Linearly and Unconditionally Energy Stable Schemes for Phase-Field Vesicle Membrane Model

Yang He, Yuting Zhang, Lingzhi Qian*, Huiping Cai* and Haiqiang Xiao*

***Abstract*—In this work, we present two linearly and unconditionally energy stable numerical schemes for a vesicle membrane model that exactly satisfies the conservation of volume constraint and penalizes the surface area constraint. First of all, we introduce some auxiliary variables to transform the original model into an equivalent system using the energy quadratization(EQ) technique. Moreover, with the transformed free energy a quadratic functional with respect to the new variables and the modified energy dissipative law is satisfied. Then an implicit-explicit(IMEX) time discrete technique for the semi-discrete scheme is used to construct two linearly and unconditionally energy stable fully discrete schemes for the model. Finally, numerical experiments are presented to demonstrate the accuracy and unconditionally energy-stability of the proposed schemes.**

***Index Terms*—Vesicle membrane, Phase-field model, Unconditionally energy stable, Stability analysis.**

I. Introduction

BIOLOGICAL vesicle membranes have been widely studied in biology, biophysics and bioengineering for the past several decades ([4] and the references therein). It is a great challenge to model and simulate the morphological changes accurately due to the variety of equilibrium shapes assumed by vesicles in biological experiments [7].

In recent times, many works have focused on designing and studying numerical approximations of this type of models. In the pioneering work of Canham, Evans and Helfrich [3], [5], [10], [16] in which the sharp interface method is derived. On the other hand the phase field method had been used in the field base on a phase field variable [8], [9]. The evolution equations then resulted from the variation of the action function of the free energy.

From the numerical point of view, the main challenge in designing efficient and accurate schemes for the models is to preserve the thermodynamically consistent law at the discrete level while imposing physical constraints such as conservation of volume and surface area. There are many popular strategy to design energy stable time discretization schemes for phase-field models. Such as convex splitting method [13], [14], [17], stabilization approach [2], [6], [17], IEQ or EQ method [12], [19], [20], SAV method [1], [7], [18] and Runge-Kutta method [1], [18]. It is remarkable that, when dealing with the phase-field vesicle membrane model we have some essential difficulties, including that (i) the sixth order derivative for spatial variable, (ii) the semi-discrete scheme is nonlinear, (iii) the constraint of mass and surface area are desired for designing numerical schemes. [7], [12].

Therefore, the main purpose of this work is to develop two linearly and unconditionally energy stable schemes for the phase field vesicle membrane model. First of all, using the new unknown variables, we can write the sixth-order model as a second-order system. Second, reformulate the system to an equivalent quadratic form by using EQ method. Then IMEX time discrete technique is used to construct completely linear schemes. The proposed schemes are proven rigorously to be unconditionally energy stable.

The rest of the work is organized as follows. In section 2, we briefly introduce the model. Section 3 is devoted to design two linear numerical schemes and show their unconditionally energy stability. In section 4, we present numerical tests to validate the accuracy and efficiency of the numerical schemes. Finally, some conclusions of our work are given in sections 5.

II. The model

In the phase-field vesicle membrane model, the location of the membrane is described by a phase function ϕ. The corresponding interface motion is derived through the energetic variational approach with respect to the bending energy:

$$E_b(\phi) := \frac{\varepsilon}{2}\int_\Omega \Big(\Delta\phi - \frac{1}{\varepsilon^2}G(\phi)\Big)^2 dx = \frac{\varepsilon}{2}\int_\Omega w^2 dx\ , \quad (1)$$

where

$$w := -\Delta\phi + \frac{1}{\varepsilon^2}G(\phi)\ , \quad G(\phi) := F' - \varepsilon k(x)H'(\phi),$$

with

$$F(\phi) := \frac{1}{4}(\phi^2-1)^2\ , \quad H(\phi) := \frac{1}{3}\phi^3 - \phi\ ,$$

and $k(x)$ is a given function representing the spontaneous curvature. The vesicle volume and surface area defined as

$$A(\phi) := \int_\Omega \phi dx \quad \text{and} \quad B(\phi) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi)\Big)dx.$$

The model to study deformation of vesicle membranes can be derived from the bending energy (1) as the Cahn-Hilliard equation

$$\phi_t - \nabla\cdot\Big(\gamma\nabla(\frac{\delta E_b}{\delta\phi})\Big) = 0, \quad (2)$$

Manuscript received December 9, 2021, revised May 8, 2023. This work was supported in part by the the NSF of China (No. 11861054), Natural Science Foundation of Guangxi (No. 2020GXNSFAA297223), Innovation Project of Guangxi Guaduate Education(No. JGY2021028) and English curriculum construction project of Guangxi Normal University (No. 2021XJQYW04).

