Dual Mixed Orlicz-Brunn-Minkowski Inequality and The General Dual Orlicz Mixed Volume

Ping Zhang, Xiaohua Zhang

Abstract—**In this paper, we initially establish the dual mixed Orlicz-Brunn-Minkowski inequalities for the dual mixed volume, and then introduce the definition of the general dual Orlicz mixed volume based on the Orlicz radial sum. Subsequently, we derive the dual Orlicz-Minkowski inequalities for the general dual Orlicz mixed volume. Finally, we derive the dual mixed Log-Minkowski inequality.**

Index Terms—**Orlicz radial addition, dual Orlicz mixed volume, the general dual Orlicz mixed volume, dual Orlicz-Brunn-Minkowski inequality, dual Orlicz-Minkowski inequality.**

I. Introduction

THE combination of Minkowski sums with volumes gives rise to the extensive and powerful classical Brunn-Minkowski theory of convex bodies, which refers to compact convex subsets with nonempty interiors([12]). Similarly, Lutwak introduced the dual Brunn-Minkowski theory of star bodies, providing further details on this topic([6], [7]). In order to expand upon the renowned Brunn-Minkowski theory and its duality, Firey introduced a modification by replacing the linear function $\phi(u) = u$ with $\phi(u) = u^p$([1]). Furthermore, in order to extend the renowned Brunn-Minkowski theory into the Orlicz-Brunn-Minkowski theory, Lutwak made a substitution of a homogeneous function $\phi(u) = u^p$ with a nonhomogeneous function $\phi(u)$([9],[10]). Recently, Gardner, Hug, and Weil (2014) developed a comprehensive framework for the Orlicz-Brunn-Minkowski theory, encompassing both Orlicz addition and Minkowski addition([3]). In a more recent study, Ye introduced an Orlicz $\phi-$radial addition of multiple star bodies and established the corresponding dual Orlicz-Brunn-Minkowski inequalities([15]). Additionally, in [18], Zhu, Zhou, and Xu established the dual Orlicz-Brunn-Minkowski theory for star bodies. Wang, Shi, and Ye derived the dual Orlicz-Brunn-Minkowski inequalities for dual quermassintegrals([14]). Moreover, Ma and Wang had successfully established the dual Brunn-Minkowski inequality for the novel dual Orlicz harmonic mixed quermassintegrals([11]). Furthermore, Gardner, Hug, Weil, and Ye provided a broader conceptual framework and more comprehensive findings for two or more star sets([4]).

The first objective of this paper is to establish the dual mixed Orlicz-Brunn-Minkowski inequality for the dual mixed volume of the Orlicz radial sum, as stated in Theorem 2.1: Let $H, L, H_{i+1}, \cdots, H_n \in \mathcal{S}_o^n$, real $i \neq 0$, $\phi \in \Phi_2$ or $\phi \in \Psi_2$, and $G_\phi(u,v) = \phi(\frac{1}{u^{\frac{1}{i}}}, \frac{1}{v^{\frac{1}{i}}})$. If G_ϕ is convex, then

$$\phi\left[\left(\frac{\hat{V}(H\tilde{+}_\phi L,\cdots,H\tilde{+}_\phi L,H_{i+1},\cdots,H_n)}{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}\right)^{\frac{1}{i}},\left(\frac{\hat{V}(H\tilde{+}_\phi L,\cdots,H\tilde{+}_\phi L,H_{i+1},\cdots,H_n)}{\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)}\right)^{\frac{1}{i}}\right] \leq 1,$$

the equality holds if and only if H and L are dilates, provided that G_ϕ is strictly convex. In the case of concave G_ϕ, the inequality is reversed.

Definition 3.1 introduces the general dual Orlicz mixed volume by utilizing the Orlicz radial sum as its foundation: For $H, L, H_{i+1}, \cdots, H_n \in \mathcal{S}_o^n$, if i is real, $\phi : (0,\infty) \to \mathbb{R}$, the general dual Orlicz mixed volume, $\hat{V}_{\phi,i}(H,\cdots,H,L,H_{i+1},\cdots,H_n)$, of $H, L, H_{i+1}, \cdots, H_n$ is defined by

$$\hat{V}_{\phi,i}(H,\cdots,H,L,H_{i+1},\cdots,H_n) = \frac{1}{n}\int_{S^{n-1}} \phi(\frac{\rho(H,w)}{\rho(L,w)}) \cdot\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w).$$

