On Positive Cone and Partial Order in a Generalized Algebraic System

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Abstract—**One of the extensions of a nearring and a gamma ring is the concept of a gamma nearring, which allows for a more general multiplication operation. In this paper, we aim to establish the concept of a partial order in a Γ-nearring, thereby extending the notion of partial order observed in a nearring. We introduce several key concepts such as partial order, positive cone, convex ideal, and others, within the context of a Γ-nearring. Additionally, we provide proofs for various classical results pertaining to these notions. Moreover, we investigate different types of prime ideals within a lattice-ordered Γ-nearring and examine their properties. By exploring the characteristics and behavior of these prime ideals, we enhance our understanding of lattice-ordered Γ-nearrings and their structural properties.**

Index Terms—**Nearring; Gamma nearring; Partial order; Prime ideal.**

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I. Introduction and Preliminaries

The notion of Γ-ring is originated from a ring, as its generalization, which was defined by Nobusawa [1], and later studied by Barnes [2]. In Bhavanari [24], [15], the notions of nearring and the ring, taken together to generalize a new notion, namely Γ-Nearring. Bhavanari [15]; Booth [13], Booth and Groenewald [14] studied various radical properties of Γ-nearrings.

The concept of a Γ-ring builds upon the foundation of a ring, providing a generalization that was initially introduced by Nobusawa [1] and further extended by Barnes [2]. In Bhavanari's work [24], [15], the notions of nearrings and rings are combined to introduce a novel concept known as a Γ-nearring. The exploration of Γ-nearrings has attracted considerable research interest. The authors such as Bhavanari [15], Booth [13], and Booth and Groenewald [14] have focused on investigating various radical properties associated with Γ-nearrings.

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Let $(M, +)$ be a group (not necessarily abelian) and Γ be a non-empty set. Then M is said to be a Γ-nearring if there exists a mapping $M \times \Gamma \times M \to M$ satisfying the following: (i) $(m_1 + m_2)\alpha m_3 = m_1\alpha m_3 + m_2\alpha m_3$; (ii) $(m_1\alpha_1 m_2)\alpha_2 m_3 = m_1\alpha_1(m_2\alpha_2 m_3)$, for all $m_1, m_2, m_3 \in M$ and for all $\alpha_1, \alpha_2 \in \Gamma$.

Example I-A: Let $M = \mathbb{Z}_6$, $\Gamma = \{\gamma_1, \gamma_2\}$ where

$$a\gamma_1 b = \begin{cases} a & \text{if } b = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$a\gamma_1 b = \begin{cases} a & \text{if } b = 2 \\ 0 & \text{otherwise} \end{cases}$$

Then (M, Γ) is a Γ-nearring.

M is zero symmetric (resp. constant) if $M = M_0 = \{m \in M : m\alpha 0 = 0,\ \forall \alpha \in \Gamma\}$ (resp. $M = M_c = \{m \in M : m\alpha m' = m,\ \forall m' \in M,\ \alpha \in \Gamma\}$). A normal subgroup $(K, +)$ of $(M, +)$ is called a left (resp. right) ideal if $m_1\alpha(m_2 + k) - m_1\alpha m_2 \in K$ (resp. $k\alpha m_1 \in K$), for all $m_1, m_2 \in M$, $\alpha \in \Gamma$, and $k \in K$. Let M and M' be Γ-nearrings. A homomorphism $\psi : M \to M'$ is called a Γ-homomorphism if: (i) $\psi(m + m_1) = \psi(m) + \psi(m_1)$; (ii) $\psi(m\alpha m_1) = \psi(m)\alpha\psi(m_1)$, for all $m, m_1 \in M$, $\alpha \in \Gamma$, and ψ is called Γ-isomorphism if ψ is one-one and onto.

Throughout, M stands for a gamma nearring.

For necessary definitions and results in nearrings, we refer to [3], [10]; and for Γ-nearrings, we refer to [13], [14], [15], [21], [25], [26]. For partial order and lattice order aspects of rings, nearrings, and modules, we refer to [9], [11], [12], [20], [29], [30].

II. Partial order in a Gamma nearring

We introduce the notion of partial order in gamma nearrings.

Definition II-A: A gamma nearring M is called an ordered Γ-nearring if $\leq$ is a partial ordering on M satisfying the following conditions: If $a \leq b$ and $c \leq d$, then (i)

1) $a + c \leq b + d$ and $c + a \leq d + b$;
2) $a\alpha c \leq b\alpha d$;
3) $c\alpha a \leq d\alpha b$,

for all $a, b, c, d \in M$, $\alpha \in \Gamma$.

Note II-B: When $\Gamma = \{\cdot\}$, then the notion p.o. Γ-nearring becomes the notion of p.o. nearrings given by Pilz [10].

Note II-C: For a Γ-nearring $(M, +, \Gamma)$, in the following we show some examples that indicate either $(M, +, \Gamma_1)$ is a p.o. Γ-nearring for some $\Gamma_1 \subset \Gamma$ or $(M, +, \Gamma)$ is a p.o. Γ-nearring.

Example II-D: Let G be a p.o. group. Then $M(G) = \{f : G \to G\}$ with pointwise addition and composition of mappings forms a nearring. Define $\leq$ on $M(G)$ as

$$f \leq g \iff g(x) - f(x) \geq 0$$

satisfying:

$$f \leq g \text{ and } h \geq 0 \Rightarrow \begin{cases} f+h \leq g+h, & h+f \leq h+g \\ f \circ h \leq g \circ h, & \text{(right monotone)} \\ h \circ f \leq h \circ g, & \text{(left monotone).} \end{cases}$$

Then $(M(G), +, \Gamma, \leq)$ is a p.o. Γ-nearring with $\Gamma = \{\circ\}$.

Example II-E: Let G be a p.o. group. Then $(M(G), +, \Gamma)$ is a Γ-nearring, where $\Gamma = \{\star_1, \star_2\}$ defined as follows:
$\star_1 : (f \star_1 g)(x) = f(g(x))$,
$\star_2 : (f \star_2 g)(x) = f(x)$,
for all $f, g \in M(G)$ and $x \in G$.
Define partial order on $M(G)$ as follows:

$$f \leq g \iff f(x) \leq g(x), \text{ for all } x \in G.$$

Then $\leq$ satisfies the following: (i)

1) $f \leq g$ and $h \geq 0 \Rightarrow f+h \leq g+h$ and $h+f \leq h+g$.
2) $f \leq g$ and $h \geq 0 \Rightarrow (f \star_i h)(x) \leq (g \star_i h)(x)$ and $(h \star_i f)(x) \leq (h \star_i g)(x)$, for $i \in \{1, 2\}$.

