Method for Solving Variational Inequality Problems and Fixed Point Problems without Some Well-known Condition

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Abstract—**This study presents an innovative method for approximating solutions to the variational inequality and fixed-point problems. The proposed approach deviates from traditional methods by employing different conditions and techniques drawn from [18] [20] [21]. Uniquely, our work circumvents the utilization of a commonly used lemma (see [10]) that forms the basis for most proofs related to strong convergence theorems. As part of our investigation, we provide a comprehensive numerical example to substantiate our findings, thus enhancing the practical relevance and applicability of our research.**

Index Terms—**Fixed-point problem, Iteration, nonexpansive mapping, Variational inequality problem.**

I. Introduction

THE Fixed-point theory is widely recognized as an exceptionally potent and fundamental tool in the research of nonlinear phenomena, spanning applications in diverse fields such as computer science, chemistry, economics, biology, engineering, game theory, physics, image processing, and geometry. Fixed point techniques have emerged as a vital cornerstone of modern scientific research. For example, [1] [2] [3] [4] [5] have all utilized these techniques. At its core, the view of a fixed point describes a point that remains invariant under a given transformation such as a map, a system of differential equations, or other mathematical operations. Specifically, a fixed point of a mapping $T(x)$ is a point $x_0 \in D(T)$ that satisfies $T(x_0) = x_0$, and the set of solutions to the fixed point problem associated with T is denoted by $F(T)$.

In this paper, we focus on the variational inequality problem which involves solving an inequality for all possible values of a given variable. This functional is typically solved over a convex set and has been widely used in numerous scientific fields, including game theory, optimization, finance, and economics.

In 1959, Signorini introduced a challenge that became the first instance of a variational inequality, known as the Signorini problem. This was later addressed and resolved by Fichera in 1963 [6]. Subsequently, in 1964, Stampacchia introduced an extension to the Lax-Milgram theorem [7]. This extension was aimed at exploring the consistency issues related to partial differential equations. He introduced the term "variational inequality" for all problems involving inequalities of this kind.

In the context of our study, we designate H as a real Hilbert space characterized by the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. Additionally, we identify C as a closed convex set within H that is not devoid of elements.

Let $G : C \to H$ be a given mapping. The task associated with the variational inequality challenge is to ascertain an element $\mu \in C$ such that

$$\langle G\mu, \nu - \mu\rangle \geq 0,$$

is satisfied for all $\nu \in C$. We use $VI(C,G)$ to denote the collection of solutions pertinent to the aforementioned variational inequality. It is established in the literature that, under the conditions of G being both strongly monotone and Lipschitzian within C, the set $VI(C,G)$ yields a unique solution.

Given a mapping $W : C \to C$, we term it *β-Lipschitz continuous* when a $\beta > 0$ can be found such that

$$\|W\mu - W\nu\| \leq \beta\|\mu - \nu\|$$

for every μ, ν within C. The constant β represents the Lipschitz constant. When β lies between 0 and 1, W is described as a *β-contractive mapping*. In the special case where $\beta = 1$, W is designated as nonexpansive.

Given the mapping $G : C \to H$, it is deemed *α-strongly monotone* when there exists an $\mu \geq 0$ such that

$$\langle Gx - Gy, x - y\rangle \geq \mu\|x - y\|^2,$$

for every $x, y \in C$. When $\mu = 0$, we refer to G as simply *monotonic*.

For a mapping J taking values from C to H, it is defined as *Φ-inverse strongly monotone* [6] when a positive real constant Φ can be identified, satisfying

$$\langle x - y, Jx - Jy\rangle \geq \Phi \left\|Jx - Jy\right\|^2,$$

for every x, y belonging to C.

Xu [10] introduced one of the most fundamental and widely used methods for proving strong convergence theorems in order to approximate nonlinear and inverse problems. For examples, see [16], [17], and [19]. The details are presented below.

Lemma 1: Let $\{Q_n\}$ be a sequence of real numbers satisfying

$$Q_{n+1} \leq (1 - p_n)Q_n + q_n,$$

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for all $n \geq 0$ with a sequence $\{p_n\}$ in (0,1) and $\{q_n\}$ is a sequence satisfying

(i) $\sum_{n=1}^{\infty} p_n = \infty$,

(ii) $\limsup_{n\to\infty} q_n/p_n \leq 0$ or $\sum_{n=1}^{\infty} q_n < \infty$.

