The M/M/1 Repairable Queueing System with Two Types of Server Breakdowns and Negative Customers

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***Abstract*—We analyze the affect of server breakdowns and negative customers on queueing systems. Using the quasi-brith-and-death (QBD) process and matrix geometry solution method, the M/M/1 repairable queueing system with two types of server breakdowns and negative customers was studied. In the model, arriving at a negative customer will offset one positive customer, simultaneously, it may reduce the service rate or stop working of the system. The stationary conditions are given, the steady-state probability vectors are obtained, and some steady-state queueing and reliable measures are calculated. The results suggest that the system's mean queue length decreases as the increase of the negative customers arrival rate. Finally, the affect of parameter changes on the system performance measures are shown by graphs, which can provide an actual reference for practical applications.**

***Index Terms*—two types of server breakdowns, negative customers, repairable queueing system, matrix geometry solution.**

I. Introduction

IN queueing theory, considering negative customers is an important research director. In 1991, Gelenbe et al. [1] firstly introduced a queueing strategy with negative customers in queueing system. Negative customers were usually considered as a virus of the system. Their impact is producing different offsetting effects in the normal service period. The offsetting rule was generally that negative customers offset the first customer or the last customer in the queueing system. Zhang and Niu [2] analyzed the M/G/1 system with negative customers and a single working vacation strategy. They gave the distribution of queue length, the probability of the system being in each state and mathematical expectation under stationary conditions by the matrix analysis method. Yan and Yang [3] derived infinitesimal generating elements of Markov chains by establishing a QBD process.They derived the steady-state measures using the matrix geometric solution. Finally, the results were verified by numerical simulations. Atencia and Moreno [4] discussed a Geo/Geo/1 model with negative customers' strategy. They used state transfer to get the PGF of the system waiting queue length. Xu et al. [5] discussed the M/M/1 fluid model with negative customers and working vacation. They used the Laplace transform method to get the probability of the two buffers of the queue respectively.

The negative customer queueing model has been further extended after extensive research. Wang and Zhang [6] analyzed the queueing model with negative customers in discrete time. In the model, negative customer arrival can cause the server to breakdown during normal service periods, with no impact during repairs. Zhou [7] studied the pre-emptive feedback M/G/1 retrial model with N-strategy. Pan [8] analyzed a priority queueing model with variable input rate, feedback, negative customers, and service time obeying general distribution. They pointed out the matters needing attention in the application of the model and the problems to be further solved. Yang et al. [9] have introduced setup time, working vacation and working breakdown into the M/M/1/N model. They established the finite state QBD process of the model. The system performance measures were calculated, such as steady-state availability, system variance, throughput rate of the system. Using embedded Markov chain and PGF methods, Chen and Jia [10] derived the steady-state mean queue length of the system during departure periods. He also obtained the probability of the server in each state. Qu et al. [11] firstly combined negative customers and repairable systems. The model was analyzed using the state transfer equation, and the queueing and reliability measures of this model were obtained. The above has greatly enriched the theoretical system of negative customer queueing models.

In telecommunication and computer network systems, a virus invasion damage the system files, the system will be back as new after the repair of the damaged files is completed. Kalidass and Kasturi [12] first analyzed the working breakdown strategy in M/M/1 queueing model. Prevalent in many practical queueing systems, working breakdown strategy means that the system continued to provide service at a lower service rate during a breakdown period. For example, a virus in computer systems may degrade the performance of the system rather than cause it to stop completely. Kim and Lee [13] introduced the M/M/1 queueing model with working breakdown to the M/G/1 model. They studied queueing systems with standby servers and obtained the queue length. Liu and Song [14] introduced the working breakdown strategy in the M/M/1 model with batch arrivals. They analyzed the PGF of the steady-state queue length and its random decomposition. Li and Zhang [15] considered working breakdown in the M/G/1 retrial model with negative customers. They get the probability distribution of the retrial

Manuscript received March 17, 2023; revised October 24, 2023.

This work was supported by the National Nature Science Foundation of China (No. 71971189).

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queue length. Li and Li [16] analyzed an M/G/1 G-queue with working vacations and Bernoulli vacation interruption, server breakdown, and negative customers. They calculate the PGF of the queue length and the server state using matrix geometry solutions and supplementary variable methods. Tsai et al. [17] studied the open queueing network with server breakdown and verified the correctness of the method and the validity of the model. Lv [18] studies a repairable queuing system with two repairmen and limited space, and deduces the main system performance indicators. Li and Li [19] studied the retrial queuing system and calculated the steady-state conditions and probability distribution. Ramasamy et al. [20] analyzed discrete time Geo/G/2 queue under a serial and parallel queue disciplines and obtained steady state results.

In real life, the queueing system will be affected inevitably by information interference. For instance, virus intrusion in computer systems leading to system crashes, the deterioration of computer hardware results in computer lag. Further, the cancellation of orders has an impact on inventory levels, the malfunction of production systems restricts production schedules and so on. Those similar interference phenomenons in the aforementioned examples are known as "negative customers". Their major impact is to produce various offsetting effects during normal service periods, or lead to system breakdown and lower service rate of the system. System breakdowns are further divided into working breakdown (or called as incomplete breakdown) and complete breakdown.