Hence, $(M(G), +, \Gamma, \leq)$ is a p.o. Γ-nearring.

Example II-F: Let $N = (\mathbb{Z}, +, \cdot)$ a nearring. Write $M = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} : x, y, z \in N \right\}$, $\Gamma = \{\mathbb{B}, \star\}$, where $\mathbb{B} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in N \right\}$. Ternary operation is defined as: $(A, \alpha, C) \to A\alpha C$, for all $A, C \in M$ and $\alpha \in \mathbb{B}$; and $\star$: usual matrix multiplication. Then $(M, +, \Gamma)$ is a Γ-nearring. We define $A \leq B$ if and only if $a_{ij} \leq b_{ij}$, for all i, j. Here $(M, +, \{\star\}, \leq)$ is a p.o. Γ-nearring, whereas $(M, +, \mathbb{B}, \leq)$ is not a p.o. Γ-nearring, for let $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \in M$, $A \geq B$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \geq 0$, and $X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{B}$. Then $AXC = \begin{pmatrix} -1 & 0 \\ -2 & 6 \end{pmatrix}$ and $BXC = \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}$. But $AXC \not\geq BXC$.

Example II-G: Let $M = (\mathbb{Z}, +)$; a group. Define $\Gamma = \{\star_1, \star_2\}$ by

$$a \star_1 b = \begin{cases} a, & \text{if } b \neq 0 \\ 0, & \text{if } b = 0; \end{cases}$$

and $\star_2$: usual multiplication.
Then $(M, +, \Gamma)$ is a p.o. Γ-nearring with the usual order $\leq$.

Example II-H: Let $M = (\mathbb{Z} \times \mathbb{Z}, +)$ and $\mathbb{B} = (0) \times \mathbb{Z}$; be groups (additive). Let $\Gamma = \{\mathbb{B}, \star\}$. We define the ternary operation as: $(a, \alpha, b) \to a\alpha b$, for all $a, b \in M$ and $\alpha \in \mathbb{B}$; and $\star$: usual multiplication. We define $(a_1, a_2) \leq (b_1, b_2)$ if and only if $a_i \leq b_i$, for all i. Then $(M, +, \Gamma, \leq)$ is a Γ-nearring. Here, $(M, +, \{\star\})$ is a p.o. Γ-nearring, whereas $(M, +, \mathbb{B}, \leq)$ is not a p.o. Γ-nearring, for let $a = (-1, 2)$, $b = (3, 4) \in M$, $c = (2, 3) \geq 0$, and $x = (0, -2) \in \mathbb{B}$. Then $axc = (0, -12)$ and $bxc = (0, -24)$. Here, $a \leq b$ but $axc \not\leq bxc$.

Definition II-I: Let M be a Γ-nearring. M is called fully ordered by $\leq$ if (i)

1) $(M, +)$ is fully ordered;
2) for all $m, m' \in M$ and $\alpha \in \Gamma$, $m \geq 0$, $m' \geq 0$ implies $m\alpha m' \geq 0$ and $m'\alpha m \geq 0$.

Definition II-J: Let M be a p.o. Γ-nearring. We define the positive cone of M as $\{m \in M : m \geq 0\}$, we denote it as P or M^+.

Lemma II-K: P satisfies:

1) $P + P = P$;
2) $P \cap -P = \{0\}$; where $-P = \{m \in M : m \leq 0\}$
3) $m + P = P + m$, for all $m \in M$;
4) $P\Gamma P \subseteq P$.

Conversely, suppose that $P \subseteq M$ satisfying all the above four conditions. Then the relation '$\leq_P$' defined by $m_1 \leq_P m_2 \Leftrightarrow m_2 - m_1 \in P$, is a partial order on M, for which P is the positive cone.

Proof: The verification of (1)-(3) is straightforward.
(4) Let $a, b \in P$. Then $a \geq 0$ and $b \geq 0$. Now by definition, we have $a\alpha b \geq 0$, for all $\alpha \in \Gamma$. This implies $a\alpha b \in P$, for all $\alpha \in \Gamma$, and hence $P\Gamma P \subseteq P$.
Conversely, to show $(G, \leq_P)$ is a p.o. Γ-nearring. Clearly, '$\leq_P$' is a p.o. relation on M.

(i) Let $m_1, m_2 \in M$ such that $m_1 \geq 0$, $m_2 \geq 0$. Then $m_1, m_2 \in P$. Now by (1), we get $m_1 + m_2 \in P + P \subseteq P$, implies $m_1 + m_2 \geq_P 0$. Also, $m_2 + m_1 \in P + P \subseteq P$ implies $m_2 + m_1 \geq_P 0$.
(ii) Let $m_1, m_2 \in M$ and $x \in M$ such that $m_1 \leq_P m_2$ and $0 \leq_P x$. That is, $m_2 - m_1 \in P$ and $x \in P$. Now by (4), we have $(m_2 - m_1)\alpha x \in P$, for all $\alpha \in \Gamma$. This implies $m_2\alpha x - m_1\alpha x \in P$, and hence $m_1\alpha x \leq m_2\alpha x$. Similarly, we get that $x\alpha m_1 \leq x\alpha m_2$.

Therefore, M is a p.o. Γ-nearring with respect to '$\leq_P$'. Now to show $P = \{x \in M : x \geq_P 0\}$, let $x \in P$. Then $x - 0 \in P$ implies $0 \leq_P x$. Hence $x \in M^+$. Conversely, let $m \in M^+$. Then $m - 0 \in P$ implies $m \in P$. Therefore, P is a positive cone of $(M, \leq_P)$. ■

Proposition II-L: Let M be ordered by P.

1) $\leq_P$ is fully ordered if and only if $P \cup (-P) = M$.
2) $\leq_P$ is trivial (that is, $m \leq_P m' \Leftrightarrow m = m'$) if and only if $P = \{0\}$.

Definition II-M: Let M, M' be Γ-nearrings ordered by the positive cones P and P' respectively. A map $f : M \to M'$ is order preserving if $f(P) \subseteq P'$ (we use $\backsimeq_o$ to denote order preserving isomorphism).