The dual Orlicz-Minkowski inequality is established in Theorem 3.2 based on the definition of the general dual Orlicz mixed volume, which can be formulated as follows: Let $H, L, H_{i+1}, \cdots, H_n \in \mathcal{S}_o^n, i \neq 0$, $\phi : (0,+\infty) \to \mathbb{R}$, and $G(u) = \phi(\frac{1}{u^{\frac{1}{i}}}), u > 0$. If G is convex, then

$$\begin{aligned}&\hat{V}_{\phi,i}(H,\cdots,H,L,H_{i+1},\cdots,H_n)\\ &\geq \hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)\\ &\cdot\phi\left[\left(\frac{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}{\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)}\right)^{\frac{1}{i}}\right],\end{aligned}$$

the equation holds if and only if H and L are dilates, provided that G is strictly convex. Conversely, if G is concave, the preceding inequality is reversed.

Furthermore, Theorem 3.3 establishes the general dual mixed Log-Minkowski inequality.

II. The Dual Mixed Orlicz-Brunn-Minkowski Inequality

Let $\mathbb{R}^n$ denote the Euclidean space, where B represents the unit ball centered at the origin and its surface is denoted by S^{n-1}. The radial function $\rho(H,w)$ of a compact set $H \in \mathbb{R}^n$ (where H is star-shaped with respect to the origin) is defined by(see [2])

$$\rho(H,w) = \max\{\lambda \geq 0 : \lambda w \in H\}, \qquad w \in S^{n-1},$$

the term “star body (about the origin) ” will be assigned to H if $\rho(H,w)$ is positive and continuous. If $\rho(H,w)/\rho(L,w)$

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does not depend on $w \in S^{n-1}$, we say that H and L are dilates with each other. Let $\mathcal{S}_o^n \in \mathbb{R}^n$ denote the set of star bodies centered at the origin, $\hat{V}(H_1, H_2, \cdots, H_n)$ represent the dual mixed volume of $H_1, H_2, \cdots, H_n \in \mathcal{S}_o^n$, while $V(H) = \hat{V}(H, H, \cdots, H)$ denotes the n-dimensional volume of a star body $H \in \mathcal{S}_o^n$. The integral representation $\hat{V}(H_1, \cdots, H_n)$ of $H_1, H_2, \cdots, H_n \in \mathcal{S}_o^n$ is given as follows([7]):

$$\hat{V}(H_1, \cdots, H_n) = \frac{1}{n} \int_{S^{n-1}} \rho(H_1, w) \cdots \rho(H_n, w) dS(w),$$

where $dS(w)$ is the standard spherical Lebesgue measure on S^{n-1}.

Let Φ_m (m is a finite positive integer) be the set of functions $\phi : [0, \infty)^m \to [0, \infty), \forall x \in [0, \infty)^m \backslash \{o\}$, and ϕ satisfies([4], [15]):

(1) $\phi \in \Phi_m$ and ϕ is continuous on $[0, \infty)^m \backslash \{o\}$;
(2) $\phi \in \Phi_m$ and ϕ is strictly increasing on $[0, \infty)^m \backslash \{o\}$;
(3) $\phi(o) = 0, \lim_{t \to \infty} \phi(tx) = \infty$.

Let Ψ_m (m is a finite positive integer) be the set of functions $\psi : (0, \infty)^m \to (0, \infty), \forall x \in (0, \infty)^m$, and ψ satisfies:

(1) $\psi \in \Psi_m$ and ψ is continuous on $(0, \infty)^m$;
(2) $\psi \in \Psi_m$ and ψ is strictly decreasing on $(0, \infty)^m$;
(3) $\lim_{t \to 0} \psi(tx) = \infty, \lim_{t \to \infty} \psi(tx) = 0$.

The definition of the Orlicz radial sum of two star bodies was previously stated in ([4],[14],[15]) as follows:

Definition 2.A. *For $H, L \in \mathcal{S}_o^n$, $\phi \in \Phi_2$ or $\phi \in \Psi_2$, the Orlicz radial sum, $H \tilde{+}_\phi L \in \mathcal{S}_o^n$, of H and L is defined implicitly by*

$$\phi\left(\frac{\rho(H \tilde{+}_\phi L, w)}{\rho(H, w)}, \frac{\rho(H \tilde{+}_\phi L, w)}{\rho(L, w)} \right) = 1, \forall w \in S^{n-1}. \quad (2.1)$$