This paper was to study various system measures of the M/M/1 model with two types of server breakdowns and negative customers. It provided theoretical references for practical applications. The queueing model was that two types of server breakdowns and negative customers were considered comprehensively. Then, the system measures of the model were calculated. Afterwards, numerical examples were given, the characters of the system measures of the model was verified by numerical examples. Finally, the MATLAB software was used to analyze the model numerically, and the affect of parameter changes on the system performance measures was showed.

II. Model description

The queueing system that is studied in this paper has one server that may breakdown incompletely or completely at any time. Customers are divided into two types.

1) Customer arrival: the customers includes positive customers and negative customers. The arrival rate of positive customers is λ^+. The arrival rate of negative customers is λ^-. Positive customers form a waiting queue, but negative customers do not form a waiting queue.

2) If there are positive customers in the system during the normal service period, the arriving negative customers will offset the positive customers receiving service one by one. Meanwhile, the server will be working under an incomplete breakdown state with probability α, and the server will be breakdown completely with probability $1-\alpha$. When the system is in incomplete breakdown state, the arrival of negative customers will cause the server to breakdown completely. If there are no positive customers in the system, the negative customers will automatically disappear, and the negative customers does not accept the service.

3) The service time of a positive customer obeys the negative exponential distribution of μ during the normal service period. The service time of a customer obeys the negative exponential distribution of μ_0 during incomplete breakdown state. During complete breakdown state, the server stops the service completely.

4) The server may breakdown with no negative customers. The arrival of the breakdowns obey Poisson process. The incomplete breakdown rate is ε_1. The complete breakdown rate is ε_2.

5) When the server is in breakdown state, the server is repaired immediately and the server is repaired as well as new. During incomplete breakdown state, the repair time is an exponential distribution with parameter ξ_1. During complete breakdown state, the service time of a customer is an exponential distribution with parameter ξ_2.

6) The service rule of the server to customers in the system is first-come-first-served (FCFS). Assuming the customers waiting in line will not leave the system without serviced. The arrival interval, service time and repair time are independent of each other.

Let $L(t)$ be the number of customers in the queueing system at moment t, and $J(t)$ be the state of the sever at moment t:

$$J(t)=\begin{cases} 0, \text{ in the normal service period at time t,} \\ 1, \text{ in the incomplete failure period at time t,} \\ 2, \text{ in the complete failure period at time t.} \end{cases}$$

Then $\{L(t),\ J(t),\ t\geq 0\}$ is a Markov process whose state space is $\Omega=\{(k,\ j),\ k=0,\ 1,\ 2,\cdots;j=0,\ 1,\ 2\}$. Figure 1 shows the state transition diagram of the three-dimensional Markov chain.

$\boldsymbol{Q}$ is the following matrix:

$$\boldsymbol{Q}=\begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{C} & & & \\ \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} & & \\ & \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$\boldsymbol{A}_0=\begin{pmatrix} -\lambda^+-\varepsilon_1-\varepsilon_2 & \varepsilon_1 & \varepsilon_2 \\ \xi_1 & -\lambda^+-\xi_1 & 0 \\ \xi_2 & 0 & -\lambda^+-\xi_2 \end{pmatrix},$$

$$\boldsymbol{C}=\begin{pmatrix} \lambda^+ & 0 & 0 \\ 0 & \lambda^+ & 0 \\ 0 & 0 & \lambda^+ \end{pmatrix},\boldsymbol{A}=\begin{pmatrix} A_{11} & \varepsilon_1 & \varepsilon_2 \\ \xi_1 & A_{22} & 0 \\ \xi_2 & 0 & A_{33} \end{pmatrix},$$

$$\boldsymbol{B}=\begin{pmatrix} \mu & \alpha\lambda^- & (1-\alpha)\lambda^- \\ 0 & \mu_0 & \lambda^- \\ 0 & 0 & \lambda^- \end{pmatrix},$$

$A_{11}=-\lambda^+-\lambda^--\mu-\varepsilon_1-\varepsilon_2$, $A_{22}=-\lambda^+-\lambda^--\mu_0-\xi_1$, $A_{33}=-\lambda^+-\lambda^--\xi_2$.

The matrix $\boldsymbol{Q}$ shows that $\{L(t),\ J(t),\ t\geq 0\}$ is a QBD process.

III. Steady-state conditions

According to the related theory of the QBD process, a sufficient and necessary condition to ensure the existence of the steady-state probability distribution of $X(t) = \{L(t),\ J(t),\ t \geq 0\}$ is $\boldsymbol{PCe} < \boldsymbol{PBe}$, where $\boldsymbol{P} = \begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix}$ is the steady-state probability vector of $\boldsymbol{H}$, and
$\boldsymbol{H} = \boldsymbol{B} + \boldsymbol{A} + \boldsymbol{C} =$

$$\begin{pmatrix} -(\lambda^- + \varepsilon_1 + \varepsilon_2) & \alpha\lambda^- + \varepsilon_1 & (1-\alpha)\lambda^- + \varepsilon_2 \\ \xi_1 & -(\lambda^- + \xi_1) & \lambda^- \\ \xi_2 & 0 & -\xi_2 \end{pmatrix}.$$

$\boldsymbol{H}$ is an irreducible generating element, substituting $\boldsymbol{P}$ and $\boldsymbol{H}$ into the normalization condition yields the following results:

$$\begin{cases} \boldsymbol{PH} = \boldsymbol{0}, \\ \boldsymbol{Pe} = 1, \end{cases} \tag{1}$$

where $\boldsymbol{e} = \begin{bmatrix} 1, & 1, & 1 \end{bmatrix}^\top$.