Definition II-N: A subset T of an ordered Γ-nearring M is called convex if for any $t_1, t_2 \in T$, and $m \in M$, $t_1 \leq m \leq t_2$, then $m \in T$.

Lemma II-O: Let M be an ordered Γ-nearring and I be a convex ideal of M. Define a relation on the quotient Γ-nearring M/I by

$$x + I \leq_{M/I} y + I \text{ if } x \leq_M y + i, \text{ for some } i \in I.$$

Then $(M/I, \leq_{M/I})$ is a p.o. Γ-nearring.

Proof: Clearly, $\leq_{M/I}$ is a p.o. relation on M/I. To show $(M/I, \leq_{M/I})$ is a p.o. Γ-nearring, we show the monotonicity.
(i) Let $m_1 + I, m_2 + I \in M/I$ such that $0 + I \leq_{M/I} m_1 + I$ and $0 + I \leq_{M/I} m_2 + I$. Then $0 \leq_M m_1 + i$ and

$0 \leq_M m_2 + i_1$, for some $i, i_1 \in I$, implies

$$\begin{aligned} 0 &\leq_M (m_1+i)\alpha(m_2+i_1), \text{ for all } \alpha\in\Gamma \\ &= m_1\alpha(m_2+i_1) + i\alpha(m_2+i_1) \\ &= m_1\alpha(m_2+i_1) - m_1\alpha m_2 + m_1\alpha m_2 + i\alpha(m_2+i_1) \\ &= i_2 + m_1\alpha m_2 + i_3, \text{ for some } i_2, i_3 \in I \\ &= m_1\alpha m_2 + (i_4+i_3), \text{ for some } i_4\in I. \end{aligned}$$

This implies $0+I \leq_{M/I} m_1\alpha m_2 + I$, and so $0+I \leq_{M/I} (m_1+I)\alpha(m_2+I)$.

■

Theorem II-P: A subset I of an ordered Γ-nearring M is the kernel of an order-preserving Γ-homomorphism from M to some ordered Γ-nearring M' if and only if I is a convex ideal.

Proof: Let $f: M \to M'$ be an order preserving Γ-homomorphism. Let I be the kernel of f. Let I be the kernel of f. To show I is a convex ideal, let $i_1, i_2 \in I$ and $m \in M$ such that $i_1 \leq m \leq i_2$. Since $i_1 \leq m$, we have $m - i_1 \in P$. Now f is order preserving implies $f(m-i_1) \in f(P) \subseteq P'$. Hence, $f(m) - f(i) \in P'$ so that $f(m) - f(i) \geq 0$ and since I is the kernel of f, we have $f(i) = 0$. Therefore, $f(m) \geq 0$. Similarly, when $m \leq i_2$, we get $f(m) \leq 0$. Hence, $f(m) = 0$ implies that $m \in I$. This proves that I is a convex ideal. Conversely, suppose that I be a convex ideal of M. Then, by Lemma II-O, M/I is partial order nearring. Define $\phi: M \to M/I$ by $\phi(m) = m + I$. Clearly M is a Γ-homomorphism and $ker\ \phi = I$. To show ϕ is order preserving, let $t \in \phi(P)$. Then there exists $u \in P$ such that $t = \phi(u) = u + I$. Now, $u \in P$ implies $u \geq 0$. Then, by monotonicity, $u + m \geq o + m$ for every $m \in M$. In particular, $u + i \geq i$, for every $i \in I$, and so $u + I \geq I$, where $I = 0 + I \in M/I$. This shows that $u + I \in P'$. Therefore $f(P) \subseteq P'$. ■

Theorem II-Q: Let M and M' be Γ-nearrings, ordered by P and P'. Let $h: M \to M'$ be an order-preserving epimorphism (i.e. $h(M) = M'$ and $h(P) = P'$). If I' is a convex ideal of M' then $h^{-1}(I') := I$ is a convex ideal of M and $M/I \cong_0 M'/I'$.

Proof: Let $h: M \to M'$ be an order preserving Γ-epimorphism. Suppose I' be a convex ideal of M'. To show, $h^{-1}(I') = I$ is a convex ideal of M. Clearly, I is an ideal of M. Now let $x, y \in I$ and $m \in M$ such that $x \leq m \leq y$. Then there exists $s, t \in I'$ such that $x = h^{-1}(s)$, $y = h^{-1}(t)$ and $h^{-1}(s) \leq m \leq h^{-1}(t)$. Since h is order Γ-epimorphism, we get $h(h^{-1}(s)) \leq h(m) \leq h(h^{-1}(t))$. This implies $s \leq h(m) \leq t$, and I' is convex ideal implies $h(m) \in I'$. Thus $m \in h^{-1}(I') = I$, shows that I is convex. Since $h: M \to M'$ is an order preserving Γ-epimorphism, by 2nd Isomorphism theorem, we get $M/I \cong h(M)/h(I)$, where $ker\ h \subseteq I$. Also, since h is Γ-epimorphism and $h^{-1}(I') = I$, we get $h(I) = I'$ and $h(M) = M'$. Therefore $M/I \cong M'/I'$. Define $\phi: M/I \to M'/I'$ by $\phi(m+I) = h(m) + I'$. To show ϕ is order preserving, let $u \in \phi(P)$. Then there exists $v + I \in P$ such that $u = \phi(v+I)$. This implies $v + I \geq 0 + I$. That is, $0 \leq v + i$, for some $i \in I$. Since h is order epimorphism, we get $h(0) \leq h(v+i) = h(v) + h(i) = h(v) + i'$, for some $i' \in I' = h(I)$. This implies $0 + I' \leq h(v) + I'$, and so $h(v) + I' \in P'$. Hence $\phi(v+I) \in P'$. Therefore, $u \in P'$. ■

Remark II-R: For a Γ-nearring M, there corresponds a group G such that $M \hookrightarrow M(G)$.

Proof: Let $(G,+)$ be a group properly containing $(N,+)$. Let $m \in M$. Define $f_m: G \to G$ by

$$f_m(g) = \begin{cases} m\alpha g, & \text{if } g\in M \\ m, & \text{if } g \notin M, \end{cases}$$

for all $\alpha \in \Gamma$.
To show f_m is Γ-homomorphism.

1) Case-(i): Let $g \in M$. Then, $(f_m + f_{m'})(g) = f_m(g) + f_{m'}(g) = m\alpha g + m'\alpha g = (m+m')\alpha g = f_{m+m'}(g)$.
Case-(ii): Let $g \notin M$. Then, $(f_m + f_{m'})(g) = f_m(g) + f_{m'}(g) = m + m' = f_{m+m'}(g)$.

