Inventory Model with Weibull Ameliorating Items under Fuzzy Environment

Shusheng Wu, Jinyuan Liu, Te-Yuan Chiang, Gino Yang

Abstract—**We propose a new inventory system with an amelioration rate satisfying the Weibull distribution, purchasing cost, ordering cost, and amelioration cost under a fuzzy environment. We apply the function principle to aggregate those fuzzy costs. Then, we refer to the median rule to defuzzy our fuzzy model to a crist model to locate the optimal solution. Depending on the scale parameter, we divide our solution procedure into two Cases. For each Case, we find conditions to find the optimal solution. The same numerical example proposed by a previously published paper is examined to illustrate that our solution process is more effective than the previously published paper.**

Index Terms—**Inventory model, Amelioration, Weibull distribution, Fuzzy environment**

I. Introduction

In this study, we will develop a new inventory model with fuzzy demand and ameliorating that follows Weibull distribution. In the following, we provide a brief literature review of this inventory model. Gupta [1] studied the influence of lead time on inventory systems. Hwang [2] considered inventory systems to be ameliorating and deteriorating. For demand related to sailing price, Mondal et al. [3] examined an inventory model with ameliorating. Giri et al. [4] developed EOQ inventory systems with ramp-type demand, shortages, and decay following Weibull distribution. Wu [5] constructed economic order quantity inventory systems under partial backlogging, time-varying demand, and decay following Weibull distribution. Wu [6] analyzed inventory systems with partial backlogging, ramp-type demand rate, and decay following Weibull distribution. Chang and Dye [7] solved economic order quantity inventory systems for a temporary on-sale price with deteriorating items. Wu et al. [8] established economic order quantity inventory systems, with shortages, decay following Weibull distribution, and time-varying demand. Based on Philp's inventory system, Chakrabarty et al. [9] set up economic order quantity inventory systems, with trended demand, shortages, and decay following Weibull distribution. Jalan et al. [10] instituted inventory models under decay following Weibull distribution, trended demand, and shortages. Hwang [11] found inventory systems with ameliorating that follow Weibull distribution. In this paper, we will extend the inventory systems proposed by Hwang [11] from crisp demand to fuzzy demand to generalize his models to a more practical environment.

Manuscript received July 29, 2023; revised November 13, 2023.

Shusheng Wu is an Instructor at the School of Mathematics and Science Teaching Center, Weifang University of Science and Technology, Weifang 262799, China (email: wushusheng@wfust.edu.cn).

Jinyuan Liu is an Associate Professor in the School of General Studies, Weifang University of Science and Technology, Weifang 262799, China (email: shgljy@126.com).

Te-Yuan Chiang is an Associate Professor in the School of Intelligent Manufacturing, Weifang University of Science and Technology, Shandong, Weifang 262799, China (e-mail: kelly4921@gmail.com).

Gino Yang is a Professor in the Department of Multimedia Game Development and Application, Hungkuang University, Taichung City 43302, Taiwan (email: yangklung@yahoo.com.tw)

II. Assumptions and Notation

With several new expressions for analysis and discussion and two supplementary functions, this paper uses the notation and expressions as Hwang [11] under the fuzzy environment.

$\tilde{R}$ fuzzy demand rate, with

$$\tilde{R} = \left(R_1, R_2, R_3, R_4\right). \tag{2.1}$$

t parameter to denote the time of amelioration.

$\tilde{C}_o$ fuzzy ordering cost, with

$$\tilde{C}_o = \left(C_{o1}, C_{o2}, C_{o3}, C_{o4}\right). \tag{2.2}$$

$\tilde{C}_p$ fuzzy purchase cost, with

$$\tilde{C}_p = \left(C_{p1}, C_{p2}, C_{p3}, C_{p4}\right). \tag{2.3}$$

$\tilde{C}_a$ fuzzy amelioration cost, with $\tilde{C}_p \geq \tilde{C}_a$, where

$$\tilde{C}_a = \left(C_{a1}, C_{a2}, C_{a3}, C_{a4}\right), \tag{2.4}$$

and under the restriction,

$$C_{pi} \geq C_{ai}, \tag{2.5}$$

for $i = 1,2,3,4$.

$\tilde{C}_h$ fuzzy holding cost, with,

$$\tilde{C}_h = \left(C_{h1}, C_{h2}, C_{h3}, C_{h4}\right). \tag{2.6}$$

$f_m\left(T\right)$ defuzzied average cost per unit time that consists of holding cost, purchasing cost, ordering cost, and the ameliorating cost, by median rule.

T cycle time.

$I\left(t\right)$ the inventory level, where the initial inventory level is denoted as I_0, with

$$I\left(0\right) = I_0. \tag{2.7}$$

$A(t)$ the amelioration rate with

$$A(t) = \alpha\,\beta\, t^{\beta-1}, \tag{2.8}$$

where β is the scale parameter with $1 \geq \beta > 0$, and α is the shape parameter, with $\alpha > 0$.

$g(T)$ a supplementary function to point out the critical part of the first derivative of $f_m(T)$, with

$$\frac{df_m(T)}{dT}=\frac{1}{8T^2}g(T). \tag{2.9}$$

$h(T)$ a supplementary function to indicate the critical part of the first derivative of $g(T)$, with

$$\frac{d}{dT}g(T)=T^{\beta}e^{-\alpha T^{\beta}}h(T). \tag{2.10}$$

$T^{\&}$ the unique positive solution for $\frac{d}{dT}h(T)=0$ with

$$T^{\&}=\left(2(\lambda-1)/\alpha\right)^{\lambda}. \tag{2.11}$$

T^* the optimal solution.

