Permanence and Almost Periodic Solution for Impulsive Hematopoiesis Model with Harvesting Terms on Time Scales

Pan Wang*, Xuechen Li and Qianqian Zheng

Abstract—**This paper discusses a model of delayed hematopoiesis with impulsive effects and harvesting terms on time scales. The author derives criteria that ensure the permanence of system by using Δ-differential inequalities on time scales. Based on the permanence result, fixed point theory and Gronwall-Bellman's inequality, The author develops criteria for the existence and exponential stability of the almost periodic solutions for the studied model. The study further provides an illustrative scenario to showcase the theoretical findings that have been obtained.**

Index Terms—**Impulsive hematopoiesis model, Almost periodic solution, Exponential stability, Permanence, Time scales.**

I. Introduction

IN [1], the authors primordially put forward the following hematopoiesis model:

$$Œ'(t) = -\pi Œ(t) + \frac{\varpi}{1 + Œ^n(t-\upsilon)}, \tag{1}$$

where $Œ(t)$ indicates the density of mature cells in blood circulation at time t; π represents the rate of cell loss in circulation and υ is the time delay. Once the model (2) was proposed, it attracted the interest of many scholars and the dynamic behavior of the model has been studied in depth (see [2]–[9]). Especially, the author in [9] proposed the following model:

$$Œ'(t) = -\pi(t)Œ(t) + \sum_{R=1}^{m} \frac{\varpi_q(t)}{1 + Œ^n(t-\upsilon_R(t))}.$$

By means of the fixed-point theorem, some criteria for the existence of a unique globally attractive positive ω-periodic solution are obtained.

In the real world, a variety of natural and human forces will invariably induce either a fast decline in population or a rapid growth in it over time. Typically, an impulse can be used to mathematically represent this abrupt change (see [10], [11]). As a result, many population dynamics models for impulsive differential equations have been developed and extensively researched (see [12]–[18]).

At the same time, it is crucial to examine the discrete-time model because, in practical applications, the discrete time model defined by the difference equation is just as significant as the continuous model. The idea of time scale has been suggested by academics to integrate the study of discrete and continuous systems ([19]). Studying a system's dynamical characteristics on the time scale can help to avoid having to repeat the study of discrete and continuous systems, and as a result, this field is currently a focus of active research. Excellent results have been attained in recent years from the study of dynamical equations on diverse time scales (see [20]–[26]). Therefore, it is merit in further investigating the hematopoiesis model with impulsive effects and harvesting terms on the time scale.

In light of the aforementioned, this work examines the existence and exponential stability of almost periodic solutions for the following hematopoiesis model with impulses and harvesting terms on time scales by the following:

$$\begin{cases} Œ^{\Delta}(t) = -\pi(t)Œ(t) + \sum_{R=1}^{m} \frac{\varpi_R(t)}{1+Œ^n(t-\upsilon_R(t))} \\ \qquad\qquad -E(t, Œ(t-\varsigma(t))),\ t \neq a_k,\ t \in [a_0, +\infty)_{\mathbb{T}}, \\ \tilde{\Delta}Œ(a_k) = \gamma_k Œ(a_k) + \delta_k,\ t = a_k,\ k \in \mathbb{N}. \end{cases} \tag{2}$$

where $n > 0$, $a_0 \in \mathbb{T}$ and $\mathbb{T}$ is an almost periodic time scale. $\tilde{\Delta}Œ(a_k) = Œ(a_k^+) - Œ(a_k^-)$, $Œ(a_k^-) = Œ(a_k)$ are impulses at moments a_k and $0 \leq a_0 < a_1 < a_2 < \ldots < a_k < \ldots$ is a strictly increasing sequence such that $\lim_{t\to\infty} a_k = +\infty$, γ_k, $\delta_k \in \mathbb{R}$.

Remark 1.1: If $\mathbb{T} = \mathbb{R}$, then system (2) is reduced to the following system:

$$\begin{cases} Œ'(t) = -\pi(t)Œ(t) + \sum_{R=1}^{m} \frac{\varpi_R(t)}{1+Œ^n(t-\upsilon_R(t))} \\ \qquad\qquad -E(t, Œ(t-\varsigma(t))),\ t \neq a_k,\ t \in [a_0, +\infty)_{\mathbb{T}}, \\ \tilde{\Delta}Œ(a_k) = \gamma_k Œ(a_k) + \delta_k,\ t = a_k,\ k \in \mathbb{N}. \end{cases}$$