Substituting $\boldsymbol{P}$, $\boldsymbol{e}$ and $\boldsymbol{H}$ into Eq. (1) obtain:

$$\begin{cases} -(\lambda^- + \varepsilon_1 + \varepsilon_2)P_0 + \xi_1 P_1 + \xi_2 P_2 = 0, \\ (\alpha\lambda^- + \varepsilon_1)P_0 - (\lambda^- + \xi_1)P_1 = 0, \\ \left[(1-\alpha)\lambda^- + \varepsilon_2\right] P_0 + \lambda^- P_1 - \xi_2 P_2 = 0, \\ P_0 + P_1 + P_2 = 1. \end{cases} \tag{2}$$

Solving Eq. (2) yields the following results:

$$\begin{cases} P_0 = \frac{1}{\phi}(\lambda^- + \xi_1)\xi_2, \\ P_1 = \frac{1}{\phi}(\alpha\lambda^- + \varepsilon_1)\xi_2, \\ P_2 = 1 - \frac{1}{\phi}(\lambda^- + \xi_1 + \alpha\lambda^- + \varepsilon_1)\xi_2, \end{cases}$$

where $\phi = \lambda^{-2} + \lambda^-\varepsilon_1 + \lambda^-\varepsilon_2 + \lambda^-\xi_1 + \varepsilon_2\xi_1 + \lambda^-\xi_2 + \varepsilon_1\xi_2 + \xi_1\xi_2 - \lambda^-\xi_1\alpha + \lambda^-\xi_2\alpha$.

Theorem 1. If and only if $\rho < 1$, the system is stationary, where

$$\rho = \frac{\lambda^+[\lambda^{-2} + \varepsilon_2\xi_1 + (\varepsilon_1 + \xi_1)\xi_2 + \lambda^-(\psi + \xi_2^2 - \xi_2\alpha)]}{\lambda^-[\lambda^{-2} + \varepsilon_2\xi_1 + \lambda^-\psi + \xi_2(\varepsilon_1 + \xi_1 + \mu + \alpha\mu_0)] + \xi_2(\xi_1\mu + \varepsilon_1\mu_0)}$$

$$\psi = \varepsilon_1 + \varepsilon_2 + \xi_1 + \xi_2 - \xi_2^2 - \xi_1\alpha + 2\xi_2\alpha.$$

Proof The positive recurrent of the QBD processes $\{L(t),\ J(t),\ t \geq 0\}$ means $\boldsymbol{PCe} < \boldsymbol{PBe}$, where $\boldsymbol{e} = \begin{pmatrix} 1, & 1, & 1 \end{pmatrix}^\top$, then

$$\begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} \begin{pmatrix} \lambda^+ & 0 & 0 \\ 0 & \lambda^+ & 0 \\ 0 & 0 & \lambda^+ \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$< \begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} \begin{pmatrix} \mu & \alpha\lambda^- & (1-\alpha)\lambda^- \\ 0 & \mu_0 & \lambda^- \\ 0 & 0 & \lambda^- \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Equivalent to

$$\rho = \frac{\lambda^+[\lambda^{-2} + \varepsilon_2\xi_1 + (\varepsilon_1 + \xi_1)\xi_2 + \lambda^-(\psi + \xi_2^2 - \xi_2\alpha)]}{\lambda^-[\lambda^{-2} + \varepsilon_2\xi_1 + \lambda^-\psi + \xi_2(\varepsilon_1 + \xi_1 + \mu + \alpha\mu_0)] + \xi_2(\xi_1\mu + \varepsilon_1\mu_0)},$$

where $\psi = \varepsilon_1 + \varepsilon_2 + \xi_1 + \xi_2 - \xi_2^2 - \xi_1\alpha + 2\xi_2\alpha$.

IV. Steady-state probability

Define the steady-state probability vector as follows:

$$\boldsymbol{\Pi} = (\boldsymbol{\pi}_0,\ \boldsymbol{\pi}_1,\ \boldsymbol{\pi}_2,\ \cdots),$$

where $\boldsymbol{\pi}_k = (\pi_{k0},\ \pi_{k1},\ \pi_{k2}),\ k = 0,\ 1,\ 2,\ \cdots,$
the stead-state distribution is

$$\pi_{k,\ j} = \lim P\left\{L(t) = k,\ J(t) = j\right\},\ (k,\ j) \in \Omega.$$

Theorem 2. The stochastic matrix $\boldsymbol{B}[\boldsymbol{R}]$ is

$$\boldsymbol{B}[\boldsymbol{R}] = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{C} \\ \boldsymbol{B} & \boldsymbol{A} + \boldsymbol{RB} \end{pmatrix},$$

then:

$$\begin{cases} (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1)\boldsymbol{B}[\boldsymbol{R}] = \boldsymbol{0}, \\ \boldsymbol{\pi}_0\boldsymbol{e} + \boldsymbol{\pi}_1(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{e} = 1, \\ \boldsymbol{\pi}_k = \boldsymbol{\pi}_1\boldsymbol{R}^{k-1},\ k \geq 1, \end{cases} \tag{3}$$

where $\boldsymbol{R}$ is the minimal non-negative solution of the equation $\boldsymbol{R}^2\boldsymbol{B} + \boldsymbol{RA} + \boldsymbol{C} = \boldsymbol{0}$, and $SP(\boldsymbol{R}) < 1$, I is a 3-dimensional unit matrix, $\boldsymbol{e} = \begin{pmatrix} 1, & 1, & 1 \end{pmatrix}^\top$.