Then, we have

$$\lim_{n\to\infty} Q_n = 0.$$

However, many researchers have endeavoured to prove strong convergence theorems without employing Lemma 1. For example, see [13], [14], and [15].

Throughout this research paper, our objective is to establish the theorem of strong convergence by modifying the Mann iteration of our proposed sequence without the need for Lemma 1. Additionally, we illustrate how to tackle the variational inequality problem utilizing our primary conclusions within the context of applicational theory.

II. Preliminaries

In this part, we outline key lemmas that will play a significant role in validating the principal findings of Section 3.

Let C be a closed convex subset of a real Hilbert space H, and let $P_C : H \overset{onto}{\rightarrow} C$ be the metric projection. That is, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

The following lemmas explore the properties of P_C and the set of inequality problems.

Lemma 2: (See [10]) Let C be a nonempty closed convex subset of a Hilbert space H, and L be a mapping of C into H. Let w be an element of C, then for $\sigma > 0$,

$$w \in VI(C, L) \text{ if and only if } w = P_C(I - \sigma L)w$$

.

Lemma 3: (See [10]) Let C be a closed convex subset of a Hilbert space H, and let S and T be operators from C to H that are δ and ε-inverse strongly monotone, respectively. If $VI(C, S) \cap VI(C, T) \neq \emptyset$, then

$$VI(C, dS + (1 - d)T) = VI(C, S) \cap VI(C, T),$$

for all $d \in (0, 1)$. Moreover, if $0 < \Omega < \min\{2\delta, 2\varepsilon\}$, we have $I - \Omega(dS + (1 - d)T)$ is a nonexpansive operator.

Lemma 4: (See [11]) Let $\{\iota_n\}_{n=0}^{\infty}$ and $\{\vartheta_n\}_{n=0}^{\infty}$ be sequences of non-negative numbers satisfying

$$\iota_{n+1} \leq \iota_n + \vartheta_n,$$

for all $n \geq 0$.

(i) If $\sum_{n=0}^{\infty} \vartheta_n < \infty$, then $\lim_{n\to\infty} \iota_n$ exists.

(ii) If $\sum_{n=0}^{\infty} \vartheta_n < \infty$ and $\{\iota_n\}_{n=0}^{\infty}$ has a subsequence that converges to zero, then $\lim_{n\to\infty} \iota_n = 0$.

Lemma 5: (See [12]) Every Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{\mu_n\} \subset H$ with $\mu_n \rightharpoonup \mu$, the inequality

$$\liminf_{n\to\infty} \|\mu_n - \mu\| < \liminf_{n\to\infty} \|\mu_n - \nu\|$$

holds for every $\nu \in H$ with $\nu \neq \mu$.

III. Main result

Theorem 1: Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\Gamma : C \to C$ be a nonexpansive mapping. Let $Q : C \to H$ be ε-strongly monotone and Υ-Lipschitzian. Let $\{q_n\}$ be a sequence generated by $q_0 \in C$ and

$$q_{n+1} = aq_n + b\Gamma q_n + cP_C(I - \lambda Q)q_n,$$

for all $n \geq 1$ with $0 < \Upsilon \leq \varepsilon < 1$, $\lambda \in (0, 1)$, and $a+b+c = 1$. Then, the following are equivalent:

(i) The sequence $\{q_n\}$ converges strongly to $q^\star \in F(\Gamma) \cap VI(C, Q)$.

(ii) $\lim_{n\to\infty} \|q_n - \Gamma q_n\| = 0$.

Proof: Assuming that condition (i) holds, we show that (i) $\to$ (ii). Let q_n be the sequence generated as defined in Theorem 1. Then we have

$$\begin{aligned}
\|q_n - \Gamma q_n\| &= \|q_n - q^\star + q^\star - \Gamma q_n\| \\
&\leq \|q_n - q^\star\| + \|q^\star - \Gamma q_n\| \\
&= \|q_n - q^\star\| + \|\Gamma q^\star - \Gamma q_n\| \\
&\leq 2\|q_n - q^\star\|.
\end{aligned}$$