If $\mathbb{T} = \mathbb{Z}$, then system (2) is reduced to the following system:

$$\begin{cases} \Delta Œ(\Lambda) = -\pi(\Lambda)Œ(\Lambda) + \sum_{R=1}^{m} \frac{\varpi_R(\Lambda)}{1+Œ^n(\Lambda-\upsilon_R(\Lambda))} \\ \qquad\qquad -E(\Lambda, Œ(\Lambda-\varsigma(\Lambda))),\ \Lambda \neq \Lambda_k,\ \Lambda \in [a_0, +\infty)_{\mathbb{Z}}, \\ \tilde{\Delta}Œ(\Lambda_k) = \gamma_k Œ(\Lambda_k) + \delta_k,\ \Lambda = \Lambda_k,\ k \in \mathbb{N}. \end{cases}$$

For convenience, we denote

$$f^l = \inf_{t\in\mathbb{T}} |f(t)|, \quad f^u = \sup_{t\in\mathbb{T}} |f(t)|.$$

In the context of this study, it is assumed that

(H_1) The functions $\pi, \varpi_R, \upsilon_R, \varsigma \in PC_{rd}(\mathbb{T}, \mathbb{R}^+)$ are positive bounded almost periodic functions on $\mathbb{T}$ such that $\pi^l > 0$, $\varpi_R^l > 0$, $\upsilon_R^l > 0$, $\varsigma^l > 0$, $-\pi^u \in \mathcal{R}^+$ and there exists a constant $\lambda > 0$ such that $\pi(t) \geq \lambda$, for $t \in \mathbb{T}$, $t - \upsilon_R(t) \in \mathbb{T}$, $t - \varsigma(t) \in \mathbb{T}$, $R = 1, 2, \ldots, m$.

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(H_2) The function $E \in PC_{rd}(\mathbb{T}\times\mathbb{R},\mathbb{R}^+)$ is bounded and satisfies the Lipschitz condition: $\exists\ \alpha,\beta,\aleph>0$ such that $\alpha \le E(t,Œ) \le \beta$ and

$$|E(t,Œ)-E(t,\tilde{Œ})| \le \aleph|Œ-\tilde{Œ}|,\ t\in\mathbb{T},\ Œ,\tilde{Œ}\in\mathbb{R}.$$

(H_3) $\{\gamma_k\}$, $\{\delta_k\}$ are almost periodic sequences with $-1<\gamma_k\le 0$, where $b <\le \prod_{a_0<a_k<t}(1+\gamma_k)\le B$ and $\sup_{k\in\mathbb{N}}|\delta_k|\le L$ for $k\in\mathbb{N}$;

(H_4) The set of sequences $\{a_k^j\}$, $a_k^j=a_{k+j}-a_k$, $k,j\in\mathbb{N}$ are equipotentially almost periodic and $\inf_k a_k^1 = Œ > 0$.

where $PC_{rd}(\mathbb{T},\mathbb{X}) = \{Œ(t)|Œ(t)$ is a rd-piecewise continuous functions from the time scale $\mathbb{T}$ to a Banach space $\mathbb{X}\}$.

For $a_0\in\mathbb{T}$, and denote

$$Œ(t)=E(t;a_0,\phi_0)$$

is the solution of system (2), satisfying

$$\begin{cases} E(t;a_0,\phi_0)=\phi_0(t),\ t\in(a_0-\bar{\upsilon},a_0)_{\mathbb{T}},\\ E(a_0^+;a_0,\phi_0)=\phi_0(a_0),\end{cases} \tag{3}$$

where $\phi_0\in PC_{rd}(a_0)$ is a rd-piecewise continuous function with respects to the sequence $\{a_k\}$, $k\in\mathbb{N}$, $\bar{\upsilon}=\max\left\{\max_{1\le R\le m}\sup_{t\in\mathbb{T}}\{\upsilon_R(t)\},\sup_{t\in\mathbb{T}}\{\varsigma(t)\}\right\}$.

II. Permanence

Permanence is one of the important indicators of an ecosystem, which refers to the balance between the amount of resource use and the carrying capacity of the natural environment within a certain period of time, thus maintaining the dynamic stability of the ecosystem. In order to obtain the permanence of system (2), it is necessary to introduce the following lemma:

Lemma 2.1: Supposed that $(H_1)-(H_4)$ hold and let $Œ(t)$ be any solution of system (2), then

$$\bar{M}\le\liminf_{t\to+\infty}Œ(t)\le\limsup_{t\to+\infty}Œ(t)\le M.$$

Proof: Let $Œ(t)$ denote an arbitrary solution of system (2), then

$$\begin{cases} Œ^{\Delta}(t)\le\sum_{R=1}^m\varpi_R^u-\pi^lŒ(t),\ t\ne a_k,\ t\in[a_0,+\infty)_{\mathbb{T}},\\ Œ(a_k^+)\le(1+\gamma_k)Œ(a_k)+\delta_k,\ t=a_k,\ k\in\mathbb{N}.\end{cases}$$