Yang He is a graduate student of the College of Sciences, Shihezi University, Shihezi 832003, P.R. China (e-mail:yanghe@163.com).

Yuting Zhang is a graduate student of the College of Mathematics and Statistics in Guangxi Normal University, Guilin 541006, P.R. China, (e-mail:2268610841@qq.com).

*Lingzhi Qian is a professor of the College of Mathematics and Statistics in Guangxi Normal University, Guilin 541006, P.R. China, Department of Mathematics, College of Sciences, Shihezi University, Shihezi 832003, P.R. China (corresponding author, e-mail: qianlzc1103@sina.cn).

* Huiping Cai is an associate professor of the College of Mathematics and Statistics in Guangxi Normal University, Guilin 541006, P.R. China, (corresponding author, e-mail:caihp1103@sina.com).

* Haiqiang Xiao is a lecturer in the Department of Mathematics, College of Sciences in Shihezi University, Shihezi 832003, P.R. China, (corresponding author, e-mail: xiaohaiqiang@shzu.edu.cn.)

where $\gamma > 0$ is a mobility parameter and

$$\frac{\delta E_b}{\delta\phi} = \varepsilon\Big(- \Delta w + \frac{1}{\varepsilon^2}G'(\phi)w\Big).$$

A penalty term is added to the elastic bending energy $E_b(\phi)$ in order to enforce the surface area constraints. Then a modified energy is given by

$$E_{bp}(\phi) := E_b(\phi) + \frac{1}{2\eta}(B(\phi) - \beta)^2 \ , \tag{3}$$

where $\eta > 0$ is the penalization parameter, and $\beta > 0$ is the desired surface area. Then, we introduce the (chemical potential) unknown

$$\mu := \frac{\delta E_{bp}}{\delta\phi} = -\varepsilon\Delta w + \frac{1}{\varepsilon}G'(\phi)w + \frac{1}{\eta}(B(\phi) - \beta)\Big(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi)\Big).$$

Using the new unknown, it is possible to write the corresponding sixth-order Cahn-Hilliard equation as the second-order system of (μ, w, ϕ) :

$$\begin{cases} \phi_t - \gamma\Delta\mu = 0, \\ -\varepsilon\Delta w + \frac{1}{\varepsilon}G'(\phi)w + \frac{1}{\eta}(B(\phi) - \beta)\Big(- \varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi)\Big) \\ \qquad -\mu = 0, \\ \varepsilon w + \varepsilon\Delta\phi - \frac{1}{\varepsilon}G(\phi) = 0. \end{cases} \tag{4}$$

Supplemented by the initial condtion

$$\phi|_{t=0} = \phi_0, \quad \text{in} \quad \Omega.$$

The boundary condtions can be either one of the following types:

$$\begin{cases} \nabla\phi \cdot \mathbf{n}|_\Omega = 0, \quad \nabla w \cdot \mathbf{n}|_\Omega = 0, \quad \nabla\mu \cdot \mathbf{n}|_\Omega = 0, \\ \phi|_\Omega = -1, \quad w_\Omega = 0, \quad \nabla\mu \cdot \mathbf{n}|_\Omega = 0, \\ \phi|_\Omega = -1, \quad \nabla w \cdot \mathbf{n}|_\Omega = 0, \quad \nabla\mu \cdot \mathbf{n}|_\Omega = 0. \end{cases} \tag{5}$$

Lemma 2.1: [12] System (4) complemented with one of the boundary conditions proposed in (5) satisfies the following dissipative energy law,

$$\frac{d}{dt}E_{bp}(\phi) + \gamma\|\nabla\mu\|^2 = 0 \ . \tag{6}$$

III. Numerical schemes

In order to construct the linearly and unconditionally energy stable numerical schemes for the model, we introduce the auxiliary variables.