Proof 1) Verifing that $\boldsymbol{\pi}_k = \boldsymbol{\pi}_1\boldsymbol{R}^{k-1},\ k \geq 1$ holds.

Expanding the equilibrium equation $\boldsymbol{\Pi Q} = \boldsymbol{0}$ yields

$$\begin{cases} \boldsymbol{\pi}_0\boldsymbol{A_0} + \boldsymbol{\pi}_1\boldsymbol{B} = \boldsymbol{0}, \\ \boldsymbol{\pi}_{k-1}\boldsymbol{C} + \boldsymbol{\pi}_k\boldsymbol{A} + \boldsymbol{\pi}_{k+1}\boldsymbol{B} = \boldsymbol{0},\ k \geq 1. \end{cases} \tag{4}$$

When $k = 1$, $\boldsymbol{\pi}_k = \boldsymbol{\pi}_1\boldsymbol{R}^{k-1}$ clearly holds, and it is only necessary to prove whether $\boldsymbol{\pi}_k = \boldsymbol{\pi}_1\boldsymbol{R}^{k-1}$ holds when $k \geq 2$. Substituting $\boldsymbol{\pi}_k = \boldsymbol{\pi}_1\boldsymbol{R}^{k-1}$ into Eq. (4) gives:

$$\begin{aligned} &\boldsymbol{\pi}_{k-1}\boldsymbol{C} + \boldsymbol{\pi}_k\boldsymbol{A} + \boldsymbol{\pi}_{k+1}\boldsymbol{B} \\ &= \boldsymbol{\pi}_1\boldsymbol{R}^{k-2}\boldsymbol{C} + \boldsymbol{\pi}_1\boldsymbol{R}^{k-1}\boldsymbol{A} + \boldsymbol{\pi}_1\boldsymbol{R}^{k}\boldsymbol{B} \\ &= \boldsymbol{\pi}_1\boldsymbol{R}^{k-2}(\boldsymbol{C} + \boldsymbol{RA} + \boldsymbol{R}^2\boldsymbol{B}) \\ &= \boldsymbol{0}, \end{aligned}$$

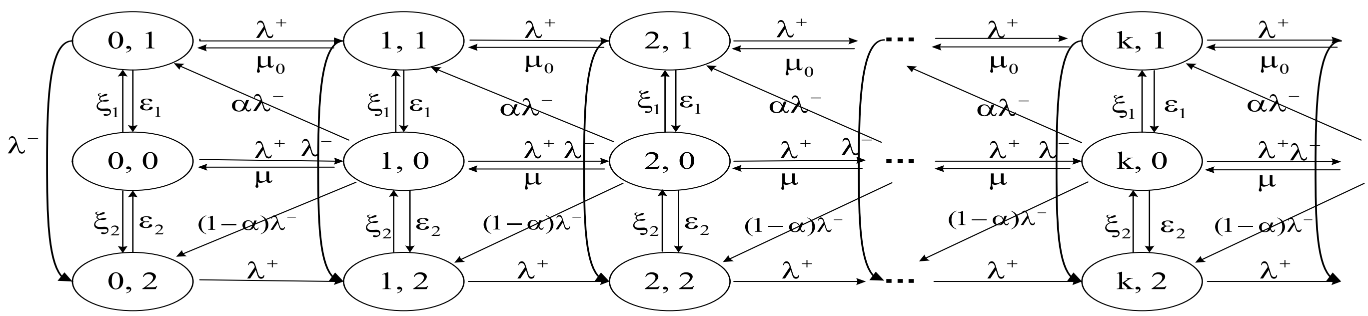

Fig. 1. State transfer diagram.

so $\boldsymbol{\pi}_k = \boldsymbol{\pi}_1 \boldsymbol{R}^{k-1},\ k \geq 1$ holds.

2) Verifing that $\begin{pmatrix} \boldsymbol{\pi}_0, & \boldsymbol{\pi}_1 \end{pmatrix} \boldsymbol{B}[\boldsymbol{R}] = \mathbf{0}$ holds.

$$\begin{aligned} &\begin{pmatrix} \boldsymbol{\pi}_0, & \boldsymbol{\pi}_1 \end{pmatrix} \boldsymbol{B}[\boldsymbol{R}] \\ &= \begin{pmatrix} \boldsymbol{\pi}_0 \boldsymbol{A_0} + \boldsymbol{\pi}_1 \boldsymbol{B}, & \boldsymbol{\pi}_0 \boldsymbol{C} + \boldsymbol{\pi}_1 \boldsymbol{A} + \boldsymbol{\pi}_1 \boldsymbol{R} \boldsymbol{B} \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

To convert it into a groups of equations, the form is:

$$\begin{cases} \boldsymbol{\pi}_0 \boldsymbol{A_0} + \boldsymbol{\pi}_1 \boldsymbol{B} = \mathbf{0}, \\ \boldsymbol{\pi}_0 \boldsymbol{C} + \boldsymbol{\pi}_1 \boldsymbol{A} + \boldsymbol{\pi}_1 \boldsymbol{R} \boldsymbol{B} = \mathbf{0}. \end{cases} \tag{5}$$