In view of the Lemma 2.11 in [27], we have

$$\limsup_{t\to+\infty}Œ(t)\le\frac{\sum_{R=1}^m\varpi_R^uB}{\pi^l}=M. \tag{4}$$

For any positive constant ϵ small enough, it follows from (4) that there exists enough large $T_1(\ge a_0)$ such that

$$Œ(t)\le M+\epsilon,\ \text{for } t\ge T_1. \tag{5}$$

By system (2) and (5), we arrive at

$$\begin{cases} Œ^{\Delta}(t)\ge\frac{\sum_{R=1}^m\varpi_R^l}{1+(M+\epsilon)^n}-\beta-\pi^uŒ(t),\ t\ne a_k,\\ Œ(a_k^+)\ge(1+\gamma_k)Œ(a_k)+\delta_k,\ t=a_k,\ k\in\mathbb{N}.\end{cases}$$

By applying the Lemma 2.11 in [27], we have

$$\liminf_{t\to+\infty}Œ(t)\ge\frac{b}{\pi^u}\left(\frac{\sum_{R=1}^m\varpi_R^l}{(1+(M+\epsilon)^n)}-\beta\right).$$

letting $\epsilon\to0$ in the above inequality, we have

$$\liminf_{t\to+\infty}Œ(t)\ge\frac{b}{\pi^u}\left(\frac{\sum_{R=1}^m\varpi_R^l}{(1+M^n)}-\beta\right)=\bar{M}.$$

The proof is completed. ■

Theorem 2.1: Assume that (H_1)-(H_4) hold, then system (2) is permanence.

III. Existence and exponential stability

By use of the system (2), we consider the linear system

$$\begin{cases} Œ^{\Delta}(t)=-\pi(t)Œ(t),\ t\ne a_k,\ t\in[a_0,+\infty)_{\mathbb{T}},\\ \tilde{\Delta}Œ(a_k)=\gamma_kŒ(a_k),\ t=a_k,\ k\in\mathbb{N}.\end{cases} \tag{6}$$

Let the equation

$$Œ^{\Delta}(t)=-\pi(t)Œ(t),\ a_{k-1}<t\le a_k,$$

and the solution

$$Œ(t)=Œ(s)e_{(-\pi)}(t,s)$$

for $a_{k-1}<s\le t\le a_k$. Then the Cauchy matrix of the linear system (6) is

$$¥[t,s]=\begin{cases} e_{(-\pi)}(t,s),\ a_{k-1}<s\le t\le a_k,\\ \prod_{i=m}^{k+1}(1+\gamma_i)e_{(-\pi)}(t,s),\\ a_{m-1}<s\le a_m<a_k<t\le a_{k+1},\end{cases} \tag{7}$$

and the solutions of (6) are in the form

$$Œ(t;a_0,Œ_0)=¥[t,Œ_0]Œ_0,\ a_0\in\mathbb{T},\ Œ_0\in\mathbb{R}.$$

Similar to the proof of Lemma 3.1 and Lemma 3.2 in [24], one can easily show the following:

Lemma 3.1: For system (2), let (H_1)-(H_4) hold. Then for each $\varepsilon>0$ there exist $\varepsilon_1>0$, $0<\varepsilon_1<\varepsilon$, $\Omega\subset\Xi=\{r\in\mathbb{R}: t\pm r\in\mathbb{T},\forall t\in\mathbb{T}\}\ne\{0\}$ is a relative dense set of real numbers, and $\tilde{P}$ is a integer numbers set, such that the following relations are fulfilled:

(I) the following hold:

$$|\pi(t+r)-\pi(t)|<\varepsilon,\ |\varpi_R(t+r)-\varpi_R(t)|<\varepsilon$$

$$|\upsilon_R(t+r)-\upsilon_R(t)|<\varepsilon,\ |E(t+r,Œ(\cdot))-E(t,Œ(\cdot))|<\varepsilon,$$

$$|\varsigma(t+r)-\varsigma(t)|<\varepsilon,\ |t-a_k|>\varepsilon,\ t\in\mathbb{T},\ r\in\Omega;$$

(II) the following hold:

$$|\gamma_{k+p}-\gamma_k|<\varepsilon,\ |\delta_{k+p}-\delta_k|<\varepsilon,\ p\in\tilde{P},\ k\in\mathbb{Z};$$

(III) $|a_k^p-r|<\varepsilon_1,\ p\in\tilde{P},\ r\in\Omega,\ k\in\mathbb{Z}$.