Let $B(\phi) - \beta = U, \quad \phi^2 - 1 = V, \quad \phi^3 - \phi = g.$

$$\begin{aligned} E_{bp}(\phi) &= E_b(\phi) + \frac{1}{2\eta}(B(\phi) - \beta)^2 \\ &= \frac{\varepsilon}{2}\int_\Omega w^2 dx + \frac{1}{2\eta}U^2 := E(w, U). \end{aligned} \tag{7}$$

Then we obtain a new, but equivalent PDE system by taking the time derivative for the new variables:

$$\begin{cases} \phi_t - \gamma\Delta\mu = 0, \\ -\varepsilon\Delta w + \frac{1}{\varepsilon}((3V + 2) - 2\varepsilon k(x)\phi)w \\ +\frac{1}{\eta}\Big(- \varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi)\Big)U - \mu = 0, \\ \varepsilon w + \varepsilon\Delta\phi - \frac{1}{\varepsilon}(g - \varepsilon k(x)V) = 0, \\ U_t = \int_\Omega(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi))\phi_t dx, \\ V_t = 2\phi\phi_t, \\ g_t = (3\phi^2 - 1)\phi_t. \end{cases} \tag{8}$$

Supplemented by the initial condition

$$\phi|_{t=0} = \phi_0, \quad U|_{t=0} = B(\phi_0) - \beta, \quad V|_{t=0} = \phi_0^2 - 1. \tag{9}$$

and one of the admissible boundary conditions given in (5).

It is clear that the new transformed system still retains a similar energy dissipative law.

Lemma 3.1: System (8) complemented with one of the boundary conditions proposed in (5) satisfies the following dissipative energy law,

$$\frac{d}{dt}E_{bp}(\phi) = \frac{d}{dt}E(w, U) = -\gamma\|\nabla\mu\|^2 \le 0 \ . \tag{10}$$

Proof. Testing $(8)_1$ by μ, $(8)_2$ by ϕ_t and taking the time derivative of $(8)_3$ by w, $(8)_4$ by U, $(8)_5$ by V and adding these relations, we can easily derive the energy dissipation law of the new system.

Let $\mathcal{T}_h$ be a triangulation of Ω, and h be the mesh parameter of $\mathcal{T}_h$. The unknowns (ϕ, μ, w) are approximated by the conforming finite element spaces:

$$(\Phi_h, M_h, W_h) \subset (H^1(\Omega), H^1(\Omega), H^1(\Omega)) \ . \tag{11}$$

Then, we give the semi-discrete scheme as follows. Find $(\phi(t), \mu(t), w(t)) \in (\Phi_h, M_h, W_h)$, such that

$$\begin{cases} (\phi_t, \bar\mu) + \gamma(\nabla\mu, \nabla\bar\mu) = 0, \\ (\varepsilon\nabla w, \nabla\bar\phi) + \frac{1}{\varepsilon}\Big(((3V + 2) - 2\varepsilon k(x)\phi)w, \bar\phi\Big) \\ +\frac{1}{\eta}\Big((- \varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi))U, \bar\phi\Big) - (\mu, \bar\phi) = 0, \\ \varepsilon(w, \bar w) - \varepsilon(\nabla\phi, \nabla\bar w) - \frac{1}{\varepsilon}\Big((g, \bar w) - (\varepsilon k(x)V, \bar w)\Big) = 0, \\ U_t = \int_\Omega(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi))\phi_t dx, \\ V_t = 2\phi\phi_t, \\ g_t = (3\phi^2 - 1)\phi_t. \end{cases} \tag{12}$$