2) Case-(i): Let $g \in M$. Then, $(f_m \circ f_{m'})(g) = f_m(f_{m'}(g)) = f_m(m'\alpha g) = m\alpha(m'\alpha g) = (m\alpha m')\alpha g = f_{m\alpha m'}(g)$.
Case-(ii): Let $g \notin M$. Then, $(f_m \circ f_{m'})(g) = f_m(f_{m'}(g)) = f_m(m') = m\alpha m' = f_{m\alpha m'}(g)$.
Thus, $h: M \to M(G)$ defined by $h(m) = f_m$ is a Γ-homomorphism. If $h(m) = h(m')$, then $f_m = f_{m'}$. In particular, for all $g \in G \setminus M$

$$\begin{aligned} m &= f_m(g) \\ &= f_{m'}(g) \\ &= m'. \end{aligned}$$

Therefore, h is a Γ-monomorphism and an embedding map.

■

Proposition II-S: To every abelian ordered Γ-nearring M there exists an (abelian) ordered Γ-nearring $\widehat{M}$ with identity such that $M \hookrightarrow \widehat{M}$.

Proof: By Theorem 1.86 of [10], we have for every M, there exists a group G such that $M \hookrightarrow M(G)$. That is, $h: M \hookrightarrow M(G)$ is monomorphism. If M is ordered by P, take $\widehat{P} = h(P)$. Now we show that $\widehat{P}$ is a positive cone in $\widehat{M}$. In view of ([10], Theorem 9.125), it is enough to show the condition (4).
Let $\alpha \in \Gamma$ and $x, y \in \widehat{P}$. Then there exists $a, b \in P$ such that $x = h(a)$ and $y = h(b)$. Now, $x\alpha y = h(a)\alpha h(b) = h(a\alpha b) \in h(P\Gamma P) \subseteq h(P) = \widehat{P}$. Therefore, $\widehat{P}\Gamma\widehat{P} \subseteq \widehat{P}$. This shows that $\widehat{P}$ is a positive cone in $\widehat{M}$. Hence, h is an order-preserving monomorphism. ■

Theorem II-T: [5] (i)

1) If e is an idempotent in M, then we get a 'Peirce decomposition':

   $\forall m \in M$ and $\forall x_0 \in \{x \in M : x\alpha e = 0\}$, $\exists x_1 \in M\alpha e$

   such that $m = x_0 + x_1$.
2) Taking $e = 0$, we get $\forall m \in M$, $\exists m_0 \in M_0$, $\exists m_c \in M_c$ such that $m = m_0 + m_c$.
3) Hence $(M,+) = (M_0,+) + (M_c,+)$ and $M_0 \cap M_c = \{0\}$.

Theorem II-U: Let M be f.o. (by P) and $M_c \neq \{0\}$. Then $\forall m \in M$, for all $c \in P_c = P \cap M_c$: $m\alpha c = m\alpha 0$, for all $\alpha \in \Gamma$.

Proof: Suppose there are some $m \in M$, $c \in P_c$ and $\alpha \in \Gamma$ such that $m\alpha c \neq m\alpha 0$. Without loss of generality, assume that $m\alpha 0 \geq 0$.

Now, $m_0\alpha c = (m - m\alpha 0)\alpha c = m\alpha c - m\alpha(0\alpha c) = m\alpha c - m\alpha 0 \neq 0$. Therefore, $m_0 \neq 0$. If $m_0 > 0$, then $0 \leq m_0\alpha c \neq 0$, we have $m_0\alpha c > 0$.
Write $l = m - m\alpha c + m_0$.
Then,

$$\begin{aligned} l\alpha c &= (m - m\alpha c + m_0)\alpha c \\ &= m\alpha c - (m\alpha c)\alpha c + m_0\alpha c \\ &= m\alpha c - m\alpha c + m_0\alpha c \\ &= m_0\alpha c > 0 \end{aligned}$$

Since $c \in P_c$, we get $l > 0$.
On the other hand,

$$\begin{aligned} l\alpha 0 &= (m - m\alpha c + m_0)\alpha 0 \\ &= m\alpha 0 - (m\alpha c)\alpha 0 + m_0\alpha 0 \\ &= m\alpha 0 - m\alpha(c\alpha 0) + m_0\alpha 0 \\ &= m\alpha 0 - m\alpha c + m_0\alpha 0 \\ &= m\alpha 0 - m\alpha c \\ &= -m_0\alpha c < 0. \end{aligned}$$

Since $0 \in P_c$, we get $l = l\alpha 0 > 0$, a contradiction.
Similarly, if $m_0 < 0$, we get a contradiction. Therefore, $m\alpha c = m\alpha 0$, for all $m \in M$, $c \in P_c$ and $\alpha \in \Gamma$. ■

Definition II-V: A p.o. Γ-nearring M is called a weak p.o. Γ-nearring if (i)

1) $(M, +)$ is a p.o. group;
2) $a \geq 0$, $b \geq 0$ implies $a\alpha b \geq 0$;
3) $a \geq 0$, $b \leq 0$ implies $a\alpha b \leq 0$;

for all $a, b \in M$, $\alpha \in \Gamma$.
A weak p.o. Γ-nearring is called weak f.o. Γ-nearring if $\leq$ is f.o.

Remark II-W: In a weak f.o. Γ-nearring M with identity e, $m\alpha 0 = 0$ for all $m \in M$ and $\alpha \in \Gamma$.

Proof: Let $a \geq 0$. Then $a\alpha 0 \geq 0\alpha 0 = 0$. Since M is f.o., either $a\alpha 0 \geq e$ or $a\alpha 0 \leq e$. If $a\alpha 0 \geq e$, then for any $x \geq 0$, $a\alpha 0 = a\alpha(0\alpha x) = (a\alpha 0)\alpha x \geq e\alpha x = x$. In particular, by taking $x = a\alpha 0 + a\alpha 0$, we get $a\alpha 0 \geq x = a\alpha 0 + a\alpha 0 \geq a\alpha 0$. So, $a\alpha 0 \geq a\alpha 0 + a\alpha 0 \geq a\alpha 0$ implies $a\alpha 0 + a\alpha 0 = a\alpha 0$. Therefore, $a\alpha 0 = 0$. Also, $a\alpha 0 \geq x \geq a\alpha 0$ implies $x = a\alpha 0 = 0$. Therefore, $M = (0)$. If $e > a\alpha 0$, then $0 = e\alpha 0 \geq (a\alpha 0)\alpha 0 = a\alpha(0\alpha 0) = a\alpha 0 \geq 0$. Therefore, $a\alpha 0 = 0$. ■

Proposition II-X: If M is a Γ-nearring, then $(-x)\alpha y = -(x\alpha y)$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Proof: Proof is straightforward. ■

Lemma II-Y: If M is a weak f.o. Γ-nearring with identity e, then $a\alpha(-e) = -a$, for all $a \in M$ and $\alpha \in \Gamma$.

