	$0.538/1.73e-7$	$0.366/3.81e-7$	$0.697/1.59e-7$
86	EB	500	$0.631/5.36e-7$	$0.701/3.34e-7$	$0.559/3.88e-7$	$0.443/9.07e-7$	$0.354/F$
87	EB	1000	$0.716/7.58e-7$	$0.669/4.72e-7$	$0.625/5.49e-7$	$0.709/2.96e-7$	$0.528/F$
88	EB	10000	$1.368/5.92e-7$	$1.549/7.58e-7$	$1.602/1.92e-7$	$1.091/6.81e-7$	$0.756/F$
89	EB	50000	$3.613/8.84e-7$	$3.624/6.03e-7$	$5.152/4.30e-7$	$2.968/4.09e-7$	$4.11/7.24e-7$
90	EB	100000	$6.952/4.64e-7$	$6.652/8.53e-7$	$10.58/6.09e-7$	$11.36/7.37e-7$	$6.174/7.84e-7$
91	CU	2	$10.25/3.58e-7$	$3.039/5.32e-7$	$42.89/9.91e-7$	$6.246/6.05e-7$	$2.964/8.94e-7$
92	CQ	2	$0.017/0.00e+0$	$0.036/0.00e+0$	$0.025/0.00e+0$	$0.025/0.00e+0$	$0.03/0.00e+0$
93	CQ	100	$0.469/9.29e-7$	$1.448/9.41e-7$	F/F	$F/2.68e-16$	$0.047/3.1e-17$
94	CQ	500	$0.83/9.83e-7$	F/F	F/F	F/F	$0.038/3e-15$
95	CQ	1000	$1.061/9.71e-7$	F/F	F/F	$7.896/9.92e-7$	$0.047/2.4e-15$
96	CQ	10000	$2.051/9.73e-7$	F/F	F/F	F/F	$0.063/1.6e-15$

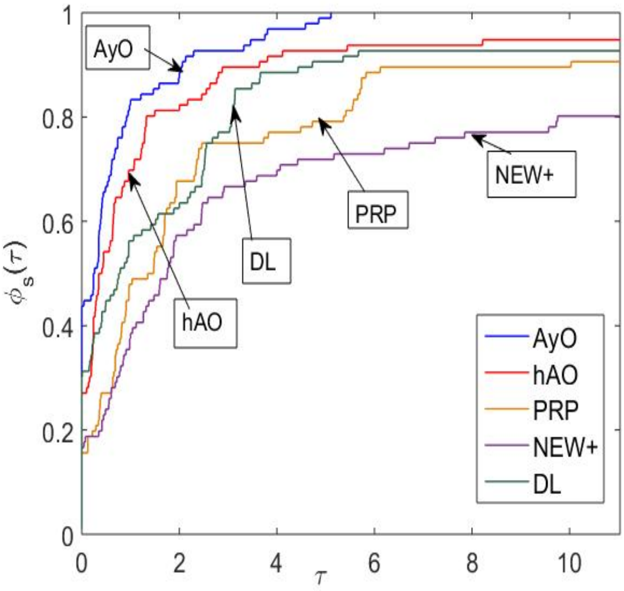

Fig. 1: Number of iterations (IT)

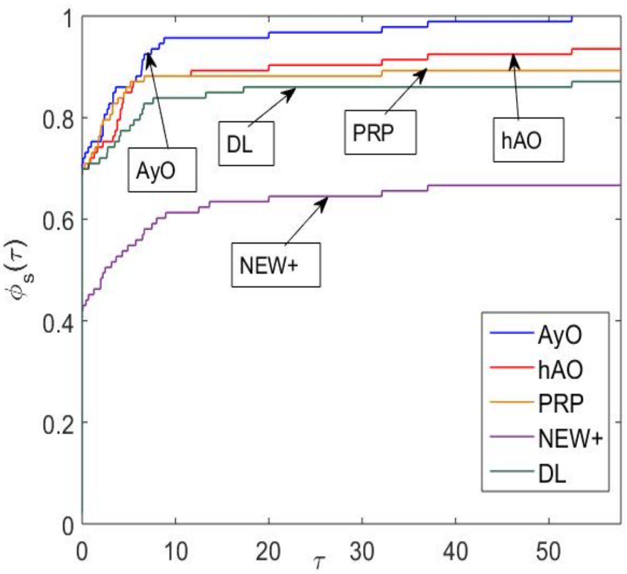

Fig. 2: Value of function f (FE)