A. First order scheme

Let $\Delta t = \frac{T}{N}, t_n = n\Delta t$. Assuming that ϕ^n, U^n, V^n, w^n, g^n are already calculated, we then compute ϕ^{n+1}, w^{n+1}, $U^{n+1}, V^{n+1}, g^{n+1}$ from the fully-discrete scheme:

$$\begin{cases} (\frac{\phi^{n+1} - \phi^n}{\Delta t}, \bar\mu) + \gamma(\nabla\mu^{n+1}, \nabla\bar\mu) = 0, \\ (\varepsilon\nabla w^{n+1}, \nabla\bar\phi) + \frac{1}{\varepsilon}\Big(((3V^n + 2) - 2\varepsilon k(x)\phi^n)w^{n+1}, \bar\phi\Big) \\ +\frac{1}{\eta}\Big((- \varepsilon\Delta\phi^n + \frac{1}{\varepsilon}F'(\phi^n))U^{n+1}, \bar\phi\Big) \\ -(\mu^{n+1}, \bar\phi) = 0, \\ \varepsilon(w^{n+1}, \bar w) - \varepsilon(\nabla\phi^{n+1}, \nabla\bar w) \\ -\frac{1}{\varepsilon}\Big((g^{n+1}, \bar w) - (\varepsilon k(x)V^{n+1}, \bar w)\Big) = 0, \\ U^{n+1} - U^n = \int_\Omega(-\varepsilon\Delta\phi^n + \frac{1}{\varepsilon}F'(\phi^n))(\phi^{n+1} - \phi^n)dx, \\ V^{n+1} - V^n = 2\phi^n(\phi^{n+1} - \phi^n), \\ g^{n+1} - g^n = ((3\phi^n)^2 - 1)(\phi^{n+1} - \phi^n). \end{cases} \tag{13}$$

Theorem 3.1: The linear scheme (13) is unconditionally energy stable, i.e. satisfies the following discrete energy dissipation law:

$$\begin{aligned} E^{n+1} &+ \frac{\varepsilon}{2}\|w^{n+1} - w^n\|^2 + \frac{1}{2\eta}(U^{n+1} - U^n)^2 \\ &+ \gamma\Delta t\|\nabla\mu^{n+1}\|^2 = E^n, \end{aligned} \tag{14}$$

where $E^n = \frac{\varepsilon}{2}\|w^n\|^2 + \frac{1}{2\eta}(U^n)^2$.

Proof. Taking $\bar{\mu} = \Delta t\mu^{n+1}, \bar{\phi} = \phi^{n+1} - \phi^n$ in $(13)_{1,2}$, adding the resulting relations, we obtain

$$\begin{aligned} &\gamma\Delta t\|\nabla\mu^{n+1}\|^2 + (\varepsilon\nabla w^{n+1}, \nabla(\phi^{n+1} - \phi^n)) \\ &+ \frac{1}{\varepsilon}\Big(((3V^n + 2) - 2\varepsilon k(x)\phi^n)w^{n+1}, \phi^{n+1} - \phi^n\Big) \\ &+ \frac{1}{\eta}\Big((- \varepsilon\Delta\phi^n + \frac{1}{\varepsilon}F'(\phi^n))U^{n+1}, \phi^{n+1} - \phi^n\Big) = 0. \end{aligned} \quad (15)$$

Subtracting $(13)_3$ and $(13)_3$ for previous time step, and taking $\bar{w} = w^{n+1}$, we obtain

$$\begin{aligned} &\varepsilon(w^{n+1} - w^n, w^{n+1}) - \varepsilon(\nabla(\phi^{n+1} - \phi^n), \nabla w^{n+1}) \\ &- \frac{1}{\varepsilon}\Big((g^{n+1} - g^n, w^{n+1}) - (\varepsilon k(x)(V^{n+1} - V^n), w^{n+1})\Big) \\ &= 0. \end{aligned} \quad (16)$$

We can easily derived

$$\begin{aligned} &- \frac{1}{\varepsilon}\Big((g^{n+1} - g^n, w^{n+1}) - (\varepsilon k(x)(V^{n+1} - V^n), w^{n+1})\Big) \\ &= -\frac{1}{\varepsilon}\Big(((3V^n + 2) - 2\varepsilon k(x)\phi^n)(\phi^{n+1} - \phi^n), w^{n+1}\Big). \end{aligned} \quad (17)$$

By taking the simple multiplication of $(13)_4$ with $\frac{1}{\eta}U^{n+1}$ and applying the following identities

$$(a - b, a) = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2). \quad (18)$$