III. Review of Previous Findings

In the beginning, we recall the classical result of a deterministic inventory model to denote the cycle length being T and assume that the inventory level is $I(t)$ for $0\le t\le T$, with the initial inventory level, $I(0)=I_0$, and then the purchasing cost is I_0C_p. The minimum inventory level is $I(T)=0$, and the maximum inventory level is I_0 such that the average inventory level is $I_0/2$ and the holding cost for one replenishment cycle is $(I_0/2)TC_h$. The setup cost is C_0. The amelioration cost for one replenishment cycle is computed by the increasing weights, $RT-I_0$, times the unit amelioration cost, C_a to imply the total cost for one replenishment cycle, $TC(T)$. The average cost per unit of time, $AC(T)$, is computed by $AC(T)=\frac{TC(T)}{T}$ and then

$$AC(T)=I_0\left(\frac{C_p-C_a}{T}+\frac{C_h}{2}\right)+RC_a+\frac{C_o}{T}. \tag{3.1}$$

With the amelioration rate satisfies the Weibull distribution, with shape parameter, α, and scale parameter, β, the differential equation for the inventory level is expressed as follows,

$$\frac{d}{dt}I(t)=\alpha\beta t^{\beta-1}I(t)-R, \tag{3.2}$$

and $I(T)=0$. To solve Equation (3.2), it derives that

$$I(t)=e^{\alpha t^{\beta}}R\int_t^T e^{-\alpha x^{\beta}}dx, \tag{3.3}$$

and the initial inventory level is

$$I_0=I(0)=R\int_0^T e^{-\alpha x^{\beta}}dx, \tag{3.4}$$

and then we rewrite Equation (3.1) as

$$\mathrm{AC}(T)=\frac{C_o}{T}+C_aR+\left(\frac{C_h}{2}+\frac{C_p-C_a}{T}\right)R\int_0^T e^{-\alpha x^{\beta}}dx. \tag{3.5}$$

IV. Our Proposed Inventory Model under Fuzzy Environment

Next, we consider the inventory model under fuzzy environment with fuzzy demand, fuzzy purchasing cost, fuzzy amelioration cost, fuzzy inventory holding cost, and fuzzy ordering cost, then

$$\widetilde{AC}(T)=\frac{\widetilde{C_o}}{T}+\widetilde{C_a}\widetilde{R}+\left(\frac{\widetilde{C_h}}{2}+\frac{\widetilde{C_p}-\widetilde{C_a}}{T}\right)\widetilde{R}\int_0^T e^{-\alpha x^{\beta}}dx. \tag{4.1}$$

Applying the function principle (Chen [12]), the membership function of $A\tilde{C}(T)$ can be expressed as,

$$A\tilde{C}(T)=\left(f_1(T),f_2(T),f_3(T),f_4(T)\right), \tag{4.2}$$

where

$$f_i(T)=\frac{C_{oi}}{T}+C_{ai}R_i+\left(\frac{C_{hi}}{2}+\frac{C_{pi}-C_{ai}}{T}\right)R_i\int_0^T e^{-\alpha x^{\beta}}dx, \tag{4.3}$$

for $i=1,2,3,4$. Next, we adopt the median rule (Chen [13]) to find the minimum of $A\tilde{C}(T)$. The median, f_m, can be obtained as follows,

$$f_m=\left(f_1+f_2+f_3+f_4\right)/4. \tag{4.4}$$

Hence, we rewrite f_m as follows

$$f_m(T)=\frac{\sum_{i=1}^4 C_{hi}R_i}{8}\int_0^T e^{-\alpha x^{\beta}}dx+\frac{\sum_{i=1}^4 C_{ai}R_i}{4}+\frac{\sum_{i=1}^4(C_{pi}-C_{ai})R_i}{4T}\int_0^T e^{-\alpha x^{\beta}}dx+\frac{\sum_{i=1}^4 C_{oi}}{4T}. \tag{4.5}$$

V. Our Solution Procedure

From Equation (4.4), we assume that

$$\frac{df_m(T)}{dT}=\frac{1}{8T^2}g(T), \tag{5.1}$$

with a supplementary function $g(T)$ where

$$g(T)=T^2e^{-\alpha T^{\beta}}\sum_{i=1}^4 C_{hi}R_i-2\sum_{i=1}^4 C_{oi}, +2\left(Te^{-\alpha T^{\beta}}-\int_0^T e^{-\alpha x^{\beta}}dx\right)\sum_{i=1}^4(C_{pi}-C_{ai})R_i. \tag{5.2}$$

Since $g(T)$ and $\frac{df_m(T)}{dT}$ have the same sign, in the following, we will study the sign of $g(T)$ that will reflect the sign of $\frac{df_m(T)}{dT}$.

We divide our solution procedure into two Cases: Case (i) $0<\beta<1$, and Case (ii) $\beta=1$.