From Eq. (4) it follows that $\boldsymbol{\pi}_0 \boldsymbol{A_0} + \boldsymbol{\pi}_1 \boldsymbol{B} = \mathbf{0}$ holds. Let $k = 2$, then $\boldsymbol{\pi}_2 = \boldsymbol{\pi}_1 \boldsymbol{R}$, which can be substituted into Eq. (5) gives:

$$\begin{aligned} &\boldsymbol{\pi}_0 \boldsymbol{C} + \boldsymbol{\pi}_1 \boldsymbol{A} + \boldsymbol{\pi}_1 \boldsymbol{R} \boldsymbol{B} \\ &= \boldsymbol{\pi}_0 \boldsymbol{C} + \boldsymbol{\pi}_1 \boldsymbol{A} + \boldsymbol{\pi}_2 \boldsymbol{B} \\ &= \mathbf{0}, \end{aligned}$$

so $\begin{pmatrix} \boldsymbol{\pi}_0, & \boldsymbol{\pi}_1 \end{pmatrix} \boldsymbol{B}[\boldsymbol{R}] = \mathbf{0}$.

3) Verifing that $\boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 (\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{e} = 1$ holds.

From the normalization condition $\boldsymbol{\Pi} \boldsymbol{e} = 1$, we know that $\boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 \boldsymbol{e} + \boldsymbol{\pi}_2 \boldsymbol{e} + \boldsymbol{\pi}_3 \boldsymbol{e} + \cdots = 1$. Substituting $\boldsymbol{\pi}_k = \boldsymbol{\pi}_1 \boldsymbol{R}^{k-1},\ k \geq 1$ into $\boldsymbol{\Pi} \boldsymbol{e} = 1$, we can obtain:

$$\begin{aligned} &\boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 \boldsymbol{e} + \boldsymbol{\pi}_2 \boldsymbol{e} + \boldsymbol{\pi}_3 \boldsymbol{e} + \cdots \\ &= \boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 \boldsymbol{e} + \boldsymbol{\pi}_1 \boldsymbol{R} \boldsymbol{e} + \boldsymbol{\pi}_1 \boldsymbol{R}^2 \boldsymbol{e} + \cdots \\ &= \boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 (\boldsymbol{I} + \boldsymbol{R} + \boldsymbol{R}^2 + \boldsymbol{R}^3 + \cdots) \boldsymbol{e} \\ &= 1. \end{aligned}$$

Since $SP(\boldsymbol{R}) < 1$, the $\boldsymbol{I} + \boldsymbol{R} + \boldsymbol{R}^2 + \boldsymbol{R}^3 + \cdots$ converges to $(\boldsymbol{I} - \boldsymbol{R})^{-1}$.

Therefore,

$$\begin{aligned} &\boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 (\boldsymbol{I} + \boldsymbol{R} + \boldsymbol{R}^2 + \boldsymbol{R}^3 + \cdots) \boldsymbol{e} \\ &= \boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 (\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{e} \\ &= 1, \end{aligned}$$

so $\boldsymbol{\pi}_0 \boldsymbol{e} + \boldsymbol{\pi}_1 (\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{e} = 1$ holds.

According to Eq. (3) , the steady-state boundary probability vector $\begin{pmatrix} \boldsymbol{\pi}_0, & \boldsymbol{\pi}_1 \end{pmatrix}$ can be obtained.

V. System steady-state performance measures

Using the matrix R and the steady-state boundary probability vector $\begin{pmatrix} \boldsymbol{\pi}_0, & \boldsymbol{\pi}_1 \end{pmatrix}$, we obtain the steady-state performance measures of the model.

1) The probability that the server is in the state j

$$P(J = j) = \sum_{i=0}^{\infty} \boldsymbol{\pi}_i \boldsymbol{e}_j = \boldsymbol{\pi}_1 [(\boldsymbol{I} - \boldsymbol{R})^{-1} - \boldsymbol{B} {\boldsymbol{A}_0}^{-1}] \boldsymbol{e}_j, \tag{6}$$

where $\boldsymbol{e}_j (0 \leq j \leq 2)$ is a column vector of dimension 3 that the element of the $j+1$ line is 1, and the other elements are 0.

2) The steady-state mean queue length of the system

$$E(L) = \sum_{i=1}^{\infty} i \boldsymbol{\pi}_i \boldsymbol{e} = \boldsymbol{\pi}_1 (\boldsymbol{I} - \boldsymbol{R})^{-2} \boldsymbol{e}. \tag{7}$$

3) The steady-state mean waiting queue length of the system

$$E(L_q) = \sum_{i=1}^{\infty} (i-1) \boldsymbol{\pi}_i \boldsymbol{e} = \boldsymbol{\pi}_1 [(\boldsymbol{I} - \boldsymbol{R})^{-2} - (\boldsymbol{I} - \boldsymbol{R})^{-1}] \boldsymbol{e}. \tag{8}$$

4) Positive customers loss rate due to negative customers arrival

$$D = \sum_{j=1}^{2} \sum_{i=0}^{\infty} \lambda^- \pi_{ij} \boldsymbol{e}_1 = \lambda^- \boldsymbol{\pi}_1 [(\boldsymbol{I} - \boldsymbol{R})^{-1} - \boldsymbol{B} {\boldsymbol{A}_0}^{-1}] \boldsymbol{e}_1, \tag{9}$$

where $\boldsymbol{e}_1 = \begin{pmatrix} 0, & 1, & 1 \end{pmatrix}^\top$.