We obtain

$$\begin{aligned} &\frac{1}{2\eta}\Big((U^{n+1})^2 - (U^n)^2 + (U^{n+1} - U^n)^2\Big) \\ &= \frac{1}{\eta}\Big((- \varepsilon\Delta\phi^n + \frac{1}{\varepsilon}F'(\phi^n))U^{n+1}, \phi^{n+1} - \phi^n\Big). \end{aligned} \quad (19)$$

Combing of above relations gives us

$$\begin{aligned} &\frac{\varepsilon}{2}\Big(\|w^{n+1}\|^2 - \|w^n\|^2 + \|w^{n+1} - w^n\|^2\Big) \\ &+ \frac{1}{2\eta}\Big((U^{n+1})^2 - (U^n)^2 + (U^{n+1} - U^n)^2\Big) \\ &+ \gamma\Delta t\|\nabla\mu^{n+1}\|^2 = 0. \end{aligned} \quad (20)$$

i.e.

$$\begin{aligned} &\frac{\varepsilon}{2}\|w^{n+1}\|^2 + \frac{1}{2\eta}(U^{n+1})^2 + \frac{\varepsilon}{2}\|w^{n+1} - w^n\|^2 \\ &+ \frac{1}{2\eta}(U^{n+1} - U^n)^2 + \gamma\Delta t\|\nabla\mu^{n+1}\|^2 \\ &= \frac{\varepsilon}{2}\|w^n\|^2 + \frac{1}{2\eta}(U^n)^2. \end{aligned} \quad (21)$$

Then, we have completed the proof.

B. Second order scheme

Assuming that ϕ^{n-1}, U^{n-1}, V^{n-1}, w^{n-1}, g^{n-1} and ϕ^n, U^n, V^n, w^n, g^n are already calculated, we then compute $\phi^{n+1}, w^{n+1}, U^{n+1}, V^{n+1}, g^{n+1}$ from the fully-discrete scheme:

$$\begin{cases} (\frac{3\phi^{n+1}-4\phi^n+\phi^{n-1}}{2\Delta t}, \bar{\mu}) + \gamma(\nabla\mu^{n+1}, \nabla\bar{\mu}) = 0 \ , \\ (\varepsilon\nabla w^{n+1}, \nabla\bar{\phi}) + \frac{1}{\varepsilon}\Big(((3V^* + 2) - 2\varepsilon k(x)\phi^*)w^{n+1}, \bar{\phi}\Big) \\ +\frac{1}{\eta}\Big((- \varepsilon\Delta\phi^* + \frac{1}{\varepsilon}F'(\phi^*))U^{n+1}, \bar{\phi}\Big) \\ -(\mu^{n+1}, \bar{\phi}) = 0 \ , \\ \varepsilon(w^{n+1}, \bar{w}) - \varepsilon(\nabla\phi^{n+1}, \nabla\bar{w}) \\ -\frac{1}{\varepsilon}\Big((g^{n+1}, \bar{w}) - (\varepsilon k(x)V^{n+1}, \bar{w})\Big) = 0 \ , \\ 3U^{n+1} - 4U^n + U^{n-1} \\ = \int_\Omega(-\varepsilon\Delta\phi^n + \frac{1}{\varepsilon}F'(\phi^n))(3\phi^{n+1} - 4\phi^n + \phi^{n-1})dx \ , \\ 3V^{n+1} - 4V^n + V^{n-1} = 2\phi^*(3\phi^{n+1} - 4\phi^n + \phi^{n-1}) \ , \\ 3g^{n+1} - 4g^n + g^{n-1} \\ = ((3\phi^*)^2 - 1)(3\phi^{n+1} - 4\phi^n + \phi^{n-1}) \ . \end{cases} \quad (22)$$

where $\phi^* = 2\phi^n - \phi^{n-1}$, $V^* = 2V^n - V^{n-1}$.