Lemma 3.2: For system (2), let $(H_1)-(H_4)$ hold. Then the Cauchy matrix $¥[t,s]$ of (7) satisfies the inequality

$$\|¥[t,s]\|\le e^{-\lambda(t-s)},\ t\ge s,\ t,s\in\mathbb{T},$$

and the matrix $¥[t,s]$ is almost periodic, i.e. for any $\varepsilon>0$, $t,s\in\mathbb{T}$, $|t-a_k|>\varepsilon$, $|s-a_k|>\varepsilon$, $k\in\mathbb{Z}$, there exists $\Omega\subset\Xi$ such that for $r\in\Omega$, we have

$$\|¥[t+r,s+r]-¥[t,s]\|\le\varepsilon\Gamma e^{-\frac{\lambda}{2}(t-s)},\ t\ge s,$$

where $\Gamma>0$ is a constant.

Theorem 3.1: Let conditions (H_1)-(H_4) hold and the following condition is satisfied:

(H_5) $\gamma < 1$, where

$$\gamma = \frac{(1+\lambda\bar{\mu})}{\lambda}\bigg(\sum_{R=1}^{m}\varpi_R^u nM^{n-1} + \aleph\bigg),\ \bar{\mu} = \sup_{t\in\mathbb{T}}\mu(t)$$

and M is defined in Lemma 2.1. Then (2) has a unique almost periodic solution.

Proof: Denote $A \subset PC_{rd}(\mathbb{T}, \mathbb{R}^n)$ the set of all almost periodic function $\hbar(t)$ with $\|\hbar\| < \bar{K}$, where

$$\|\hbar\| = \sup_{t\in\mathbb{T}}|\hbar(t)|,\ \bar{K} = \frac{1+\lambda\bar{\mu}}{\lambda}\sum_{R=1}^{m}\varpi_R^u + \frac{2NL}{1-e^{-\lambda}},$$

where N which denotes that the interval $(l, l+1)_{\mathbb{T}}$ contains no more than N terms of the sequences $\{a_k\}$, $0 < l \in \mathbb{R}$, $k \in \mathbb{Z}$.

Define an operator Υ in A such that if $\hbar \in A$,

$$\begin{aligned}\Upsilon_\hbar \ &= \ \int_{-\infty}^{t} ¥[t,\sigma(s)]\bigg(\sum_{R=1}^{m}\frac{\varpi_R(s)}{1+\hbar^n(t-\upsilon_R(s))} \\ &\quad -E(t,\hbar(t-\varsigma(t)))\bigg)\Delta s + \sum_{a_k<t}¥[t,a_k]\delta_k.\end{aligned}$$

In view of the inequality

$$\begin{aligned}\sum_{a_k<t}e^{-\lambda(t-a_k)} \ &= \ \sum_{m=0}^{\infty}\sum_{t-m-1<a_k<t-m}e^{-\lambda(t-a_k)} \\ &\le \ 2N\sum_{m=0}^{\infty}e^{-\lambda m} = \frac{2N}{1-e^{-\lambda}}\end{aligned}$$

and for an arbitrary $\hbar \in A$ it follow

$$\begin{aligned}\|\Upsilon_\hbar\| \ &\le \ \sup_{t\in\mathbb{T}}\bigg\{\bigg|\int_{-\infty}^{t}¥[t,\sigma(s)]\sum_{R=1}^{m}\frac{\varpi_R(s)}{1+\phi^n(t-\upsilon_R(s))}\Delta s \\ &\quad + \sum_{a_k<t}¥[t,a_k]\delta_k\bigg|\bigg\} \\ &\le \ \sup_{t\in\mathbb{T}}\bigg\{\int_{-\infty}^{t}e_{\ominus\lambda}(t,\sigma(s))\sum_{R=1}^{m}\varpi_R^u\Delta s \\ &\quad + \sum_{a_k<t}e^{-\lambda(t-a_k)}L\bigg\} \\ &\le \ \frac{1+\lambda\bar{\mu}}{\lambda}\sum_{R=1}^{m}\varpi_R^u + \frac{2NL}{1-e^{-\lambda}} = \bar{K}. \qquad (8)\end{aligned}$$