Since $\{q_n\}$ converges strongly to $q^\star$, then

$$\lim_{n\to\infty} \|q_n - \Gamma q_n\| = 0.$$

Conversely, suppose condition (ii) holds and employ the nonexpansiveness of P_C. We get

$$\begin{aligned}
&\|P_C(I - \lambda Q)q - P_C(I - \lambda Q)t\|^2 \\
&\leq \|(I - \lambda Q)q - (I - \lambda Q)t\|^2 \\
&= \|(q - t) - (\lambda Qq - \lambda Qt)\|^2 \\
&= \|q - t\|^2 - 2\langle q - t, \lambda Qq - \lambda Qt\rangle \\
&\quad + \|\lambda Qq - \lambda Qt\|^2 \\
&\leq \|q - t\|^2 - 2\lambda\langle q - t, Qq - Qt\rangle \\
&\quad + \lambda^2\varepsilon^2\|q - t\|^2 \\
&\leq (1 - 2\lambda\varepsilon + \lambda^2\varepsilon^2)\|q - t\|^2 \\
&= (1 - \lambda\varepsilon)^2\|q - t\|^2 \\
&= \omega^2\|q - t\|^2.
\end{aligned}$$

with $\omega = 1 - \lambda\varepsilon \in (0, 1)$.

Consequently, $P_C(I - \lambda Q)$ is an ω-contractive mapping. From Γ is a nonexpansive mapping, getting

$$\begin{aligned}
\|q_{n+1} - q_n\| &= \|(aq_n + b\Gamma q_n + cP_C(I - \lambda Q)q_n) \\
&\quad - (aq_{n-1} + b\Gamma q_{n-1} + cP_C(I - \lambda Q)q_{n-1})\| \\
&\leq b\|\Gamma q_n - \Gamma q_{n-1}\| + a\|q_n - q_{n-1}\| \\
&\quad + c\omega\|q_n - q_{n-1}\| \\
&\leq (1 - (1 - \omega)c)\|q_n - q_{n-1}\| \\
&\leq (1 - (1 - \omega)c)^2\|q_{n-1} - q_{n-2}\| \\
&\leq (1 - (1 - \omega)c)^3\|q_{n-2} - q_{n-3}\| \\
&\vdots \\
&\leq (1 - (1 - \omega)c)^n\|q_1 - q_0\|. \qquad (1)
\end{aligned}$$

For any natural numbers n and κ with using equation (1), we have

$$\begin{aligned}\|q_{n+\kappa}-q_n\| &= \|q_{n+\kappa}-q_{n+\kappa-1}+q_{n+\kappa-1}-q_{n+\kappa-2}\\ &\quad +q_{n+\kappa-2}-...-q_n\|\\ &\le \sum_{j=n}^{n+\kappa-1}\|q_{j+1}-q_j\|\\ &\le \sum_{j=n}^{n+\kappa-1}(1-(1-\omega)c)^j\|q_1-q_0\|\\ &\le \frac{(1-(1-\omega)c)^n}{(1-\omega)c}\|q_1-q_0\|. \end{aligned} \tag{2}$$

From (2) and $\lim_{n\to\infty}(1-(1-\omega)c)^n=0$, we have $\{q_n\}$ is a Cauchy sequence. Employing the completeness of H, there exists $q^\star\in H$ such that

$$\lim_{q\to\infty} q_n = q^\star. \tag{3}$$

Since C is closed, we obtain $q^\star\in C$. Assume that $q^\star\neq\Gamma q^\star$, using the fact that $\lim_{x\to\infty}\|q_n-\Gamma q_n\|=0$, Lemma 5, and nonexpansiveness of Γ. Getting,

$$\begin{aligned}\liminf_{n\to\infty}\|q_n-q^\star\| &< \liminf_{n\to\infty}\|q_n-\Gamma q^\star\|\\ &\le \liminf_{n\to\infty}(\|q_n-\Gamma q_n\|+\|\Gamma q_n-\Gamma q^\star\|)\\ &\le \liminf_{n\to\infty}\|q_n-q^\star\|.\end{aligned}$$