Theorem 3.2: The linear scheme (22) is unconditionally energy stable, i.e. satisfies the following discrete energy dissipation law:

$$\begin{aligned} &E^{n+1} + \frac{\varepsilon}{2}\|w^{n+1} - 2w^n + w^{n-1}\|^2 + \frac{1}{2\eta}(U^{n+1} - 2U^n + U^{n-1})^2 \\ &+ 2\gamma\Delta t\|\nabla\mu^{n+1}\|^2 = E^n, \end{aligned} \quad (23)$$

Where

$$E^n = \frac{\varepsilon}{2}(\|w^n\|^2 + \|2w^n - w^{n-1}\|^2) + \frac{1}{2\eta}((U^n)^2 + (2U^n - U^{n-1})^2).$$

Proof. Taking $\bar{\mu} = 2\Delta t\mu^{n+1}, \bar{\phi} = 3\phi^{n+1} - 4\phi^n + \phi^{n-1}$ in $(22)_{1,2}$, adding the resulting relations, we obtain

$$\begin{aligned} &2\gamma\Delta t\|\nabla\mu^{n+1}\|^2 + (\varepsilon\nabla w^{n+1}, \nabla(3\phi^{n+1} - 4\phi^n + \phi^{n-1})) \\ &+ \frac{1}{\varepsilon}\Big(((3V^* + 2) - 2\varepsilon k(x)\phi^*)w^{n+1}, 3\phi^{n+1} - 4\phi^n + \phi^{n-1}\Big) \\ &+ \frac{1}{\eta}\Big((- \varepsilon\Delta\phi^n + \frac{1}{\varepsilon}F'(\phi^*))U^{n+1}, \phi^{n+1} - \phi^n\Big) = 0. \end{aligned} \quad (24)$$

Reformulate $(22)_3$ and the two previous time steps, and taking $\bar{w} = w^{n+1}$, we obtain

$$\begin{aligned} &\varepsilon(3w^{n+1} - 4w^n + w^{n-1}, w^{n+1}) \\ &- \varepsilon(\nabla(3\phi^{n+1} - 4\phi^n + \phi^{n-1}), \nabla w^{n+1}) \\ &- \frac{1}{\varepsilon}\Big((3g^{n+1} - 4g^n + g^{n-1}, w^{n+1}) \\ &- (\varepsilon k(x)(3V^{n+1} - 4V^n + V^{n-1}), w^{n+1})\Big) = 0. \end{aligned} \quad (25)$$

Applying the following identities

$$\begin{aligned} &2(3a - 4b + c, a) \\ &= (|a|^2 - |b|^2 + |2a - b|^2 - |2b - c|^2 + |a - 2b + c|^2). \end{aligned} \quad (26)$$

We obtain

$$\begin{aligned} &\frac{\varepsilon}{2}(\|w^{n+1}\|^2 - \|w^n\|^2 + \|2w^{n+1} - w^n\|^2 - \|2w^n - w^{n-1}\|^2) \\ &+ \|w^{n+1} - 2w^n + w^{n-1}\|^2 \\ &- \frac{1}{\varepsilon}((3(\phi^*)^2 - 1)(3\phi^{n+1} - 4\phi^n + \phi^{n-1}), w^{n+1}) \\ &- (2\varepsilon k(x)\phi^*(3\phi^{n+1} - 4\phi^n + \phi^{n-1}), w^{n+1}) = 0. \end{aligned} \quad (27)$$

We can easily derived

$$
\begin{aligned}
&-\frac{1}{\varepsilon}((3(\phi^*)^2-1)(3\phi^{n+1}-4\phi^n+\phi^{n-1}),w^{n+1})\\
&-(2\varepsilon k(x)\phi^*(3\phi^{n+1}-4\phi^n+\phi^{n-1}),w^{n+1})=\\
&\frac{1}{\varepsilon}((3(V^*)^2+2-2\varepsilon k(x)\phi^*)(3\phi^{n+1}-4\phi^n+\phi^{n-1}),w^{n+1}).
\end{aligned} \tag{28}
$$

By taking the simple multiplication of $(22)_4$ with $\frac{1}{\eta}U^{n+1}$ and the identities (26), We obtain

$$
\begin{aligned}
&\frac{1}{2\eta}((U^{n+1})^2-(U^n)^2+(2U^{n+1}-U^n)^2\\
&-(2U^n-U^{n-1})^2+(U^{n+1}-2U^n+U^{n-1})^2)\\
&=\frac{1}{\eta}\Big((-\varepsilon\Delta\phi^*+\frac{1}{\varepsilon}F'(\phi^*))U^{n+1},3\phi^{n+1}-4\phi^n+\phi^{n-1}\Big).
\end{aligned} \tag{29}
$$