For Case (i), with $0<\beta<1$, from repeatedly using the Hospital rule, we know that

$$\lim_{T\to\infty}Te^{-\alpha T^{\beta}}=\lim_{T\to\infty}\frac{T}{e^{\alpha T^{\beta}}}=\lim_{T\to\infty}\frac{T^{1-\beta}}{\beta\alpha e^{\alpha T^{\beta}}}, =\frac{(1-\beta)}{(\beta\alpha)^2}\lim_{T\to\infty}\frac{T^{1-2\beta}}{e^{\alpha T^{\beta}}}=\frac{\prod_{j=1}^{n-1}(1-j\beta)}{(\beta\alpha)^n}\lim_{T\to\infty}\frac{T^{1-n\beta}}{e^{\alpha T^{\beta}}}. \tag{5.3}$$

since $0<\beta<1$, by the Archimedean principle, there is a natural number, say M, such that $1/\beta<M$ then it yields that

$$1-M\beta<0, \tag{5.4}$$

and then we derive that

$$\lim_{T\to\infty}\frac{T^{1-M\beta}}{e^{\alpha T^{\beta}}}=\lim_{T\to\infty}\frac{1}{e^{\alpha T^{\beta}}T^{M\beta-1}}=0. \tag{5.5}$$

Hence, we derive that

$$\lim_{T\to\infty}\left[Te^{-\alpha T^{\beta}}-\int_0^T e^{-\alpha x^{\beta}}dx\right]=\lim_{T\to\infty}-\int_0^T e^{-\alpha x^{\beta}}dx<0, \tag{5.6}$$

such that we obtain

$$\lim_{T\to\infty}g(T)=g(\infty)<0. \tag{5.7}$$

To examine the properties of $g(T)$, we know that

$$\frac{d}{dT}g(T)=T^{\beta}e^{-\alpha T^{\beta}}h(T), \tag{5.8}$$

where we assume an auxiliary function, denoted as h(T), where

$$\mathrm{h(T)}=2\sum_{i=1}^{4}\mathrm{C_{hi}R_i}\,\mathrm{T}^{1-\beta}-\beta\alpha\sum_{i=1}^{4}\mathrm{C_{hi}R_i}\,\mathrm{T}, \quad -2\beta\alpha\sum_{i=1}^{4}\left(\mathrm{C_{pi}}-\mathrm{C_{ai}}\right)\mathrm{R_i}. \tag{5.9}$$

VI. For the Case (i), $0<\beta<1$

For the Case (i), with $0<\beta<1$, it yields that

$$\frac{d}{dT}h(T)=2\sum_{i=1}^{4}R_iC_{hi}\left(\frac{1-\beta}{T^{\beta}}-\frac{\alpha\beta}{2}\right). \tag{6.1}$$

To simplify the expression of our following discussion, the unique positive root of Equation (6.1) is denoted as $T^{\&}$, to imply that

$$T^{\&}=\left(2(\lambda-1)/\alpha\right)^{\lambda}, \tag{6.2}$$

with $\lambda=1/\beta$. From Equation (6.1), we know that $\frac{d}{dT}h(T)>0$ for $T^{\&}>T>0$ and $\frac{d}{dT}h(T)<0$ for $T^{\&}<T$. Therefore, $h(T)$ is an increasing function for $T^{\&}>T>0$ and a decreasing function for $T^{\&}<T$. According to the sign of $h(T^{\&})$, the following lemmas 1-3 are provided to derive the main Theorem.

We divide our discussion into two Cases: Case (i-1), $h(T^{\&})\le 0$, and Case (i-2), $h(T^{\&})>0$.

Lemma 1. If $0<\beta<1$, and Case (i-1), with $h(T^{\&})\le 0$, then the optimal solution T^{*} occurs at infinity.

Proof. Based on

$$h(0)=-2\alpha\beta\sum_{i=1}^{4}R_i\left(C_{p\,i}-C_{a\,i}\right)<0, \tag{6.3}$$

$h(T)$ increases to $h(T^{\&})\le 0$ and then $h(T)$ decreases to $-\infty$, such that $h(T)\le 0$ for $0<T$. According to Equation (5.3), it shows that $\frac{d}{dT}g(T)\le 0$, for $0<T$, to imply that $g(T)$ decreases for $0<T$.

According to

$$g(0)=-2\sum_{i=1}^{4}C_{o\,i}<0, \tag{6.4}$$

it implies that $g(T)<0$ for $0<T$. By Equation (5.1), we know that $\frac{df_m(T)}{dT}<0$ for $0<T$. Hence, the minimum value will occur at infinite.

Next, we consider Case (i-2), with $h(T^{\&})>0$.

Under the condition of $h(T^{\&})>0$, $h(T)$ increases from,

$$h(0)=-2\alpha\beta\sum_{i=1}^{4}R_i\left(C_{pi}-C_{ai}\right)<0, \tag{6.5}$$

to

$$h(T^{\&})>0, \tag{6.6}$$

for $T^{\&}>T>0$, and then $h(T)$ decreases from $h(T^{\&})>0$ to $h(\infty)<0$. Hence, there are two points, say T_1 and T_2, with

$$0<T_1<T^{\&}<T_2, \tag{6.7}$$

satisfying

$$h(T_1)=0, \tag{6.8}$$

and

$$h(T_2)=0. \tag{6.9}$$

We obtained that $h(T)<0$, for $0<T<T_1$, $h(T)>0$ for $T_1<T<T_2$ and $h(T)<0$, for $T_2<T$.

Consequently, $g(T)$ decreases from the left-hand side boundary point,

$$g(0)=-2\sum_{i=1}^{4}C_{oi}<0, \tag{6.10}$$

to

$$g(T_1)<0, \tag{6.11}$$

for $0<T<T_1$ and then $g(T)$ increases from $g(T_1)<0$ to $g(T_2)$, for $T_1<T<T_2$. Moreover, $g(T)$ reduces from $g(T_2)$ to $g(\infty)<0$. According to the sign of $g(T_2)$, we divide the Case Case (i-2) into two sub-Cases, Case (i-2-1): $g(T_2)\le 0$, and Case (i-2-2): $g(T_2)>0$.