This is a contradiction, we obtain that $q^\star=\Gamma q^\star$. It means that $q^\star\in F(\Gamma)$. From definition of q_{n+1}, then

$$c\|P_C(I-\lambda Q)q_n-q_n\|\le\|q_{n+1}-q_n\|+b\|\Gamma q_n-q_n\|.$$

From $\lim_{n\to\infty}q_n=q^\star$ and condition (ii), we have

$$\lim_{n\to\infty}\|P_C(I-\lambda Q)q_n-q_n\|=0. \tag{4}$$

Assume that $q^\star\neq P(I-\lambda Q)q^\star$. From (4), Lemma 5, and the nonexpansiveness of $P_C(I-\lambda Q)$. Therefore, we have shown that

$$\begin{aligned}\liminf_{n\to\infty}\|q_n-q^\star\| &< \liminf_{n\to\infty}\|q_n-P_C(I-\lambda Q)q^\star\|\\ &\le \liminf_{n\to\infty}(\|q_n-P_C(I-\lambda Q)q_n\|\\ &\quad +\|P_C(I-\lambda Q)q_n-P_C(I-\lambda Q)q^\star\|)\\ &\le \liminf_{n\to\infty}\|q_n-q^\star\|.\end{aligned}$$

This is a contradiction, we obtain $q^\star=P_C(I-\lambda Q)q^\star$. From Lemma 2, then $q^\star\in VI(C,Q)$.

Hence, $\{q_n\}$ converges strongly to $q^\star\in F(\Gamma)\cap VI(C,Q)$. ■

From Theorem 1, we prove strong convergence theorem by using condition $F(\Gamma)\cap VI(C,Q)\neq\emptyset$ as follows.

Theorem 2: Let C be nonempty closed convex subset of a Hilbert space H and let $\Gamma:C\to C$ be a nonexpansive mapping. Let $Q:C\to H$ be ε-strongly monotone and Υ-Lipschitzian with $F(\Gamma)\cap VI(C,Q)\neq\emptyset$. Let $\{q_n\}$ be a sequence generated by $q_0\in C$ and

$$q_{n+1}=aq_n+b\Gamma q_n+cP_C(I-\lambda Q)q_n,$$

for all $n\ge 1$ with $0<\Upsilon\le\varepsilon<1$, $\lambda\in(0,1)$, and $a+b+c=1$. Then the sequence $\{q_n\}$ converges strongly to $q^\star\in F(\Gamma)\cap VI(C,Q)$.

Proof: Let conditions $(i)-(ii)$ hold and let $\psi\in F(\Gamma)\cap VI(C,Q)$, then

$$\begin{aligned}\|q_{n+1}-\psi\|^2 &= \|aq_n+b\Gamma q_n+cP_C(I-\lambda Q)q_n-\psi\|^2\\ &= \|a(q_n-\psi)+b(\Gamma q_n-\psi)\\ &\quad +c(P_C(I-\lambda Q)q_n-\psi)\|^2\\ &= a\|q_n-\psi\|^2+b\|\Gamma q_n-\psi\|^2\\ &\quad +c\|P_C(I-\lambda Q)q_n-\psi\|^2\\ &\quad -ab\|q_n-\Gamma q_n\|^2\\ &\quad -bc\|\Gamma q_n-P_C(I-\lambda Q)q_n\|^2\\ &\le a\|q_n-\psi\|^2+b\|\Gamma q_n-\Gamma\psi\|^2\\ &\quad +c\|P_C(I-\lambda Q)q_n-P_C(I-\lambda Q)\psi\|^2\\ &\quad -ab\|q_n-\Gamma q_n\|^2\\ &\le a\|q_n-\psi\|^2+b\|q_n-\psi\|^2+c\|q_n-\psi\|^2\\ &\quad -ab\|q_n-\Gamma q_n\|^2\\ &\le \|q_n-\psi\|^2-ab\|q_n-\Gamma q_n\|^2.\end{aligned} \tag{5}$$

From (5), it implies that

$$\|q_{n+1}-\psi\|\le\|q_n-\psi\|. \tag{6}$$

Employing Lemma 4, we have $\lim_{n\to\infty}\|q_n-\psi\|$ exists for all $\psi\in F(\Gamma)\cap VI(C,Q)$.

From (5), we get

$$ab\|q_n-\Gamma q_n\|^2\le\|q_n-\psi\|^2-\|q_{n+1}-\psi\|^2.$$

Accordingly,

$$\lim_{n\to\infty}\|q_n-\Gamma q_n\|=0. \tag{7}$$

Using (7) and Theorem 1, it follows that the sequence $\{q_n\}$ converges strongly to $q^\star\in F(\Gamma)\cap VI(C,Q)$. ■

IV. Applications

The combination of the variational inequality problem [22] is to find $z\in C$ such that

$$\left\langle w-z,\sum_{i=1}^{M}b_iB_iz\right\rangle\ge 0,$$

where $B_i:C\to K$ is a nonlinear mapping, and $b_i\in(0,1)$ with $\sum_{i=1}^{M}b_i=1$, for all $i=1,2,\ldots,M$.