Combing of above relations gives us the following results

$$
\begin{aligned}
&\frac{\varepsilon}{2}(\|w^{n+1}\|^2-\|w^n\|^2+\|2w^{n+1}-w^n\|^2\\
&-\|2w^n-w^{n-1}\|^2+\|w^{n+1}-2w^n+w^{n-1}\|^2)\\
&\frac{1}{2\eta}((U^{n+1})^2-(U^n)^2+(2U^{n+1}-U^n)^2-(2U^n-U^{n-1})^2)\\
&+(U^{n+1}-2U^n+U^{n-1})^2+2\gamma\Delta t\|\nabla\mu^{n+1}\|^2=0.
\end{aligned} \tag{30}
$$

i.e.

$$
\begin{aligned}
&\frac{\varepsilon}{2}(\|w^{n+1}\|^2+\|2w^{n+1}-w^n\|^2)+\frac{1}{2\eta}((U^{n+1})^2+(2U^{n+1}-U^n)^2)\\
&+\frac{\varepsilon}{2}(\|w^{n+1}-2w^n+w^{n-1}\|^2)\\
&+\frac{1}{2\eta}((U^{n+1}-2U^n+U^{n-1})^2)+2\gamma\Delta t\|\nabla\mu^{n+1}\|^2\\
&=\frac{\varepsilon}{2}(\|w^n\|^2+\|2w^n-w^{n-1}\|^2)\\
&+\frac{1}{2\eta}((U^n)^2+(2U^n-U^{n-1})^2).
\end{aligned} \tag{31}
$$

Then, we have completed the proof.

IV. Numerical experiments

In this sections we present some numerical experiments to illustrate the theoretical results obtained in the previous sections and and demonstrate the accuracy and stability of the proposed schemes. All the simulations have been carried out in 2D domains using Freefem++ software[15].

Assume $\Omega=[0,1]\times[0,1]$, and $h=\frac{1}{20}$, the right-hand side function f_1 is chosen to ensure that the given solution can satisfy the system $(4)_1$. The following function is assumed to be the exact solution

$$\phi(x,y,t)=cos(\pi x)cos(\pi y)t. \tag{32}$$

The physical parameters are presented as follows:

$$\varepsilon=0.01,\quad \eta=0.01,\quad \gamma=0.01,\quad T=0.1\ .$$

The L^2 and H^1 errors of the first order scheme are shown in Table 1.

Moreover, we test the unconditionally energy stability. We set the initial condition as $\phi=0.25cos(\pi x)cos(\pi y)$, Assume $\Omega=[0,1]\times[0,1]$, $h=\frac{1}{20}$ and $\Delta t=0.0001$. The energy, volume and surface area evolution using various time step sizes are summarized in Figure 1, demonstrating the energy stability, volume and surface area constraint of our proposed scheme. We observe the calculated energies are all decreasing with time, which are agreed with Theorem 3.1($E^{n+1}\le E^n$ for an arbitrary time step), i.e., unconditionally energy stable.

Table 1. Numerical results of the first order scheme

Δt	$Err(\phi)_{L^2}$	Rate	$Err(\phi)_{H^1}$	Rate
5×10^{-2}	1.97e-3	-	1.08e-2	-
2.5×10^{-2}	5.86e-4	1.75	4.16e-3	1.38
1.25×10^{-2}	1.62e-4	1.85	1.81e-3	1.20
6.25×10^{-3}	4.27e-5	1.92	8.60e-4	1.07
3.125×10^{-3}	1.10e-5	1.96	4.25e-4	1.02
1.5625×10^{-3}	2.85e-6	1.95	2.12e-4	1.00

Furthermore, we use the second order scheme to the same problem and $T=0.5$, the corresponding results are shown in Table 2.