Let $r \in \Omega$, $p \in \tilde{P}$, where the sets Ω and $\tilde{P}$ are introduced in Lemma 3.1. Then

$$\begin{aligned}&\|\Upsilon_\hbar(t+r) - \Upsilon_\hbar(t)\| \\ \le \ &\sup_{t\in\mathbb{T}}\bigg\{\int_{-\infty}^{t}\|¥[t+r,\sigma(s+r)] - ¥[t,\sigma(s)]\| \\ &\times\sum_{R=1}^{m}\frac{\varpi_R^u}{1+\hbar^n(s+\tau-\upsilon_R(s+r))}\Delta s \\ &+\int_{-\infty}^{t}\|¥[t,\sigma(s)]\|\sum_{R=1}^{m}\bigg|\frac{\varpi_R(s+r)}{1+\hbar^n(s+r-\upsilon_R(s+r))} \\ &-\frac{\varpi_R(s)}{1+\hbar^n(s-\upsilon_R(s))}\bigg|\Delta s + \int_{-\infty}^{t}\|¥[t+r,\sigma(s+r)] \\ &-¥[t,\sigma(s)]\|\bigg|E(s+r,\hbar(s+r-\varsigma(s+r)))\bigg|\Delta s \\ &+\int_{-\infty}^{t}\|¥[t,\sigma(s)]\|\bigg|E(s+r,\hbar(s+r-\varsigma(s+r))) \\ &-E(s,\hbar(s-\varsigma(s)))\bigg|\Delta s + \sum_{a_k<t}\|¥[t+r,a_{k+p}] \\ &-¥[t,a_k]\|\|\delta_{k+p}| + \sum_{a_k<t}\|¥[t,a_k]\|\|\delta_{k+p}-\delta_k|\bigg\} \\ \le\ &\sup_{t\in\mathbb{T}}\bigg\{\int_{-\infty}^{t}\varepsilon\Gamma e^{-\frac{\lambda}{2}(t-\sigma(s))}\sum_{R=1}^{m}\varpi_R^u\Delta s \\ &+\int_{-\infty}^{t}e^{-\lambda(t-\sigma(s))}\sum_{R=1}^{m}\bigg[\varpi_R^u\bigg|\frac{1}{1+\hbar^n(s+r-\upsilon_R(s+r))} \\ &-\frac{1}{1+\hbar^n(s-\upsilon_R(s))}\bigg| + |\varpi_R(s+r)-\varpi_R(s)| \\ &\times\frac{1}{1+\hbar^n(s-\upsilon_R(s))}\bigg]\Delta s + \int_{-\infty}^{t}\varepsilon\Gamma e^{-\frac{\lambda}{2}(t-\sigma(s))}\beta\Delta s \\ &+\int_{-\infty}^{t}e^{-\lambda(t-\sigma(s))}\bigg[|E(s+r,\hbar(s+r-\varsigma(s+r))) \\ &-E(s,\hbar(s+r-\varsigma(s+r)))| \\ &+|E(s,\hbar(s+r-\varsigma(s+r))) - E(s,\hbar(s-\varsigma(s)))|\bigg]\Delta s \\ &+\sum_{a_k<t}\varepsilon\Gamma e^{-\frac{\lambda}{2}(t-a_k)}L + \sum_{a_k<t}\varepsilon e^{-\lambda(t-a_k)}\bigg\} \\ \le\ &\sup_{t\in\mathbb{T}}\bigg\{\int_{-\infty}^{t}\varepsilon\Gamma e^{-\frac{\lambda}{2}(t-\sigma(s))}\bigg(\sum_{R=1}^{m}\varpi_R^u+\beta\bigg)\Delta s \\ &+\int_{-\infty}^{t}\varepsilon e^{-\lambda(t-\sigma(s))}\bigg[\sum_{R=1}^{m}(\varpi_R^u nM^{n-1}+1)+\aleph+1\bigg]\Delta s \\ &+\sum_{a_k<t}\varepsilon\Gamma e^{-\frac{\lambda}{2}(t-a_k)}L + \sum_{a_k<t}\varepsilon e^{-\lambda(t-a_k)}\bigg\} \\ \le\ &\varepsilon\sup_{t\in\mathbb{T}}\bigg\{\int_{-\infty}^{t}\Gamma e_{\ominus\frac{\lambda}{2}}(t,\sigma(s))\bigg(\sum_{R=1}^{m}\varpi_R^u+\beta\bigg)\Delta s \\ &+\int_{-\infty}^{t}e_{\ominus\lambda}(t,\sigma(s))\bigg[\sum_{R=1}^{m}(\varpi_R^u nM^{n-1}+1)+\aleph+1\bigg]\Delta s \\ &+\sum_{a_k<t}\Gamma e^{-\frac{\lambda}{2}(t-a_k)}L + \sum_{a_k<t}e^{-\lambda(t-a_k)}\bigg\} \\ \le\ &\varepsilon\bigg\{\frac{\Gamma(2+\lambda\bar{\mu})}{\lambda}\bigg(\sum_{R=1}^{m}\varpi_R^u+\beta\bigg) \\ &+\frac{(1+\lambda\bar{\mu})}{\lambda}\bigg[\sum_{R=1}^{m}(\varpi_R^u nM^{n-1}+1)+\aleph+1\bigg] \\ &+\frac{2N\Gamma L}{1-e^{-\frac{\lambda}{2}}} + \frac{2N}{1-e^{-\lambda}}\bigg\}. \qquad (9)\end{aligned}$$

Consequently, by (8) and (9), we obtain that $\Upsilon_\hbar \in A$.