The set of solutions of combination of this variational inequality problem is given by $VI(C,\sum_{i=1}^{M}b_iB_i)=\left\{v\in C:\left\langle w-z,\sum_{i=1}^{M}b_iB_iz\right\rangle\ge 0,\forall w\in C\right\}$.

This problem is called the variational inequality problem if $B_i=B$ for all $i=1,2,\ldots,M$.

In the research conducted by Kangtunyakarn [22], a robust convergence theorem is established to identify the solution sets pertaining to the common element within the fixed-point sets of a finite family of nonspreading mappings. The theorem, demonstrating convergence, is intricately connected with two split variational inequality problems and is substantiated through the application of outcomes derived from Lemma 3. This foundational theorem is articulated as follows:

Theorem 3: Let C be nonempty closed convex subset of a real Hilbert space H, and let $Q^{\heartsuit},G^{\heartsuit}:C\to H$ be ε,δ-inverse strongly monotone, respectively. Let $Q:C\to H$ be ε-strongly monotone and Υ-Lipschitz operator, with

$VI(C,Q) \cap VI(C,Q^{\heartsuit}) \cap VI(C,G^{\heartsuit}) \neq \emptyset$. Let $\{q_n\}$ be a sequence generated by $q_0 \in C$ and

$$q_{n+1} = aq_n + bP_C(I - \gamma(aQ^{\heartsuit} + (1-a)G^{\heartsuit}))q_n + cP_C(I - \lambda A)q_n,$$

for all $n \geq 1$ with $0 < \Upsilon \leq \varepsilon < 1$, $\lambda \in (0,1)$, $\gamma \in (0, \min\{2\varepsilon, 2\delta\})$, and $a + b + c = 1$. Then, the sequence $\{q_n\}$ converges strongly to $q^\star \in VI(C,Q) \cap VI(C,Q^{\heartsuit}) \cap VI(C,G^{\heartsuit})$.

Proof: From Lemma 4, we have $P_C(I - \gamma(aQ^{\heartsuit} + (1-a)G^{\heartsuit})$ is a nonexpansive mapping. Using Theorem 2 and Lemma 3, we obtain that $\{q_n\}$ converges strongly to $q^\star \in VI(C,Q) \cap VI(C,Q^{\heartsuit}) \cap VI(C,G^{\heartsuit})$. ■

From Theorem 3, the following are the direct results.

Corollary 1: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $Q^{\heartsuit} : C \to H$ be ε-inverse strongly monotone, and $Q : C \to H$ be ε-strongly monotone, and Υ-Lipschitz operator with $VI(C,Q) \cap VI(C,Q^{\heartsuit}) \neq \emptyset$. Let $\{q_n\}$ be a sequence generated by $q_0 \in C$ and

$$q_{n+1} = aq_n + bP_C(I - \gamma Q^{\heartsuit})q_n + cP_C(I - \lambda Q)q_n,$$

for all $n \geq 1$ with $0 < \Upsilon \leq \varepsilon < 1$, $\lambda \in (0,1)$, $0 < \gamma < 1$, and $a+b+c = 1$. Then, the sequence $\{q_n\}$ converges strongly to $q^\star \in VI(C,Q) \cap VI(C,Q^{\heartsuit})$.

Remark:

(1) If $G : C \to H$ is a nonexpansive mapping with $F(G) \neq \emptyset$, then $VI(C, I - G) = F(G)$.

(2) If G is a nonexpansive mapping, then $I - G$ is $\frac{1}{2}$-inverse strongly monotone.

We use the above two remarks and Theorem 2 to prove the following results.