5) The steady-state availability of the server

$$A = \sum_{j=0}^{1} \sum_{i=0}^{\infty} \pi_{ij} \boldsymbol{e}_2 = \boldsymbol{\pi}_1 [(\boldsymbol{I} - \boldsymbol{R})^{-1} - \boldsymbol{B} {\boldsymbol{A}_0}^{-1}] \boldsymbol{e}_2, \tag{10}$$

where $\boldsymbol{e}_2 = \begin{pmatrix} 1, & 1, & 0 \end{pmatrix}^\top$.

VI. Numerical example

Assuming $\lambda^+ = 2,\ \lambda^- = 0.1,\ \alpha = 0.6,\ \mu = 4,\ \mu_0 = 3,\ \varepsilon_1 = 3,\ \varepsilon_2 = 2,\ \xi_1 = 3$ and $\xi_2 = 4$, easily come to $\rho < 1$, and the corresponding subblock matrix is

$$\boldsymbol{A}_0 = \begin{pmatrix} -7 & 3 & 2 \\ 3 & -5 & 0 \\ 4 & 0 & -6 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 4 & 0.06 & 0.04 \\ 0 & 3 & 0.1 \\ 0 & 0 & 0.1 \end{pmatrix},$$

$$\boldsymbol{A} = \begin{pmatrix} -11.1 & 3 & 2 \\ 3 & -8.1 & 0 \\ 4 & 0 & -6.1 \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let the matrix $\boldsymbol{R}$ is

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix},$$

and substituting $\boldsymbol{R}$ into the equation $\boldsymbol{R}^2 \boldsymbol{B} + \boldsymbol{R} \boldsymbol{A} + \boldsymbol{C} = \mathbf{0}$, which can be calculated:

$$\boldsymbol{R} = \begin{pmatrix} 0.3423 & 0.1887 & 0.1179 \\ 0.1885 & 0.3937 & 0.0675 \\ 0.3202 & 0.2074 & 0.4427 \end{pmatrix}.$$

Substituting $\boldsymbol{R}$ into Eq. (3):

$$\begin{aligned} \boldsymbol{\pi}_0 &= \begin{pmatrix} \pi_{00}, & \pi_{01}, & \pi_{02} \end{pmatrix} \\ &= \begin{pmatrix} 0.1205, & 0.1207, & 0.0427 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{\pi}_1 &= \begin{pmatrix} \pi_{10}, & \pi_{11}, & \pi_{12} \end{pmatrix} \\ &= \begin{pmatrix} 0.0777, & 0.0791, & 0.0412 \end{pmatrix}. \end{aligned}$$

Plug in the expression, we get:

1) The probability that the server is in state $0, 1$ and 2 are $P(J = 0) = 0.3976, P(J = 1) = 0.3940$ and $P(J = 2) = 0.2084$.

2) The steady-state mean queue length of the system is $E(L) = 2.6176$.

3) The steady-state mean waiting queue length of the system is $E(L_q) = 1.3833$.

4) Positive customers loss rate due to negative customers arrival is $D = 0.0602$.

5) The steady-state availability of the server is $A = 0.7916$.

VII. NUMERICAL EXPERIMENTS

We analyzes the affect of variations of each parameter on the steady-state mean queue length, positive customers loss rate due to negative customers arrival, and the availability of the server of the system through numerical experiments.

Assuming that $\lambda^+ = 2,\ \alpha = 0.6,\ \mu = 4,\ \mu_0 = 3,\ \varepsilon_1 = 3,\ \varepsilon_2 = 2,\ \xi_1 = 3,\ \xi_2 = 4$ and λ^- takes values of 0.1 to 0.5, we analyze the influence of λ^- on the mean queueing length $E(L)$ of Eq. (7), positive customers loss rate due to negative customers arrival D of Eq. (9), and the availability of the server A of Eq. (10), respectively. The results are as follows:

In Figure 1, when $\lambda^+,\ \alpha,\ \mu,\ \mu_0,\ \varepsilon_1,\ \varepsilon_2,\ \xi_1$ and ξ_2 are constant, the mean queueing length $E(L)$ shows a decreasing trend with the increase of the λ^-.

From Figure 2, when $\lambda^+,\ \alpha,\ \mu,\ \mu_0,\ \varepsilon_1,\ \varepsilon_2,\ \xi_1$ and ξ_2 are constant, positive customers loss rate due to negative customers arrival D is increasing with the increase of the λ^-, which is consistent with the actual situation.

From Figure 3, when $\lambda^+,\ \alpha,\ \mu,\ \mu_0,\ \varepsilon_1,\ \varepsilon_2,\ \xi_1$ and ξ_2 are constant, the system steady-state availability of the server A is increasing gradually with the increase of the λ^-.

Assuming that $\alpha = 0.6,\ \varepsilon_1 = 3,\ \varepsilon_2 = 2,\ \xi_1 = 3,\ \xi_2 = 4,\ \lambda^+$ takes values of 1.5 to 2.5, λ^- varies in the range of 0.1 to 0.5, μ takes values of 3 to 5 and μ_0 varies in the range of 2 to 4.