Under the condition of Case (i-2-1) with $g(T_2)\le 0$, we derive the next lemma.

Lemma 2. Under the following conditions, $0<\beta<1$, $h(T^{\&})>0$ and $g(T_2)\le 0$, then the optimal solution T^{*} occurs at infinity.

Proof. Owing to $g(T)\le 0$ for $T>0$, by Equation (5.1), it shows that $\frac{df_m(T)}{dT}<0$ for $T>0$. Hence, the minimum value will occur at infinity.

In the following, we consider Case (i-2-2) with $g\left(T_2\right)>0$.

Under the condition of $g\left(T_2\right)>0$, there are two points, say T_3 and T_4, that satisfy

$$T_1 < T_3 < T_2 < T_4, \quad (6.12)$$

with

$$g\left(T_3\right)=0, \quad (6.13)$$

and

$$g\left(T_4\right)=0. \quad (6.14)$$

It follows that $g(T)<0$ for $T_3 > T > 0$, $g(T)>0$ for $T_3 < T < T_4$, and $g(T)<0$ for $T_4 < T$.

By Equation (5.1), the objective function, $f_m(T)$, decreases for $0<T<T_3$ and increases for $T_3 < T < T_4$, such that T_3 is a local minimum. Moreover, T_4 is a local maximum as $f_m(T)$ decreases for $T_4 < T$. On the other hand, when $T \to \infty$ that is another local minimum. We summarize our results in the following lemma.

Lemma 3. If $0<\beta<1$, $h\left(T^{\&}\right)>0$ and $g\left(T_2\right)>0$, then there are two local minimum for $f_m(T)$ as T_3 and $T \to \infty$ where $T_3 < T_2$ and $g\left(T_3\right)=0$.

Based on our proposed three lemmas, the first main result for $0<\beta<1$ is derived in next theorem.

Theorem 1. Under the condition $0<\beta<1$, there are three Cases for the minimum point, T^*, and minimum value, $f_m\left(T^*\right)$:

(a) If $h\left(T^{\&}\right)>0$ and $g\left(T_2\right)>0$, then

$$f_m\left(T^*\right)=\min\left\{f_m(T_3), f_m(\infty)\right\}. \quad (6.15)$$

(b) If $h\left(T^{\&}\right)>0$ and $g\left(T_2\right)\le 0$, then $T^*=\infty$.

(c) If $h\left(T^{\&}\right)\le 0$, then $T^*=\infty$.

VII. For the Case $\beta=1$

Here, we begin to consider the Case (ii), with $\beta=1$. By Equation (6.1), it yields that

$$\frac{d}{dT}h(T)=-\alpha\beta\sum_{i=1}^{4}R_i C_{hi}<0, \quad (7.1)$$

so $h(T)$ is a decreasing function. According to the sign of $h(0)$, say

$$h(0)=2\sum_{i=1}^{4}R_i C_{hi}-2\alpha\beta\sum_{i=1}^{4}R_i\left(C_{pi}-C_{ai}\right), \quad (7.2)$$

we divide into two sub-Cases: (ii-1) $h(0)\le 0$, and (ii-2) $h(0)>0$.

First, we consider the Case (ii-1) with $h(0)\le 0$. Owing to $h(T)$ decreases from $h(0)\le 0$, we have that $h(T)<0$ for $T>0$ such that by Equation (5.8), $g(T)$ is a decreasing function. Since

$$g(0)=-2\sum_{i=1}^{4}C_{oi}<0, \quad (7.3)$$

it follows that $g(T)<0$ for $T>0$. Moreover, using Equation (5.1), $f_m(T)$ is a decreasing function to attain its minimum when $T\to\infty$. We summarize our findings in the following lemma.

Lemma 4. If $\beta=1$, and Case (ii-1), $h(0)\le 0$, then the optimal solution occurs at infinite.

Next, we consider the Case (ii-2) with $h(0)>0$. Since $h(T)$ is a decreasing function from $h(0)>0$ decreasing to $h(\infty)<0$ such that there is a point, say $T^{\%}$, with $h\left(T^{\%}\right)=0$. It indicates that

$$T^{\%}=\frac{2}{\alpha}-\frac{2\sum_{i=1}^{4}R_i\left(C_{pi}-C_{ai}\right)}{\sum_{i=1}^{4}R_i C_{hi}}=\frac{h(0)}{\alpha\sum_{i=1}^{4}R_i C_{hi}}. \quad (7.4)$$

For $0<T<T^{\%}$, it yields that $h(T)>0$ and for $T>T^{\%}$, we obtain that $h(T)<0$. Hence, from Equation (5.8), for $0<T<T^{\%}$, $g(T)$ increases, and for $T>T^{\%}$, $g(T)$ decreases.

According to the value of $g\left(T^{\%}\right)$, we divide the problem for $\beta=1$ and $h(0)>0$ into two sub-Cases, Case (ii-2-1): $g\left(T^{\%}\right)>0$ and Case (ii-2-2): $g\left(T^{\%}\right)\le 0$.

For Case (ii-2-2) with $g\left(T^{\%}\right)\le 0$, we derive that $g(T)\le 0$ for $T>0$ such that by Equation (5.1), $f_m(T)$ is a decreasing function to attain its minimum at infinite to imply the followimg lemma.