Table 2. Numerical results of the second order scheme

Δt	$Err(\phi)_{L^2}$	Rate	$Err(\phi)_{H^1}$	Rate
1×10^{-1}	3.56e-1	-	1.71e-0	-
9×10^{-2}	2.55e-1	3.17	1.20e-0	3.36
8×10^{-2}	1.83e-1	2.82	8.43e-1	3.00
7×10^{-2}	1.31e-1	2.50	5.94e-1	2.62
6×10^{-2}	9.26e-2	2.25	4.16e-1	2.31
5×10^{-2}	6.39e-2	2.03	2.86e-1	2.06

V. Conclusion

In this work, we present a linearly and unconditionally energy stable scheme for a vesicle membrane model that satisfies exactly the conservation of volume constraint and penalizes the surface area constraint. First of all, we introduce auxiliary variables to transform the original model into an equivalent system using the energy quadratization(EQ) technique. Moreover, with the transformed free energy a quadratic functional with respect to the new variables and the modified energy dissipative law is conserved. Then an implicit-explicit(IMEX) time discrete scheme for the semi-discrete scheme to construct a linearly and unconditionally energy stable fully discrete schemes for the model. Numerical examples are given to demonstrate the efficiency of the proposed schemes. Further study is underway to improve the simulation by extending to more realistic problems.

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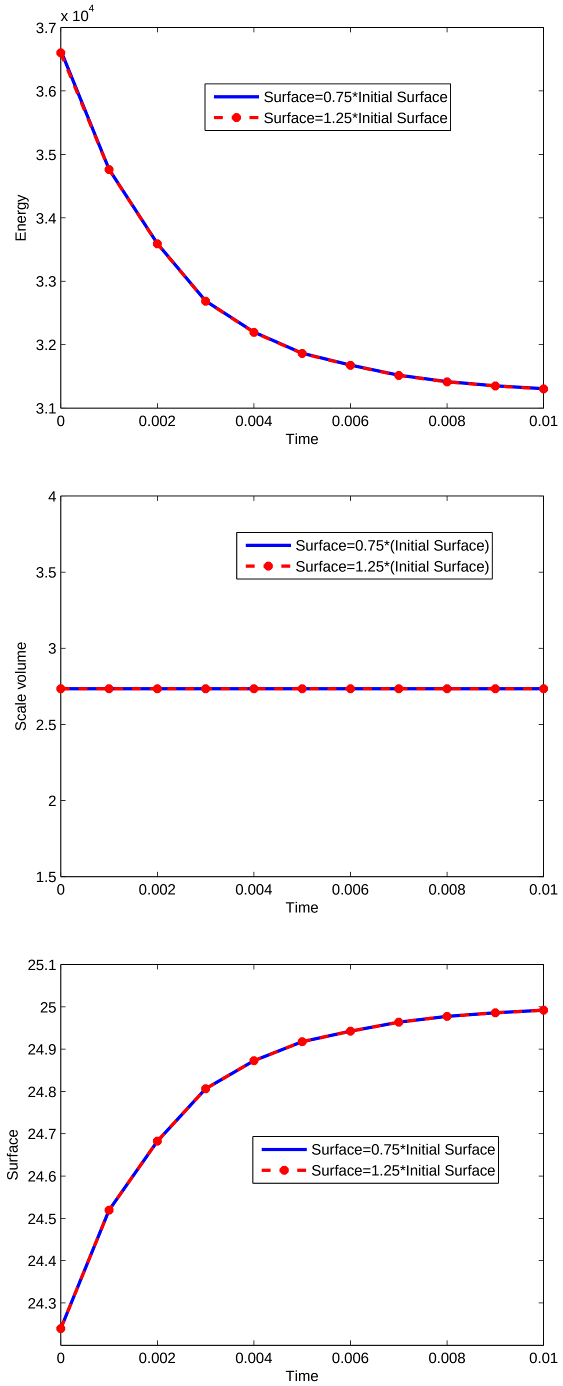

Fig. 1: (a) Evolution in time of energy; (b) Evolution in time of mass; (c) Evolution in time of surface

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Ling-zhi Qian was born in Lingbi, Anhui Province, china, in 1980. The author received his Ph. D. in computational mathematics at Nanjing Normal University, Nanjing, Jiangsu Province, China, in June 2016.

The author current research interests are in numerical solutions of partial differential equations, fluid-fluid interaction problems and two-phase incompressible fluids.