Let $\hbar_1, \hbar_2 \in A$, then

$$\begin{aligned}&\|\Upsilon_{\hbar_1} - \Upsilon_{\hbar_2}\| \\ \le\ &\sup_{t\in\mathbb{T}}\bigg\{\int_{-\infty}^{t}\|¥[t,\sigma(s)]\|\sum_{R=1}^{m}\varpi_R^u\bigg|\frac{1}{1+\hbar_1^n(s-\upsilon_R(s))} \\ &-\frac{1}{1+\hbar_2^n(s-\upsilon_R(s))}\bigg|\Delta s + \int_{-\infty}^{t}\|¥[t,\sigma(s)]\| \\ &\times\bigg|E(s,\hbar_1(s-\varsigma(s))) - E(s,\hbar_2(s-\varsigma(s)))\bigg|\Delta s\bigg\}\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in\mathbb{T}}\bigg\{\int_{-\infty}^{t} e_{\ominus\lambda}(t,\sigma(s))\sum_{R=1}^{m}\varpi_R^u\bigg|\frac{1}{1+\hbar_1^n(s-\upsilon_R(s))} \\
&\quad -\frac{1}{1+\hbar_2^n(s-\upsilon_R(s))}\bigg|\Delta s+\int_{-\infty}^{t} e_{\ominus\lambda}(t,\sigma(s)) \\
&\quad \times\Big|E(s,\hbar_1(s-\varsigma(s)))-E(s,\hbar_2(s-\varsigma(s)))\Big|\Delta s\bigg\} \\
&\leq \frac{(1+\lambda\bar{\mu})}{\lambda}\bigg(\sum_{R=1}^{m}\varpi_R^u nM^{n-1}+\aleph\bigg)\|\hbar_1-\hbar_2\| \\
&= \gamma\|\hbar_1-\hbar_2\|. \qquad (10)
\end{aligned}
$$

Subsequently, based on the condition (H_5), it can be deduced that the operator Υ exhibits the property of contraction inside the set A. Furthermore, there exists a unique almost periodic solution of (2). ■

Theorem 3.2: Let conditions $(H_1)-(H_5)$ hold and the following conditions are satisfied:

(H_6) $(\ominus\lambda)\oplus\tilde{p}<0$, where

$$
\tilde{p}=(1+\lambda\bar{\mu})\bigg(\sum_{R=1}^{m}\varpi_R^u nM^{n-1}+\aleph\bigg),
$$

where $\bar{\mu}=\sup_{t\in\mathbb{T}}\mu(t)$. Then the unique positive almost periodic solution of (2) is exponentially stable.

Proof: Let $W(t)$ and $Œ(t)$ be represented as an arbitrary solution and a unique positive almost periodic solution of equation (2), respectively. It follows that

$$
\begin{aligned}
& W(t)-Œ(t) \\
= \ & Œ[t,a_0](W(a_0)-Œ(a_0)) \\
& +\int_{a_0}^{t}¥[t,\sigma(s)]\bigg[\sum_{R=1}^{m}\varpi_R(s)\bigg(\frac{1}{1+W^n(s-\upsilon_R(s))} \\
& -\frac{1}{1+Œ^n(s-\upsilon_R(s))}\bigg)+(E(s,W(s-\varsigma(s))) \\
& -E(s,Œ(s-\varsigma(s))))\bigg]\Delta s.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \|W(t)-Œ(t)\| \\
\leq \ & e^{-\lambda(t-a_0)}\|W(a_0)-Œ(a_0)\| \\
& +\int_{a_0}^{t}e^{-\lambda(t-\sigma(s))}\sum_{R=1}^{m}\varpi_R^u\bigg\|\frac{1}{1+W^n(s-\upsilon_R(s))} \\
& -\frac{1}{1+Œ^n(s-\upsilon_R(s))}\bigg\|\Delta s+\|E(s,W(s-\varsigma(s))) \\
& -E(s,Œ(s-\varsigma(s)))\|\bigg]\Delta s \\
\leq \ & e_{\ominus\lambda}(t,a_0)\|W(a_0)-Œ(a_0)\|+(1+\lambda\bar{\mu})\int_{a_0}^{t}e_{\ominus\lambda}(t,s) \\
& \times\bigg(\sum_{R=1}^{m}\varpi_R^u nM^{n-1}+\aleph\bigg)\|W(s)-Œ(s)\|\Delta s.
\end{aligned}
$$