Lemma 5. If $\beta=1$, and $h(0)>0$, and $g\left(T^{\%}\right)\le 0$, then the optimal solution is attained when $T^*=\infty$.

Now, we consider the Case (ii-2-1) with $g\left(T^{\%}\right)>0$. According to $g(T)$ increases from $g(0)<0$ to $g\left(T^{\%}\right)>0$, and then $g(T)$ decreases from $g\left(T^{\%}\right)>0$

Table 1. The computation results from Taylor's series expansion

	$n=0$	$n=1$	$n=2$	$n=3$
$TC_n(T^*=T_3)$	1.049115×10^7	1.018879×10^7	1.019611×10^7	1.019598×10^7
$TC_n(3)$	1.070000×10^7	1.036796×10^7	1.037633×10^7	1.037619×10^7

to $g(\infty)<0$ such that there are two points, say T_5 and T_6 that satisfying

$$T_5 < T^{\%} < T_6, \tag{7.5}$$

with

$$g(T_5)=0, \tag{7.6}$$

and

$$g(T_6)=0. \tag{7.7}$$

Similar to our previous approach, it shows that T_6 is a local maximum, T_5 is a local minimum, and $T\to\infty$ is also a local minimum. We summarize our findings in the following lemma.

Lemma 6. For Case (ii), with $\beta=1$, Case (ii-2), with $h(0)>0$, and Case (ii-2-1) with $g(T^{\%})>0$, then there are two local minimum for $f_m(T)$ as T_5 and $T\to\infty$ where $T_5<T^{\%}$, and $g(T_5)=0$.

We combine our previous results for Case (ii), with $\beta=1$ into the following theorem.

Theorem 2. Under the condition of Case (ii), with $\beta=1$, there are three scenarios for the minimum point, T^*, and minimum value, $f_m(T^*)$:

(a) If $h(0)>0$, and $g(T^{\%})>0$, then

$$f_m(T^*)=\min\{f_m(T_5), f_m(\infty)\}. \tag{7.8}$$

(b) If $h(0)>0$ and $g(T^{\%})\le 0$, then $T^*=\infty$.

(c) If $h(0)\le 0$, then $T^*=\infty$.

VIII. Numerical Example

To compare with the results in Hwang [11], the same numerical example as Example 1 of his paper is considered where $R=1000$, $C_a=4000$, $C_h=400$, $C_o=300000$, $C_p=10000$, $\alpha=0.05$, $\beta=0.05$, and $\lambda=1/\beta$. It can be computed so that

$$T^{\&}=4.133\times10^{57}, \tag{8.1}$$

with $g(T^{\&})>0$, and then

$$T_1=3.158\times10^{-2}, \tag{8.2}$$

and

$$T_2=1.153\times10^{58}. \tag{8.3}$$

Since it is verified that $f(T_2)>0$, it can be determined that

$$T_3=1.316. \tag{8.4}$$

After comparing the values of $TC(T_3)$ and $\lim_{T\to\infty}TC(T)$, the minimum value occurs at T_3 with

$$TC(T^*=T_3)=1.019598\times10^7. \tag{8.5}$$

If the total cost is computed with the optimal solution by Hwang [11] with $T^*=3$, the result is that

$$TC(T=3)=1.037619\times10^7. \tag{8.6}$$

However, Hwang [11] claimed that the minimum solution will occur at $T^*=3$ with

$$TC(T^*=3)=0.9948303\times10^7. \tag{8.7}$$

To illustrate our solution is indeed the minimum value, Taylor's series expansion, as mentioned in Hwang [11], is referred to for computing the total cost

$$TC(T)=(C_p-C_a)R\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!(k\beta+1)}T^{k\beta} + \frac{C_h}{2}R\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!(k\beta+1)}T^{k\beta+1}+RC_a+\frac{C_o}{T}. \tag{8.8}$$

Motivated by Equation (8.8), it is assumed that for $n=0,1,2,3$, the partial sum for the first *n* terms in Equation (8.8) is denoted as $TC_n(T)$, which means

$$TC_n(T)=(C_p-C_a)R\sum_{k=0}^{n}\frac{(-\alpha)^k}{k!(k\beta+1)}T^{k\beta} + \frac{C_h}{2}R\sum_{k=0}^{n}\frac{(-\alpha)^k}{k!(k\beta+1)}T^{k\beta+1}+RC_a+\frac{C_o}{T}. \tag{8.9}$$

The computation results are listed in Table 1.

Table 1 shows that Taylor's series expansion of the total cost is an alternating series converging to its limit very fast. It also illustrates that the results in Equations (8.5) and (8.6) are correct, and the minimum cost is improved by our developed approach, and then the saving rate is 1.8%. The discrepancy in the result in Equation (8.7) from Hwang [11] may be caused by the inaccuracy resulting from the graphical procedure.

IX. A related Inventory System

In this section, we will examine an inventory system under outsourcing decision and production with two competitive manufacturers, manufacturer A and manufacturer B, to deliver partially substitutable items studied by Xiao et al. [14].

First, we introduce their notation:

c: The unit cost of producing the item for both manufacturers.

a_A: The retail price of the item delivered by manufacturer A, with $a_A > c$.

q_A: The quantity produced by manufacturer A.

d: The substitutability coefficient between items of manufacturers A and B, with $0 \le d \le 1$.

q_B: The quantity produced by manufacturer B.
a_B: The retail price of the item delivered by manufacturer B, with $a_B > c$.