Proof: Consider $(-e)\alpha a = -a$ and $a\alpha(-e)$. Since $(-e)\alpha(-e) = -(-e) = e = e\alpha e$, we have $e \geq 0$. If $e = 0$, $M = (0)$. So let $e > 0$. As M is f.o., either $(-a) \geq a\alpha(-e)$ or $a\alpha(-e) \geq -a$. Suppose $a\alpha(-e) \geq -a$. Then $a\alpha(-e) + a \geq 0$. Now, $(a + a\alpha(-e))\alpha(-e) = a\alpha(-e) + a\alpha(-e)\alpha(-e) = a\alpha(-e) + a\alpha e = a\alpha(-e) + a \geq 0$. Whereas, $a + a\alpha(-e) \geq 0$ and $-e < 0$, implies $(a + a\alpha(-e))\alpha(-e) \leq 0$. So $(a + a\alpha(-e))\alpha(-e) = a\alpha(-e) + a\alpha(-e)\alpha(-e) = a\alpha(-e) + a\alpha e = a\alpha(-e) + a \leq 0$. Therefore, $a\alpha(-e) + a = 0$. In the same way, we get $-a \geq a\alpha(-e)$ implies $a\alpha(-e) = -a$. ■

Theorem II-Z: If M is a weak f.o. Γ-nearring with identity e, then $(M, +)$ is abelian, and $x\alpha(-y) = -(x\alpha y)$, for all $x, y \in M$, and $\alpha \in \Gamma$.

Proof: By Lemma II-Y, we have $a\alpha(-e) = -a$, for all $a \in M$ and $\alpha \in \Gamma$. Now, for any $x, y \in M$, $0 = (x + y)\alpha(-e) + (x + y) = x\alpha(-e) + y\alpha(-e) + x + y = -x - y + x + y$. This implies $x + y = y + x$. Also, $x\alpha(-y) = x\alpha(y\alpha(-e)) = (x\alpha y)\alpha(-e) = -(x\alpha y)$. ■

III. L-IDEALS OF Γ-NEARRING

Definition III-A: A p.o. Γ-nearring is called l.o. Γ-nearring if $\leq$ is l.o.

Definition III-B: A subset I of a l.o. Γ-nearring is said to be $L\Gamma$-ideal if (i)

1) I is an Γ-ideal of M;
2) I is a convex sublattice of M.

Definition III-C: Let I and K be $L\Gamma$-ideals of a l.o. nearring M. Then

$$I\Gamma K = \{x \in M : |x| \leq a\alpha b, 0 < a \in I, 0 < b \in K, \alpha \in \Gamma\}.$$

$$I\Gamma I = \{x \in M : |x| \leq a\alpha b, 0 < a \in I, 0 < b \in I, \alpha \in \Gamma\}.$$

Definition III-D: An $L\Gamma$-ideal P of a l.o. Γ-nearring M is said to be $L\Gamma$-prime if (i)

1) I and K are $L\Gamma$-ideals of M;
2) $I\Gamma K \subseteq P$ implies $I \subseteq P$ or $K \subseteq P$.

A l.o. Γ-nearring is said to be $L\Gamma$-prime if the $L\Gamma$-ideal $\langle 0 \rangle$ is $L\Gamma$-prime.

Definition III-E: Let $m \in M$. The $L\Gamma$-ideal $\langle m \rangle$ is defined as

$$\langle m \rangle = \{a \in M : |a| \leq \sum_{i=1}^{n}(-x_i + y_i + x_i), x_i \in M,$$

$$y_i = \pm m \text{ or } m\gamma z_i, \gamma \in \Gamma, z_i \in M\}.$$

Lemma III-F: $\langle m_1 \rangle \gamma \langle m_2 \rangle \subseteq \langle m_1\gamma m_2 \rangle$, for all $m_1, m_2 \in M$ and $\gamma \in \Gamma$.

Proof: Clearly $\langle m_1 \rangle \subseteq \langle m_1\gamma m_2 \rangle$. For let $a \in \langle m_1 \rangle$. Then $|a| \leq \sum_{i=1}^{n}(-x_i + y_i + x_i)$ or $m_1\gamma z_i$. Put $y_i = m_1$. Then,

$$\begin{aligned} |a| &\leq \sum_{i=1}^{n}(-x_i + y_i + x_i) \\ &= \sum_{i=1}^{n}(-x_i + m_1 + x_i) \\ &\leq \sum_{i=1}^{n}(-x_i + m_1\gamma m_2 + x_i). \end{aligned}$$

Also, if $|a| \leq m_1\gamma z_i$, then, $|a| \leq m_1\gamma z_i \leq m_1\gamma m_2\gamma z_i$. Therefore, $a \in \langle m_1\gamma m_2 \rangle$. Similarly, $\langle m_2 \rangle \subseteq \langle m_1\gamma m_2 \rangle$. Therefore, $\langle m_1 \rangle \gamma \langle m_2 \rangle \subseteq \langle m_1\gamma m_2 \rangle$. ■

Lemma III-G: Let P be an $L\Gamma$-ideal of Γ-nearring M. The following are equivalent. (i)

1) P is is $L\Gamma$-prime;
2) For every two $L\Gamma$-ideals I, J of M, we have $I \nsubseteq P$ and $J \nsubseteq P$ implies $I\Gamma J \nsubseteq P$;
3) For every two elements i, j of M, $i \notin P$ and $j \notin P$ implies $\langle i \rangle \Gamma \langle j \rangle \nsubseteq P$;

4) For every two Γ-ideals I, J of M we have $P \subset I$ and $P \subset J$ implies $I\Gamma J \nsubseteq P$.