Let $\mathfrak{X}(t)=\|W(t)-Œ(t)\|e_{\lambda}(t,a_0)$ and it follows

$$
\mathfrak{X}(t)\leq\mathfrak{X}(a_0)+(1+\lambda\bar{\mu})\bigg(\sum_{R=1}^{m}\varpi_R^u nM^{n-1}+\aleph\bigg)\int_{a_0}^{t}\mathfrak{X}(s)\Delta s.
$$

By means of the Gronwall-Bellman's inequality on time scales (see [29]), we have

$$
\mathfrak{X}(t)\leq\mathfrak{X}(a_0)e_{\tilde{p}}(t,a_0),
$$

this is

$$
\|W(t)-Œ(t)\|\leq\|W(a_0)-Œ(a_0)\|e_{(\ominus\lambda)\oplus\tilde{p}}(t,a_0),
$$

where

$$
\tilde{p}=(1+\lambda\bar{\mu})\bigg(\sum_{R=1}^{m}\varpi_R^u nM^{n-1}+\aleph\bigg).
$$

■

Therefore, based on condition (H_6), it can be deduced that the unique positive almost periodic solution of system (2), which exhibits exponential stability. This completes the proof.

IV. An example

Consider the following the following hematopoiesis model with impulsive effects and harvesting terms on time scales:

$$
\begin{cases}
Œ^{\Delta}(t)=-\pi(t)Œ(t)+\sum_{R=1}^{m}\frac{\varpi_R(t)}{1+Œ^n(t-\upsilon_R(t))} \\
\qquad -E(t,Œ(t-\varsigma(t))),\ t\neq a_k,\ t\in[0,+\infty)_{\mathbb{T}}, \\
\tilde{\Delta}Œ(a_k)=\gamma_k Œ(a_k)+\delta_k,\ t=a_k,\ k\in\mathbb{N},
\end{cases} \qquad (11)
$$

where $m=2$, $n=\frac{1}{10}$, and

$$
\pi(t)=0.82-0.02\sin(\sqrt{4}t),\ \varpi_1(t)=0.23+0.01\cos(\sqrt{3}t),
$$

$$
\varpi_2(t)=0.18+0.04\cos(\sqrt{6}t),\ \upsilon_1(t)=0.38+0.06\cos(\sqrt{5}t),
$$

$$
\upsilon_2(t)=0.44+0.06\cos(\sqrt{4}t),\ \varsigma(t)=0.2+0.01\sin 2t,
$$

$$
E(t,Œ)=0.05(|0.5\sin t-\cos\sqrt{2}t|)\frac{|Œ(t)|}{1+Œ^2(t)},
$$

$$
\gamma_k=(2)^{\frac{0.4}{2^k}}-1,\ \delta_k=0.1,\ a_k=k,\ k\in\mathbb{N}.
$$

Assume model (11) has a solution $Œ(t)$ with the initial conditions as following

$$
\begin{cases}
E(t;a_0,\phi_0)=e^{2t},\ t\in(-0.5,0)_{\mathbb{T}}, \\
E(a_0^+;a_0,\phi_0)=1.
\end{cases} \qquad (12)
$$

Moreover, assume model (11) has a solution $W(t)$ with another initial conditions as following

$$
\begin{cases}
E(t;a_0,\phi_0)=3t,\ t\in(-0.5,0)_{\mathbb{T}}, \\
E(a_0^+;a_0,\phi_0)=2.
\end{cases} \qquad (13)
$$

By calculating, for $0\leq\mu\leq 1$, $\lambda=0.6$ and

$$
\pi^u=0.84,\ \pi^l=0.8,\ \varpi_1^u=0.24,\ \varpi_1^l=0.22,
$$

$$
\varpi_2^u=0.21,\ \varpi_2^l=0.15,\ \aleph=0.06,
$$

$$
\alpha=0.025,\ \beta=0.06,\ b=2^{0.2},\ B=2^{0.4},\ L=0.1,
$$

so we obtain

$$
M=\frac{\sum_{R=1}^{2}\varpi_i^u B}{a^l}\approx 0.742,
$$

$$
\bar{M}=\frac{b}{\pi^u}\bigg(\frac{\sum_{R=1}^{2}\varpi_R^l}{(1+M^n)}-\beta\bigg)\approx 0.249,
$$

then

$$
\gamma=\frac{(1+\lambda\bar{\mu})}{\lambda}\bigg(\sum_{R=1}^{2}\varpi_R^u nM^{n-1}+\aleph\bigg)\leq 0.317<1,
$$

and

$$\tilde{p} - \lambda = (1+\lambda\bar{\mu})\left(\sum_{R=1}^{2} \varpi_R^u n M^{n-1} + \aleph\right) - 0.6 \leq -0.409 < 0.$$

Thus, $(\ominus\lambda)\oplus\tilde{p} < 0$, based on Theorem 3.1 and Theorem 3.2, it can be concluded that the (11) possesses a unique almost periodic solution that exhibits exponential stability.