Figure 4 illustrates the effects of λ^+ and λ^- on $E(L)$ of the system with other parameters. As can be seen from Figure 4, when λ^+ is a constant value, $E(L)$ is decreasing with the increase of λ^-. When λ^- is constant, $E(L)$ is increasing with the increase of λ^+ .

Figure 5 describes the affect of λ^+ and μ on $E(L)$ of the system. As can be seen from Figure 5, when μ is a constant value, $E(L)$ is increasing with the increase of λ^+. When λ^+ is constant, $E(L)$ is decreasing with the increase of μ_0, which is consistent with the actual situation.

Figure 6 describes the affect of λ^- and μ on $E(L)$ of the system with other parameters. As can be seen from Figure 6, when μ is a constant value, $E(L)$ is decreasing with the increase of λ^-. When λ^- is constant, $E(L)$ is increasing with the increase of μ and the rising trend is slowing down.

Figure 7 describes the influences of ε_1 and ε_2 on A with other parameters. As can be seen from Figure 7, when ε_1 is a constant value, A is decreasing with the increase of ε_2. When ε_2 is constant, A is increasing with the increase of ε_1.

VIII. CONCLUSION

The paper investigated a M/M/1 model with two types of server breakdowns and negative customers. Firstly, the model was transformed into a QBD process. Using some relevant theories of the QBD process, the conditions required for the steady-state balance of the system were obtained. Secondly, the system steady-state probability vector was derived by using the matrix geometry solution method. The matrix R and steady-state boundary probability vector $(\boldsymbol{\pi}_0,\ \boldsymbol{\pi}_1)$ were used to obtain the probability of the system being in each state, the mean queue length of system and other performance measures. Numerical examples were given and the availability of the model was verified by them. Finally, the affect of parameter changes on the system performance

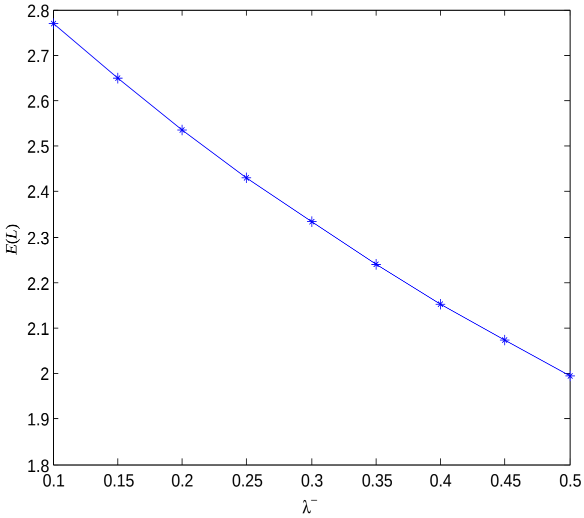

Fig. 2. The trend of $E(L)$ versus λ^-.

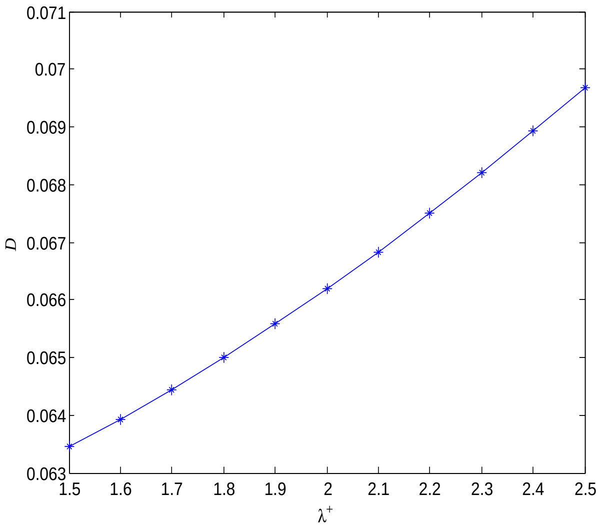

Fig. 3. The trend of D versus λ^-.

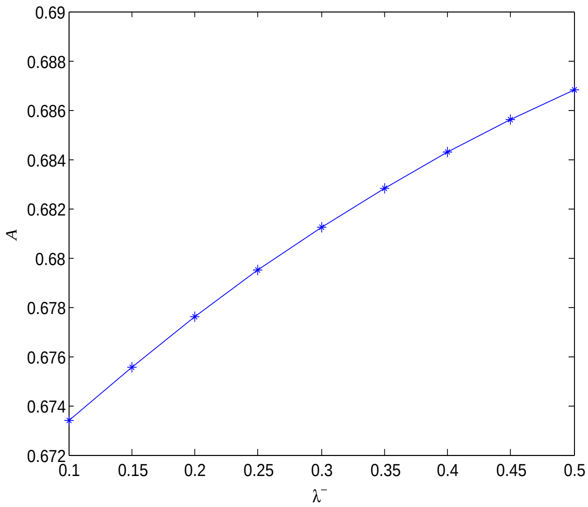

Fig. 4. The trend of A versus λ^-.