Proof: (i)$\Rightarrow$ (ii): Follows by the contraposition of (i).
(ii)$\Rightarrow$ (i): Follows by the contraposition of (ii).
(ii)$\Rightarrow$ (iii): We suppose that (ii) is true. Let $i, j \in M$ such that $i \notin P$ and $j \notin P$. Then $\langle i\rangle \nsubseteq P$ and $\langle j\rangle \nsubseteq P$. Hence by (ii), we get $\langle i\rangle\Gamma\langle j\rangle \nsubseteq P$.
(iii)$\Rightarrow$ (iv): Let $P \subset I$ and $P \subset J$. Then there exists $i \in I\setminus P$ and $j \in J \setminus P$. So $i \notin P$ and $j \notin P$. Then by (iii), we have $\langle i\rangle\Gamma\langle j\rangle \nsubseteq P$. Therefore, $I\Gamma J \nsubseteq P$.
(iv)$\Rightarrow$ (ii): Let (iv) is true. If $I \nsubseteq P$ and $J \nsubseteq P$, then there exists $i \in I \setminus P$ and $j \in J \setminus P$. Therefore, $P \subset \langle i\rangle + P$ and $P \subset \langle j\rangle + P$. Then by (iv), we have

$$(\langle i\rangle + P)\Gamma(\langle j\rangle + P) \nsubseteq P.$$

So there exists $i' \in \langle i\rangle$ and $j' \in \langle j\rangle$, $p, p' \in P$ and $\gamma \in \Gamma$ such that $(i' + p)\gamma(j' + p') \notin P$. This implies $i'\gamma(j' + p') + p\gamma(j' + p')$= $i'\gamma(j' + p') - i'\gamma j' + i'\gamma j' + p\gamma(j' + p') \notin P$, whereas, $i'\gamma(j'+p')+p\gamma(j'+p') \in P$ and $p\gamma(j'+p') \in P$, and hence $i'\gamma j' \notin P$. Therefore, $I\Gamma J \nsubseteq P$. ■

Definition III-H: A $L\Gamma$-ideal I of a l.o. Γ-nearring M is called

1) c-prime if for any a, b of M^+,
$$a\gamma b \in I \text{ implies } a \in I \text{ or } b \in I;$$
2) 3-prime if for any a, b of M^+,
$$a\Gamma M^+\Gamma b \subseteq I \text{ implies } a \in I \text{ or } b \in I;$$
3) e-prime if for any $a \in M^+ \setminus I$ and $x, y \in M$,
$$a\gamma_1 n\gamma_2 x - a\gamma_1 n\gamma_2 y \in I \text{ implies } x - y \in I \text{ for all } n \in M^+.$$

Proposition III-I: Let I be an $L\Gamma$-ideal of M. Then I is

1) c-prime implies 3-prime;
2) 3-prime implies prime;
3) e-prime implies 3-prime.

Proof: 1. Let $a, b \in M^+$ such that $a\Gamma M^+\Gamma b \subseteq I$. On a contrary, $a \notin I$ and $b \notin I$. Since I is c-prime, we have $a\gamma b \notin I$, whereas, $|a\gamma b| \le a\gamma n\gamma b$, since $n \in M^+$. Therefore, $a\gamma b \in a\Gamma M^+\Gamma b$, and hence $a\Gamma M^+\Gamma b \nsubseteq I$, a contradiction.
2. Let A and B be two $L\Gamma$-ideals of M such that $A \nsubseteq I$ and $B \nsubseteq I$. Then there exists $a \in A \setminus I$ and $b \in B \setminus I$. Since I is 3-prime, we have $a\Gamma M^+\Gamma b \nsubseteq I$. Now to show $a\Gamma M^+\Gamma b \subseteq A\Gamma B$, let $x \in a^+b$. Then $|x| \le a\gamma n\gamma b$, for all $n \in M^+$. Now, $|x| \le a\gamma n\gamma b = a'\gamma b$, where $0 < a' = a\gamma n \in A$, being a right $L\Gamma$-ideal. Therefore $x \in A\Gamma B$, which implies $a\Gamma M^+\Gamma b \subseteq A\Gamma B$. Now since $a\Gamma M^+\Gamma b \nsubseteq I$, we get $A\Gamma B \nsubseteq I$. Hence I is prime.
3. Suppose I is e-prime, and let $a\Gamma M^+\Gamma b \subseteq I$. If $a \notin I$, then $a \in M^+ \setminus I$, and $a\gamma n\gamma b \in a\Gamma M^+\Gamma b \subseteq I$ and $a\gamma n\gamma 0 = 0 \in I$, as $M = M_0$. Therefore, $a\gamma n\gamma b - a\gamma n\gamma 0 \in I$. Since I is e-prime, we have $b = b - 0 \in I$. Therefore, I is 3-prime. ■

Definition III-J: (M, Γ) is commutative if $a\alpha b = b\alpha a$, for all $\alpha \in \Gamma$ and $a, b \in M$.

Proposition III-K: If (M, Γ) is commutative, then $(M^2, +)$ is abelian.

Proof: Let $m, m' \in M^2$. Then there exist $a, b, c, d \in M$ such that $m = d\alpha a$ and $m' = b\alpha c$, for all $\alpha \in \Gamma$. To show $(M^2, +)$ is abelian, $(a+c)\alpha(b+d) = a\alpha(b+d)+c\alpha(b+d) = (b + d)\alpha a + (b + d)\alpha c = b\alpha a + d\alpha a + b\alpha c + d\alpha c$ and $(b + d)\alpha(a + c) = b\alpha(a + c) + d\alpha(a + c) = (a + c)\alpha b + (a + c)\alpha d = a\alpha b + c\alpha b + a\alpha d + c\alpha d = b\alpha a + b\alpha c + d\alpha a + d\alpha c$. Since $(a + b) \cdot (c + d) = (c + d) \cdot (a + b)$, we have $b\alpha a + d\alpha a + b\alpha c + d\alpha c = b\alpha a + b\alpha c + d\alpha a + d\alpha c$, implies $d\alpha a + b\alpha c = b\alpha c + d\alpha a$ shows that $m + m' = m' + m$. ■

Lemma III-L: If P is an $L\Gamma$-ideal of a l.o. Γ-nearring M such that $M^+ \setminus P$ is closed under multiplication, then P is $L\Gamma$-prime. Converse hold if (M, Γ) is commutative.