We recall that Xiao et al. [14] considered two manufacturers A and B to maximize their profit system,

$$\pi_A(q_A) = (a_A - q_A - dq_B - c)q_A, \tag{9.1}$$

and

$$\pi_B(q_B) = (a_B - q_B - dq_A - c)q_B, \tag{9.2}$$

under the condition that both manufacturers delivered the items.

Xiao et al. [14] applied an analytic approach to find the maximum profit solution for the inventory system. We will use the algebraic method to seek the optimal solution.

X. Our Algebraic Approach

We rewrite Equation (9.1) in the ascending order of q_A as follows,

$$\pi_A(q_A) = -q_A^2 + (a_A - dq_B - c)q_A. \tag{10.1}$$

We begin to complete the square of q_A to derive that

$$\pi_A(q_A) = -\left[q_A^2 - 2\left(\frac{a_A - dq_B - c}{2}\right)q_A,\right.$$

$$\left.+\left(\frac{a_A - dq_B - c}{2}\right)^2\right] + \left(\frac{a_A - dq_B - c}{2}\right)^2. \tag{10.2}$$

We simplify the expression of Equation (10.2) to derive that

$$\pi_A(q_A) = \left(\frac{a_A - c - dq_B}{2}\right)^2,$$

$$-\left(q_A - \frac{a_A - c - dq_B}{2}\right)^2. \tag{10.3}$$

We observe that the coefficient of the square term, $\left(q_A - \frac{a_A - c - dq_B}{2}\right)^2$, in Equation (10.3), is -1, that is a negative number to attain the maximum profit value, we should have that

$$q_A = \frac{a_A - c - dq_B}{2}. \tag{10.4}$$

We plug the results of Equation (10.4) into Equation (9.2) and rewrite it in the ascending order of q_B as follows,

$$\pi_B(q_B) = \frac{2-d^2}{(-2)}(q_B)^2,$$

$$+\left(a_B - c - \frac{d}{2}(a_A - c)\right)q_B. \tag{10.5}$$

We begin to complete the square of q_B to derive that

$$\pi_B(q_B) = \frac{2-d^2}{(-2)}\left[(q_B)^2,\right.$$

$$+\frac{(-2)}{2-d^2}\left(a_B - c - \frac{d}{2}(a_A - c)\right)q_B,$$

$$\left.+\left(\frac{a_B - c - d(a_A - c)/2}{2-d^2}\right)^2\right],$$

$$+\frac{2-d^2}{(2)}\left(\frac{a_B - c - d(a_A - c)/2}{2-d^2}\right)^2. \tag{10.6}$$

We simplify the expression of Equation (9.8) to derive that

$$\pi_B(q_B) = \frac{2-d^2}{(2)}\left(\frac{a_B - c - d(a_A - c)/2}{2-d^2}\right)^2,$$

$$+\frac{2-d^2}{(-2)}\left[q_B - \frac{(a_B - c - d(a_A - c)/2)}{2-d^2}\right]^2. \tag{10.7}$$

We observe that the coefficient of the square term, $\left[q_B - \frac{(a_B - c - d(a_A - c)/2)}{2-d^2}\right]^2$, in Equation (10.7), is $\frac{2-d^2}{(-2)}$, that is a negative number to attain the maximum profit value, we should have that

$$q_B = \frac{(a_B - c - d(a_A - c)/2)}{2-d^2}. \tag{10.8}$$

After we derive the optimal solution of q_B, we plug our findings from Equation (10.8) into Equation (10.4) to derive the optimal solution, denoted as q_A. It follows,

$$q_A = \frac{a_A - c - dq_B}{2},$$

$$= \frac{a_A - c}{2} - \frac{d}{2}\frac{(a_B - c - d(a_A - c)/2)}{2-d^2}. \tag{10.9}$$

We begin to simplify the results of Equation (10.9), and then we obtain that

$$q_A = \frac{(a_A - c)(2-d^2)}{2(2-d^2)} - \frac{d(a_B - c)}{2(2-d^2)},$$

$$+\frac{(d^2/2)(a_A - c)}{2(2-d^2)}. \tag{10.10}$$

Motivated by Equation (10.10), we further simplify the expression of to q_A find that

$$q_A = \frac{(a_A - c)(2 - (d^2/2))}{2(2-d^2)} - \frac{d(a_B - c)}{2(2-d^2)}. \tag{10.11}$$

XI. An Alternative Approach

Based on our findings of Equations (10.8) and (10.11), we derive that

$$\pi_A = \frac{[(4-d^2)(a_A - c) - 2d(a_B - c)]^2}{16(2-d^2)^2}, \tag{11.1}$$

and

$$\pi_B = \frac{[2(a_B - c) - d(a_A - c)]^2}{8(2-d^2)}. \tag{11.2}$$

We observe our results of Equations (10.8) and (10.11) to reveal that they are not symmetric to the index of A and B. Similarly, we examine our results of Equations (11.1) and (11.2) to realize that they are also not symmetric respect to the index A and B.

We recall that the expressions of Equations (9.1) and (9.2) are symmetric with respect to the index A and B. Consequently, it is reasonable to predict that the optimal solutions should be symmetric with respect to the index A and B. Hence, in the following, we present an alternative approach.