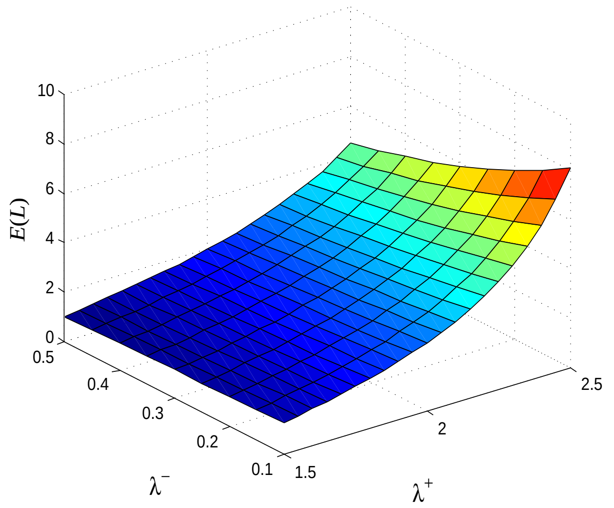

Fig. 5. The trend of $E(L)$ versus λ^+ and λ^- ($\alpha = 0.6$, $\mu = 4$, $\mu_0 = 3$, $\varepsilon_1 = 3$, $\varepsilon_2 = 2$, $\xi_1 = 3$ and $\xi_2 = 4$).

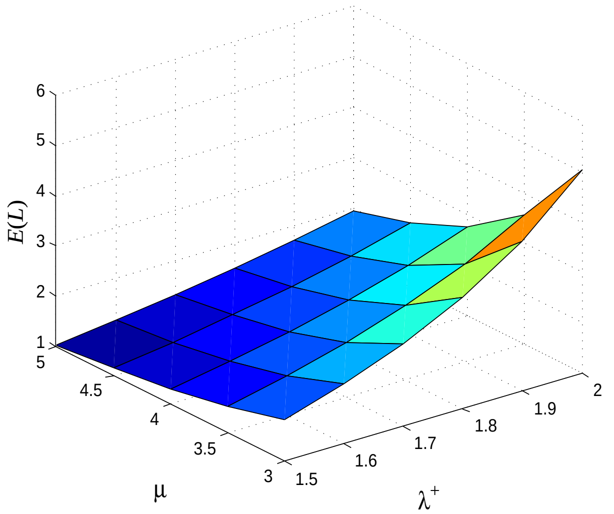

Fig. 6. The trend of $E(L)$ versus λ^+ and μ ($\lambda^- = 0.1$, $\alpha = 0.6$, $\mu_0 = 3$, $\varepsilon_1 = 3$, $\varepsilon_2 = 2$, $\xi_1 = 3$ and $\xi_2 = 4$).

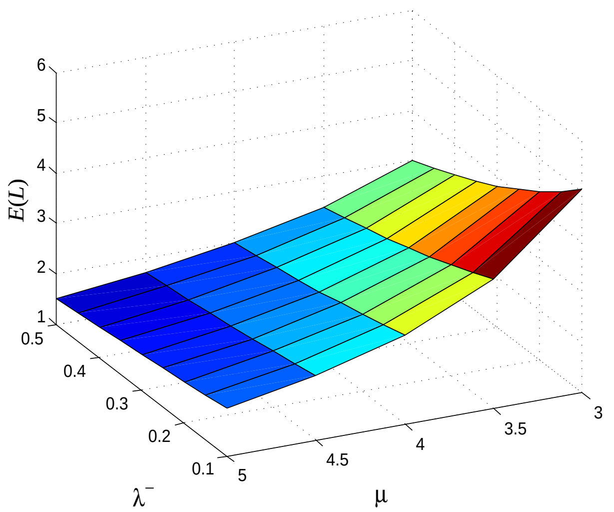

Fig. 7. The trend of $E(L)$ versus λ^- and μ ($\lambda^+ = 2$, $\alpha = 0.6$, $\mu_0 = 3$, $\varepsilon_1 = 3$, $\varepsilon_2 = 2$, $\xi_1 = 3$ and $\xi_2 = 4$).

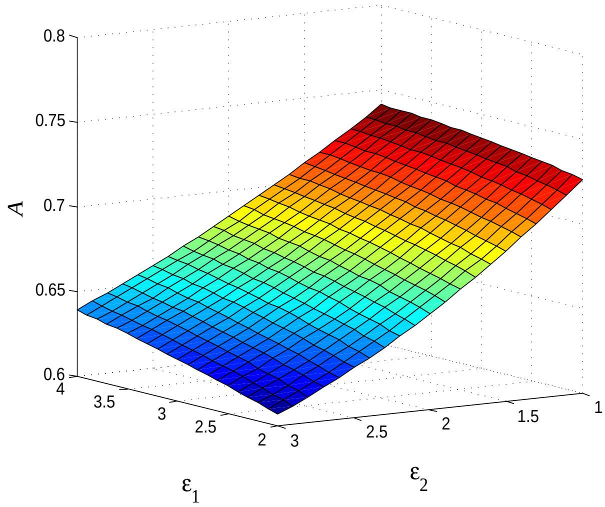

Fig. 8. The trend of A versus ε_1 and ε_2 ($\lambda^+ = 2$, $\lambda^- = 2$, $\alpha = 0.6$, $\mu = 4$, $\mu_0 = 3$, $\xi_1 = 3$ and $\xi_2 = 4$).

measures were illustrated using graphs. The results showed that different parameters can significantly affect the mean queueing length in the steady-state system. The model conditions in this paper are considerably general. It can also provide theoretical references for practical applications.

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