Proof: Let P is an $L\Gamma$-ideal of a l.o. Γ-nearring M and $M^+ \setminus P$ is closed under multiplication. To show P is $L\Gamma$-prime, let $x \notin P$ and $y \notin P$. Then $x, y \in M^+ \setminus P$. Since $M^+ \setminus P$ is multiplicative closed, we have $x\alpha y \in M^+ \setminus P$, for every $\alpha \in \Gamma$. Hence $\langle x\rangle\Gamma\langle y\rangle \subseteq M^+ \setminus P$. This shows that $\langle x\rangle\Gamma\langle y\rangle \nsubseteq P$, and so P is $L\Gamma$-prime. Conversely, let P be a $L\Gamma$-prime ideal of M. Let $x, y \in M^+ \setminus P$. That is, $x, y \notin P$. Now by Lemma III-G (iii), we have $\langle x\rangle\Gamma\langle y\rangle \nsubseteq P$. Clearly, $\langle x\rangle\Gamma\langle y\rangle \subseteq \langle x\alpha y\rangle \subseteq M^+$ for all $\alpha \in \Gamma$, and so $\langle x\rangle\Gamma\langle y\rangle \subseteq M^+ \setminus P$. This shows that $x\alpha y \in \langle x\rangle\Gamma\langle y\rangle \subseteq M^+ \setminus P$. ■

Definition III-M: A subset T of M^+ of a l.o. Γ-nearring M is called an $m\gamma$-system, if for each pair $x, y \in T$, there exists $0 < x_1 \in \langle x\rangle$, $0 < y_1 \in \langle y\rangle$ and $\gamma \in \Gamma$ such that $x_1\gamma y_1 \in T$.

Theorem III-N: An $L\Gamma$-ideal P of a l.o. Γ-nearring M is $L\Gamma$-prime if and only if $M^+ \setminus P$ is an $m\gamma$-system.

Proof: Follows by Lemma III-L. ■

Proposition III-O: Let T be an $m\gamma$-system of a l.o. Γ-nearring N and J be an $L\Gamma$-ideal with $J \cap T = \emptyset$. Then there exists a $L\Gamma$-prime ideal P, $J \subseteq P$ and $P \cap T = \emptyset$.

Proof: Let T be an $m\gamma$-system of M and J be a $L\Gamma$-ideal of M such that $J \cap T = \emptyset$. Consider $\mathcal{S} = \{k : K \text{ is a } L\Gamma\text{-ideal of } M, J \subseteq K, K \cap T = \emptyset\}$. Clearly, $J \in \mathcal{S}$, and so $\mathcal{S} \neq \emptyset$. Let $\{K\}_{i\in I}$ be a chain of $L\Gamma$-ideals of $\mathcal{S}$. This chain has an upper bound, say $\bigcup$, $I \subset K_i$, and so $\bigcup K_i \in \mathcal{S}$. Therefore, by Zorn's lemma, there exists a maximal element, say P. Since $P$is maximal, $P \neq M$. By Theorem III-N, it is enough to show that $M^+ \setminus P$ is an $m\gamma$-system. Let $x, y \in M^+\setminus P$. Then $(\langle x\rangle+P)\cap T \neq \emptyset$ and $(\langle y\rangle+P)\cap T \neq \emptyset$. Let $t \in (\langle x\rangle + P) \cap T$ and $s \in (\langle y\rangle + P) \cap T$. Since T is an $m\gamma$-system, there exists $0 < t_1 \in \langle t\rangle$ and $0 < t_2 \in \langle s\rangle$ such that $t_1\alpha s_1 \in T$ for all $\alpha \in \Gamma$. Suppose $\langle x\rangle\Gamma\langle y\rangle \subset P$ and $a = p_1 + x_1$, $b = p_2 + y_1$, where $p_1, p_2 \in P$, $x_1 \in \langle x\rangle$ and $y_1 \in \langle y\rangle$.
Then

$$\begin{aligned} a\alpha b &= (p_1 + x_1)\alpha(p_2 + y_1) \\ &= p_1\alpha(p_2 + y_1) + x_1\alpha(p_2 + y_1) - x_1\alpha y_1 + x_1\alpha y_1 \\ &\in P + \langle x\rangle\Gamma\langle y\rangle \\ &\subseteq P. \end{aligned}$$

This implies $a\alpha b \in P$. Hence $(P + \langle x\rangle)\Gamma(P + \langle y\rangle) \subseteq P$. So $0 < t_1\alpha s_1 \in \langle t\rangle\Gamma\langle s\rangle \subseteq (P + \langle x\rangle)\Gamma(P + \langle y\rangle) \subseteq P$, a contradiction to $P \cap T = \emptyset$. So there exists $x' \in \langle x\rangle$ and $y' \in \langle y\rangle$ and $\gamma \in \Gamma$ such that $x'\gamma y' \notin P$ and hence $|x'|\gamma|y'| \in M^+ \setminus P$. Thus $M^+ \setminus P$ is an $m\gamma$-system. Therefore, P is prime. ■

Definition III-P: the intersection of all $L\Gamma$-prime ideals of a l.o. Γ-nearring M is called the prime radical of M and is denoted by $\mathcal{P}(M)$.

The following proposition is a characterization of prime radicals in terms of $m\gamma$-system.

Proposition III-Q: Let M be a l.o. Γ-nearring. Then

$$\mathcal{P} = \{q \in M : \text{Every } m\gamma\text{-system containing } |q| \text{ contains } 0\}.$$

Proof: Let $q \in M$ and T be an $m\gamma$-system containing $|q|$ and not containing 0. Then by Proposition III-O, taking $J = (0)$, there exists a prime $L\Gamma$-ideal disjoint with T. So $q \notin \mathcal{P}(M)$. Conversely, let $q \notin \mathcal{P}(M)$. Then there exists a $L\Gamma$-prime ideal P such that q, and so $|q| \notin P$. This implies $|q| \notin M^+ \setminus P$. Thus $M^+ \setminus P$ is an $m\gamma$-system not containing 0, but containing $|q|$. ■

IV. Fuzzy ideals of partially ordered Γ-nearrings

Definition IV-A: Let M be a p.o. Γ-nearring. A fuzzy subset ν of M is said to be a fuzzy sub nearring of M if

1) $\nu(p-q) \geq min\{\nu(p), \nu(q)\}$
2) $\nu(p\gamma q) \geq min\{\nu(p), \nu(q)\}$
3) $p \leq q \implies \nu(p) \geq \nu(q)$ for all $p, q \in M$ and $\gamma \in \Gamma$.