We rewrite Equation (9.2) in the ascending order of q_B and then complete the square of q_B to derive that

$$\pi_B(q_B) = \left(\frac{a_B - c - dq_A}{2}\right)^2 - \left(q_B - \frac{a_B - c - dq_A}{2}\right)^2. \quad (11.3)$$

To derive the maximum value of Equation (11.3), we find that

$$q_B = \frac{a_B - c - dq_A}{2}. \quad (11.4)$$

We solve the system consisting of Equations (10.4) and (11.4) to imply that

$$q_A = \frac{2(a_A - c) - (a_B - c)d}{4 - d^2}, \quad (11.5)$$

and

$$q_B = \frac{2(a_B - c) - (a_A - c)d}{4 - d^2}. \quad (11.6)$$

We observe our new findings of Equations (11.5) and (11.6) to know that they are symmetric with respect to the index systems A and B.

We plug the results of Equations (11.5) and (11.6) into Equations (9.1) and (9.2) to yield that

$$\pi_A = \frac{[2(a_A - c) - d(a_B - c)]^2}{(4 - d^2)^2} = (q_A)^2, \quad (11.7)$$

and

$$\pi_B = \frac{[2(a_B - c) - d(a_A - c)]^2}{(4 - d^2)^2} = (q_B)^2. \quad (11.8)$$

We observe our new findings of Equations (11.7) and (11.8) to know that they are symmetric with respect to the index systems A and B. Therefore, we derive an alternative approach to obtain symmetric results for the optimal point and profit.

XII. Direction for Future Research

In this section, we provide some possible directions for future research. Wang and Chiang [15] examined the ordered weighted averaging operator to provide a further study. Chen and Cheng [16] developed a simple algorithm to evaluate the order time duration for inventory models with a linear demand. Wang et al. [17] considered the consistency test in the analytic hierarchy process to point out several questionable results in previously published papers. Cheng and Chen [18] studied contradictory pairwise comparison matrices in the analytic hierarchy process to show that those improvements contained questionable findings. Prasetyo et al. [19] examined the fuel subsidies in Indonesia using clustering large applications, partitioning around medoids, and K-Means. Aripin et al. [20] amend compound emotional text classification performance in the multichannel convolutional neural network model. Wu [21] constructed a new inventory model under fuzzy restriction and fuzzy demand to derive a formulated optimal solution. Using naive Bayes classifier and Bayesian logistic regression, Yanuar et al. [22] classified death risk for those COVID-19 patients. Referring to Covid-19 Cases, Novianti et al. [23] developed geographically weighted logistic regression models with spatial binomial data to examine whether there is non-stationarity. For the flooded passenger vehicles, Al-Qadami et al. [24] examined a 3-dimensional numerical study on the critical orientation. According to the improved Salp swarm algorithm, Long et al. [25] considered the optimal allocation of DGs in radial distribution networks. Sulistiawanti et al. [26] used multivariate exponentially weighted moving variance charts and multivariate exponentially weighted moving averages to monitor water quality under residual XGBoost regression. Based on our above discussion, researchers can find hot study spots for their future developed models and algorithms.

XIII. Conclusion

This paper presents an analytical procedure to address inventory systems designed for handling manufacturing items in an ameliorating environment. The optimality of our proposed analytical procedure is demonstrated, and a numerical example cited from Hwang [11] is referred to for comparison. Our findings on the numerical example also reveal a considerable approximation improvement compared to the graphical procedure used by Hwang [11]. Based on our developed theorems, researchers can systematically determine where to locate the optimal replenishment cycle under various conditions.

References

[1] N. K. Gupta, "Effect of lead time on inventory — a working result," *Operations Research Society*, vol. 30,1982, pp. 477-481.

[2] H. S. Hwang, "Inventory model for both deteriorating and ameliorating," *Computers and Industrial Engineering*, vol. 37, 1999, pp. 257-260.

[3] B. Mondal, A. K. Bhunia, M. Maiti, "An inventory system of ameliorating for price dependent demand rate," *Computers and Industrial Engineering*, vol. 45, 2003, pp. 443-456.

[4] R. R. Giri, A. K. Jalan, K. S. Chaudhuri, "Economic order quantity model with Weibull distribution deterioration, shortages and ramp-type demand," *International Journal of Systems Science*, vol. 34, 2003, pp. 237-243.

[5] K. S. Wu, "EOQ inventory model for items with Weibull distribution deterioration, time-varying demand and partial backlogging," *International journal of Systems Science*, vol. 33, 2002, pp. 323-329.

[6] K. S. Wu, "An EOQ inventory model for items with Weibull distribution deterioration, ramp type demand rate and partial backlogging," *Production Planning & Control*, vol. 12, 2001, pp. 787-793.

[7] H. J. Chang, C. Y. Dye, "An EOQ model with deteriorating items in response to a temporary sale price," *Production Planning & Control*, vol. 11, 2000, pp. 464-473.

[8] J. W. Wu, C. Lin, B. Tan, W. C. Lee, "An EOQ model with time-varying demand and Weibull deterioration with shortages," *International journal of Systems Science*, vol. 31, 2000, pp. 677-683.

[9] T. Chakrabarty, R. R. Giri, K. S. Chaudhuri, "An EOQ model for items with Weibull distribution deterioration, shortages and trended demand: an extension of Philp's model," *Computer and Operations Research*, vol. 25, 1998, pp. 649-657.

[10] A. K. Jalan, R. R. Giri, K. S. Chaudhuri, "EOQ model for items with Weibull distribution deterioration, shortages and trend demand," *International journal of Systems Science*, vol. 27, 1996, pp. 851-855.

[11] H. S. Hwang, "A study on an inventory model for items with Weibull ameliorating," *Computers and Industrial Engineering*, vol. 33, 1997, pp. 701-704.

[12] S. H. Chen, "Operation on fuzzy numbers with Function Principle," *Tamkang Journal of Management Science*, vol. 6, 1985, pp. 13-25.