A. Numerical Experiment and Parameter Settings

This subsection presents analysis and parameter settings for the comparison of the proposed method against selected existing methods. Test problems with their initial points given in Table I are drawn from [5]. Furthermore, the proposed algorithm is coded using conditions in (5)-(6) and implemented on MATLAB $R2015b$, HP 650 windows 10 OS and RAM 3GB. The algorithm code for the CG-methods runs with $\delta = 0.0001$, $\sigma = 0.9$. We select $t = 0.1$ for the CG-methods with parameter t while $\rho_k = 1$ is used for NEW$_+$ CG method. The iteration is stopped if

$$\|g_k\| \leq 10^{-6}.$$

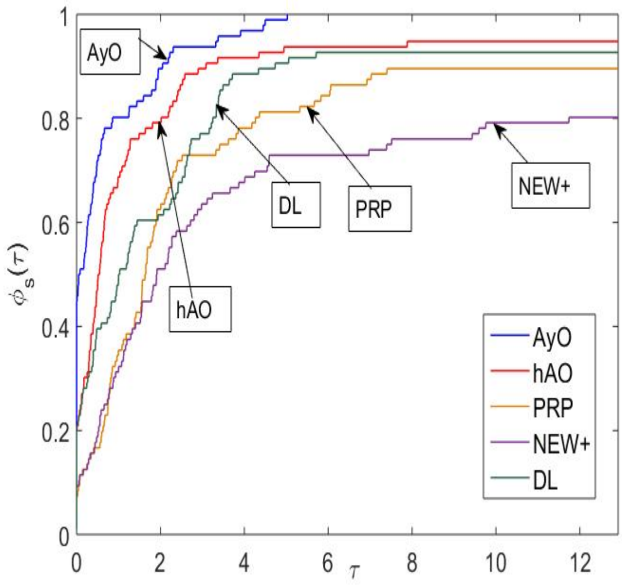

Fig. 3: CPU time

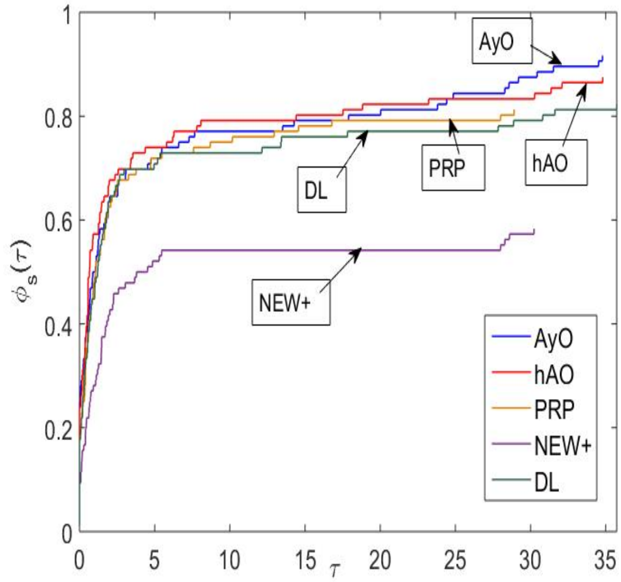

Fig. 4: Gradient norm (GN)

B. Numerical Results and Discussion

The purpose of this subsection is to report the numerical results as well as to evaluate the performance of AyO against the selected existing methods from four performance metrics-the number of iterations (IT), final values of the objective function f (FE), the CPU time consumed by the algorithms in seconds (CPU) and gradient norm (GN). In Tables II and III respectively, results are shown based on the four performance metrics. Varying dimensions $2, 4, 100, 500, 1000, 10000, 50000, 100000$ are used. F in the table indicates failure of the method to solve the test problem. Further evaluations of the performance of the five methods were done with the profiling tools of Dolan and More [11].

The Dolan and More theory is stated as follows: Suppose S is the set of n_s methods to be compared and M is the set of n_m test functions. Let $R_{m,s}$ be either IT, FE, GN or CPU time for every method S and problem M. We compare different methods based on the ratio

$$r_{m,s} = \frac{R_{m,s}}{min\{R_{m,s} : s \in S \ and \ m \in M\}} \quad (52)$$

Then the overall distribution function for $r_{m,s}$ is given by

$$\phi_s(\tau) = \frac{1}{n_m}|m \in M : logr_{m,s}| \leq \tau \quad (53)$$

for which $\tau \geq 0$. The probability that $r_{m,s}$ is within a factor $\tau \geq 1$ in relation to the method s is $\phi_s(\tau)$. When $\tau = 1$, the method has the probability $\phi_s(\tau)$ that it will outperform the other methods. The chosen method $s \in S$ fails to solve a problem if $r_i = r_{m,s}$ for some parameter r_i.

Figures (1-4) show the profiles of the five methods relative to IT, FE, CPU-time and GN respectively. Figures 1, 2 and 3 illustrate that the proposed method (AyO) outperformed all the existing methods in terms of number of iterations, values of function and CPU-time. These performance metrics actually affects the robustness and effectiveness of the methods. Meanwhile, Figure 4 presents the gradient norm where the proposed method competes favorably well with the existing methods.

V. Conclusions

A new CG method was constructed in this paper due to the global acceptability of CG method in solving (1). This construction was done based on the quasi-Newton equation. Convergence analysis and descent properties of the new method were shown. Preliminary numerical results showed that the proposed method is promising and effective as it needs the least iterations and less CPU-time consumption. However, we will attempt to extend the Wolfe conditions in the future by different values of σ.

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