Definition IV-B: Let ν be a non-empty fuzzy subset of a p.o. Γ-nearring M. Then ν is called a fuzzy ideal of M if

1) $\nu(p-q) \geq min\{\nu(p), \nu(q)\}$
2) $\nu(p\gamma q) \geq \nu(p)$ [Left ideal]
3) $\nu(p\gamma(q+r) - p\gamma q) \geq \nu(r)$ [Right ideal]
4) $p \leq q \implies \nu(p) \geq \nu(q)$ for all $p, q, r \in M$ and $\gamma \in \Gamma$.

Definition IV-C: A fuzzy subset ν of p.o. Γ-nearring M is called T-fuzzy left (resp. right) ideal if

1) $\nu(p-q) \geq T(\nu(p), \nu(q))$
2) $\nu(p\gamma(q+r) - p\gamma q) \geq \nu(r)$ $(\nu(p\gamma q) \geq \nu(p))$
3) $p \leq q \implies \nu(p) \geq \nu(q)$ for all $p, q, r \in M$ and $\gamma \in \Gamma$.

Theorem IV-D: If $\{\nu_k : k \in K\}$ is a family of T-fuzzy ideal of p.o. Γ-nearring M, then $(\bigvee_{k\in K} \nu_k)(p) = sup\{\nu_k(p) : k \in K\}$ for all $p \in M$ is also a T-fuzzy ideal of M.

Proof: Let $\{\nu_k : k \in K\}$ be a family of T-fuzzy ideal of p.o. Γ-nearring M. For any $p, q, r \in M$,

1) We have,

$$\begin{aligned}
&(\bigvee_{k\in K} \nu_k)(p-q) \\
&= sup\{\nu_k(p-q) : k \in K\} \\
&\geq sup\{T(\nu_k(p), \nu_k(q)) : k \in K\} \\
&= T\{sup\ \nu_k(p) : k \in K, sup\ \nu_k(q) : k \in K\} \\
&= T\Big((\bigvee_{k\in K} \nu_k)(p), (\bigvee_{k\in K} \nu_k)(q)\Big)
\end{aligned}$$

2) We have,

$$\begin{aligned}
&(\bigvee_{k\in K} \nu_k)(p\gamma q) \\
&= sup\{\nu_k(p\gamma q) : k \in K\} \\
&\geq sup\{T(\nu_k(p)) : k \in K\} \\
&= (\bigvee_{k\in K} \nu_k)(p)
\end{aligned}$$

and one can observe that,

$$\begin{aligned}
&(\bigvee_{k\in K} \nu_k)(p\gamma(q+r) - p\gamma q) \\
&= sup\{\nu_k(p\gamma(q+r) - p\gamma q) : k \in K\} \\
&\geq sup\{T(\nu_k(r)) : k \in K\} \\
&= (\bigvee_{k\in K} \nu_k)(r)
\end{aligned}$$

3) We have,

$$\begin{aligned}
p \leq q \implies &(\bigvee_{k\in K} \nu_k)(p) \\
&= sup\{\nu_k(p) : k \in K\} \\
&\geq sup\{\nu_k(q) : k \in K\} \\
&= \bigvee_{k\in K} \nu_k(q)
\end{aligned}$$

Hence $\bigvee_{k\in K} \nu_k$ is a T-fuzzy ideal of M. ■

Theorem IV-E: An epimorphic pre-image of a T-fuzzy ideal of a p.o. Γ-nearring M is a T-fuzzy ideal.

Proof: Let P and Q be T-fuzzy ideals of a p.o. Γ-nearring M. Let $\theta : P \to Q$ be an epimorphism. Let μ be a T-fuzzy ideals of Q and ν T-fuzzy ideals of P under θ. Then for any $p, q, r \in P$, we have

1)

$$\begin{aligned}
\nu(p-q) &= (\mu \circ \theta)(p-q) \\
&= \mu(\theta(p-q)) \\
&= \mu(\theta(p) - \theta(q)) \\
&\geq T(\mu(\theta(p)), \mu(\theta(q))) \\
&= T((\mu \circ \theta)(p), (\mu \circ \theta)(q)) \\
&= T(\nu(p), \nu(q))
\end{aligned}$$

2) We have,

$$\begin{aligned}
\nu(p\gamma q) &= (\mu \circ \theta)(p\gamma q) \\
&= \mu(\theta(p\gamma q)) \\
&= \mu(\theta(p)\gamma\theta(q)) \\
&\geq \mu(\theta(p)) \\
&= (\mu \circ \theta)(p) \\
&= \nu(p)
\end{aligned}$$

and one can observe that,

$$\begin{aligned}
\nu(p\gamma q) &= (\mu \circ \theta)(p\gamma(q+r) - p\gamma q) \\
&= \mu(\theta(p\gamma(q+r) - p\gamma q)) \\
&= \mu(\theta(p\gamma(q+r)) - \theta(p\gamma q)) \\
&= \mu(\theta(p)\gamma\theta(q+r)) - \theta(p)\gamma\theta(q)) \\
&\geq \mu(\theta(r)) \\
&= (\mu \circ \theta)(r) \\
&= \nu(r).
\end{aligned}$$

3) We have,

$$\begin{aligned}
p \leq q \implies \nu(p) &= (\mu \circ \theta)(p) \\
&= \mu(\theta(p)) \\
&\geq \mu(\theta(q)) \\
&= (\mu \circ \theta)(q) \\
&= \nu(q).
\end{aligned}$$

Hence ν is a T-fuzzy ideal of a p.o. Γ-nearring M. ■

V. Conclusion

In our research, we have extended the notion of partial order to Γ-nearrings. One of the key properties we have explored is convexity, which plays a crucial role in partially ordered nearrings. We have defined the concept of a convex ideal in a Γ-nearring, providing a framework to study and analyze this property within the context of Γ-nearrings. Additionally, we have investigated different types of prime ideals in lattice-ordered Γ-nearrings and established important properties associated with them. These findings contribute to our understanding of lattice-ordered Γ-nearrings and their structural properties. Furthermore, an avenue for further research involves extending the study of radical properties in partially ordered Γ-nearrings. Exploring the characteristics of radicals within this context can yield valuable insights into the nature of these algebraic structures. Moreover, for those interested in exploring fuzzy concepts within the framework of lattice-ordered Γ-nearrings, we suggest referring to the works cited as [27], [28]. These references delve into the application of fuzzy logic and fuzzy concepts to lattice order Γ-nearrings, offering potential avenues for future investigations.

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