Corollary 2: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $Q : C \to H$ be ε-strongly monotone, and Υ-Lipschitz. Let $\bar{S}, \hat{S} : C \to C$ be nonexpansive mappings with $VI(C,Q) \cap F(\bar{S}) \cap F(\hat{S}) \neq \emptyset$. Let $\{q_n\}$ be a sequence generated by $q_0 \in C$ and

$$q_{n+1} = aq_n + bP_C(I - \gamma((I - \bar{S})a + (1-a)(I - \hat{S}))q_n + cP_C(I - \lambda Q)q_n,$$

for all $n \geq 1$ with $0 < \Upsilon \leq \varepsilon < 1$, $\lambda \in (0,1)$, $0 < \gamma < 1$, and $a+b+c = 1$. Then the sequence $\{q_n\}$ converges strongly to $q^\star \in F(\bar{S}) \cap F(\hat{S}) \cap VI(C,Q)$.

V. Numerical method

In the field of physics, Apéry's constant is particularly intriguing and essential to the computation of the electron's gyromagnetic ratio using quantum electrodynamics. It is defined as the sum of the reciprocals of the positive cubes and represented by the Euler-Riemann zeta function:

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

This constant holds an irrational value, approximately equal to

$$1.2020569031595942853997381615114\ldots,$$

making it all the more fascinating. Furthermore, we have presented evidence of the newly proven theorem's application to support our findings, that is, the convergence trend of sample sequences.

Example: Let $C = [-100, 100]$, $Tq = \frac{1}{2}q$, and $Qq = \frac{\zeta(3)}{4}q$. Let $\{q_n\}$ be a sequence generated by $q_0 \in C$, and

$$q_{n+1} = 0.2q_n + 0.3Tq_n + 0.5P_C\left(I - \frac{\zeta(3)\pi}{20}\right)q_n,$$

for all $n \geq 1$. We know that Q is both $\frac{\zeta(3)}{4}$-strongly monotone and $\frac{\zeta(3)}{4}$-Lipschitz. Moreover, T is nonexpansive with $0 \in VI(C,Q) \cap F(T)$. Applying Theorem 2, we can conclude that the sequence $\{q_n\}$ converges strongly to 0.

The Table I and Fig 1 show the value of $\{q_n\}$ with $q_0 = -10$, 10 and n = 500.

Table I
Convergence of the iterative method from the numerical method example

n	q_n with $q_0 = -10$	q_n with $q_0 = 10$
1	-7.5559067159592303	7.5559067159592303
2	-5.709172630027780	5.709172630027780
3	-4.3137975817797525	4.3137975817797525
4	-3.2594652119458319	3.2594652119458319
5	-2.4628215085376987	2.4628215085376987
⋮	⋮	⋮
15	-0.1493821869718095	0.1493821869718095
16	-0.1128717869784972	0.1128717869784972
17	-0.0852848693273147	0.0852848693273147
18	-0.0644404516919962	0.0644404516919962
19	-0.0486906041719000	0.0486906041719000
20	-0.0367901663066572	0.0367901663066572
⋮	⋮	⋮
50	-0.0000082097395123	0.0000082097395123

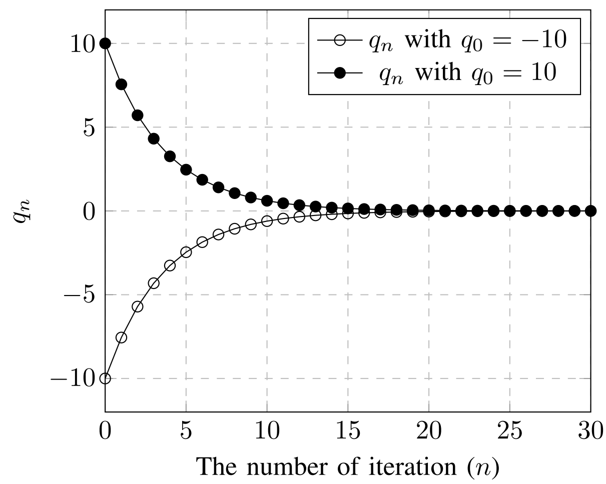

Fig 1. Graph of Iteration Convergence

VI. Conclusion

In conclusion, our research successfully implemented a novel modification of the Mann iteration to achieve strong convergence. Importantly, our approach accomplishes this without the need to depend on Lemma 1 and without assuming $F(\Gamma) \cap VI(C,Q) \neq \emptyset$. The main results of our study were subsequently employed to prove Corollary 2 and estimate the solution of the combined variational inequality problem. To substantiate our theoretical findings, we provided extensive numerical evidence. Our research underscores the potential of this modified Mann iteration technique, opening new avenues for future explorations in this field.

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