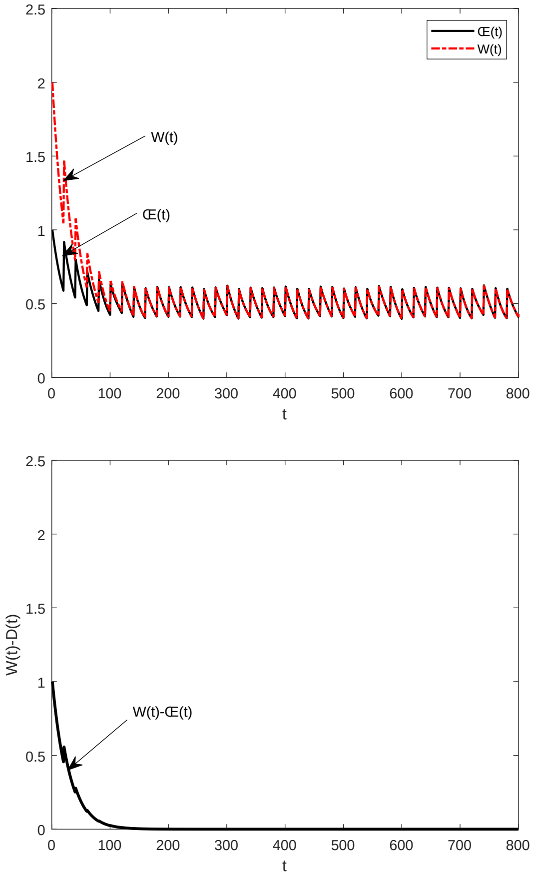

Fig. 1: The evolutions of Œ, W and $W - Œ$ with the initial conditions (12) and (13) on $\mathbb{T} = \mathbb{R}$.

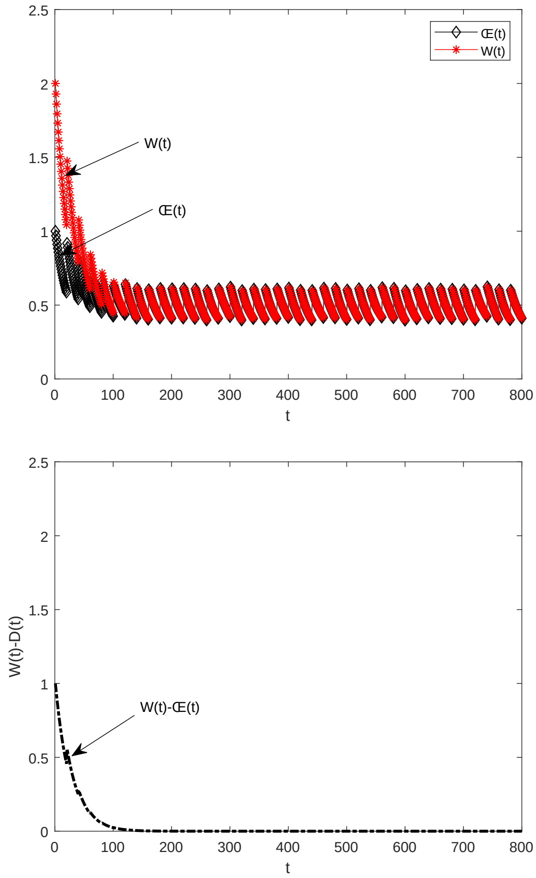

Fig. 2: The evolutions of Œ, W and $W - Œ$ with the initial conditions (12) and (13) on $\mathbb{T} = \mathbb{Z}$.

V. Conclusion

This study provides the necessary conditions for the existence and exponential stability of almost periodic solutions in a delayed hematopoiesis model with impulsive effects and harvesting terms on time scales. The aforementioned requirements are derived through the use of fixed-point theory in Banach space, along with the application of the Gronwall-Bellman's inequality approach on time scales. The main results obtained in this paper are novel, even when considering the cases where the time scale is either continuous or discrete. The findings of this paper contribute to the improvement of previously known results, highlighting the significance of the derived results.

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