[13] S. M. Chen, "Evaluating the Rate of Aggregative Risk in Software Development Using Fuzzy Set Theory," Cybernetics and Systems: International Journal, vol. 30, 1999, pp. 57-75.

[14] T. Xiao, Y. Xia, G. P. Zhang, "Strategic Outsourcing Decisions for Manufacturers that Produce Partially Substitutable Products in a Quantity-Setting Duopoly Situation," *Decision Sciences,* vol. 38, no. 1, 2007, pp. 81-105.

[15] Y. L. Wang, T. Y. Chiang, "Further Study for the Ordered Weighted Averaging Operator," IAENG International Journal of Applied Mathematics, vol. 53, no. 3, 2023, pp. 810-817.
[16] K. H. Chen, Y. N. Cheng, "A Simple Algorithm to Estimate the Order Time Interval with a Linear Demand," IAENG International Journal of Applied Mathematics, vol. 53, no. 3, 2023, pp. 985-993.
[17] C. M. Wang, Y. L. Wang, M. L. Chen, "Discussion for Consistency Test in Analytic Hierarchy Process," IAENG International Journal of Applied Mathematics, vol. 53, no. 3, 2023, pp. 1123-1131.
[18] Y. N. Cheng, K. H. Chen, "Study for Contradictory Pairwise Comparison Matrices," IAENG International Journal of Applied Mathematics, vol. 53, no. 3, 2023, pp. 1138-1147.
[19] F. A. Prasetyo, R. E. Caraka, Y. Kim, N. E. Goldameir, Sulistyowati, A. E. Tyasti, P. U. Gio, F. Anggoro, M. Ramadhani, B. Pardamean, "Revisiting Fuel Subsidies in Indonesia using K-Means, PAM, and CLARA," IAENG International Journal of Computer Science, vol. 50, no. 3, 2023, pp. 858-865.
[20] Aripin, W. Agastya, S. Huda, "Multichannel Convolutional Neural Network Model to Improve Compound Emotional Text Classification Performance," IAENG International Journal of Computer Science, vol. 50, no. 3, 2023, pp. 866-874.
[21] L. C. Wu, "Formulated Optimal Solution for EOQ Model with Fuzzy Demand," IAENG International Journal of Computer Science, vol. 50, no. 3, 2023, pp. 890-898.
[22] F. Yanuar, M. Q. Shobri, R. F. Mabrur, I. H. Putri, A. Zetra, "Classification of Death Risk for COVID-19 Patients Using Bayesian Logistic Regression and Naive Bayes Classifier," IAENG International Journal of Computer Science, vol. 50, no. 3, 2023, pp. 915-920.
[23] P. Novianti, Gunardi, D. Rosadi, "Geographically Weighted Logistic Regression Model on Binomial Data to Explore Weather Spatial Non-Stationarity in Covid-19 Cases," Engineering Letters, vol. 31, no. 3, 2023, pp. 938-947.
[24] E. H. H. Al-Qadami, M. A. M. Razi, W. S. Damanik, Z. Mustaffa, E. Martinez-Gomariz, F. Y. Teo, A. A. H. Saeed, "3-Dimensional Numerical Study on the Critical Orientation of the Flooded Passenger Vehicles," Engineering Letters, vol. 31, no. 3, 2023, pp. 961-968.
[25] H. Long, S. Pan, T. Xu, X. Liu, S. Xia, C. Yao, H. Lu, "Optimal Allocation of DGs in Radial Distribution Networks Based on Improved Salp Swarm Algorithm," Engineering Letters, vol. 31, no. 3, 2023, pp. 969-986.
[26] N. Sulistiawanti, M. Ahsan, H. Khusna, "Multivariate Exponentially Weighted Moving Average (MEWMA) and Multivariate Exponentially Weighted Moving Variance (MEWMV) Chart Based on Residual XGBoost Regression for Monitoring Water Quality," Engineering Letters, vol. 31, no. 3, 2023, pp. 1001-1008.

Shusheng Wu received his Master's degree in 2010 from the Department of Computational Mathematics, Ocean University of China. He is an Instructor at the School of Mathematics and Science Teaching Center, Weifang University of Science and Technology. The main research directions are Inventory Models, Analytical Hierarchy Process, Fuzzy Set Theorem, Algebraic Methods in Operational Research, and Pattern Recognition.

Jinyuan Liu received his Master's degree from the School of Mathematics and Systems Science, Shandong University of Science and Technology in 2009. He is an Associate Professor at the School of General Studies of Weifang University of Science and Technology. The main research directions are Pattern Recognition, Lanchester's Model, Fuzzy Set Theorem, Isolate Points, Analytical Hierarchy Process, and Inventory Models.

Te-Yuan Chiang is an Associate Professor, at the School of Intelligent Manufacturing, Weifang University of Science and Technology. He received his Ph.D. degree from the Department of Mechanical Engineering, Yuan Ze University, in 2015. His research interest includes Management Science, Mechanical and Materials Science, Management Information Systems, Artificial Intelligence, Pattern Recognition, and Image Processing.

Gino Yang received his Doctorate from the Department of Industrial Engineering and Management, National Chiao Tung University, Hsinchu, in 2004. He is a Professor at the Department of Multimedia Game Development and Application, Hungkuang University. His primary research topics are Multimedia Game Development, Multimedia Applications, Analytical Hierarchy Processes, Pattern Recognition, Inventory Systems, and Fuzzy Set Thoeram.