If $\phi(u, v) = \lambda u^q + \beta v^q (\lambda, \beta > 0, u, v > 0, q \geq 1)$, Definition 2.A yields the Lp-harmonic radial combination, $\lambda \circ H \tilde{+}_{-q} \beta \circ L$, of H and L, their radial function satisfies([14], [17]):

$$\rho(\lambda \circ H \tilde{+}_{-q} \beta \circ L, w)^{-q} = \lambda \rho(H, w)^{-q} + \beta \rho(L, w)^{-q}, \quad \forall w \in S^{n-1}.$$

If $\phi(u, v) = \lambda u^{-q} + \beta v^{-q} (\lambda, \beta > 0, u, v > 0, q > 0)$, Definition 2.A yields the Lp-radial combination, $\lambda \circ H \tilde{+}_q \beta \circ L$, of H and L, their radial function satisfies([14], [17]):

$$\rho(\lambda \circ H \tilde{+}_q \beta \circ L, w)^q = \lambda \rho(H, w)^q + \beta \rho(L, w)^q, \quad \forall w \in S^{n-1}.$$

The dual Orlicz-Brunn-Minkowski inequalities of the Orlicz radial addition were established in ([4],[14]).

In this paper, we first establish the dual mixed Orlicz-Brunn-Minkowski inequalities for dual mixed volume. To prove Theorem 2.1, we require the utilization of Jensen's inequalities ([5]):

Lemma 2.1. Suppose that μ is a probability measure on a space Y and $f : Y \to I \subset \mathbb{R}$ is a $\mu-$integrable function, where I is a possibly infinite interval.

If $\phi : I \to \mathbb{R}$ is convex, then

$$\int_Y \phi(f(y)) d\mu(y) \geq \phi\left(\int_Y f(y) d\mu(y) \right), \quad (2.2)$$

under strict convexity of ϕ, equality holds if and only if $f(y)$ is constant for $\mu-$almost all $y \in Y$.

If $\phi : I \to \mathbb{R}$ is concave, then

$$\int_Y \phi(f(y)) d\mu(y) \leq \phi\left(\int_Y f(y) d\mu(y) \right), \quad (2.3)$$

under strict convexity of ϕ, equality holds if and only if $f(y)$ is constant for $\mu-$almost all $y \in Y$.

Theorem 2.1. *Let $H, L, H_{i+1}, \cdots, H_n \in \mathcal{S}_o^n$, $\phi \in \Phi_2$ or $\phi \in \Psi_2$, real $i \neq 0$, and $G_\phi(u, v) = \phi(\frac{1}{u^{\frac{1}{i}}}, \frac{1}{v^{\frac{1}{i}}})$. If G_ϕ is convex, then*

$$\phi\left[\left(\frac{\hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)}{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)} \right)^{\frac{1}{i}}, \left(\frac{\hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)}{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n)} \right)^{\frac{1}{i}} \right] \leq 1, \quad (2.4)$$

the equality holds if and only if H and L are dilates, provided that G_ϕ is strictly convex. Conversely, when G_ϕ is concave, the inequality is reversed.

Proof. *Since $G_\phi(u, v) = \phi(\frac{1}{u^{\frac{1}{i}}}, \frac{1}{v^{\frac{1}{i}}})$, we have*

$$\phi(u, v) = G_\phi\left((\frac{1}{u})^i, (\frac{1}{v})^i \right). \quad (2.5)$$

Let $H, L, H_{i+1}, \cdots, H_n \in \mathcal{S}_o^n$, assuming the convexity of G_ϕ, by utilizing the Orlicz sum of two star bodies (2.1), the integral representation of dual mixed volume, (2.2), and (2.5), we obtain

$$\begin{aligned}
1 &= \frac{\int_{S^{n-1}} \rho^i(H \tilde{+}_\phi L, w) \rho(H_{i+1}, w) \cdots \rho(H_n, w) dS(w)}{n \hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)} \\
&= \int_{S^{n-1}} \phi\left(\frac{\rho(H \tilde{+}_\phi L, w)}{\rho(H, w)}, \frac{\rho(H \tilde{+}_\phi L, w)}{\rho(L, w)} \right) \\
&\quad \cdot \frac{\rho^i(H \tilde{+}_\phi L, w) \rho(L_{i+1}, w) \cdots \rho(H_n, w) dS(w)}{n \hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)} \\
&= \int_{S^{n-1}} G_\phi\left[\left(\frac{\rho(H, w)}{\rho(H \tilde{+}_\phi L, w)} \right)^i, \left(\frac{\rho(L, w)}{\rho(H \tilde{+}_\phi L, w)} \right)^i \right] \\
&\quad \cdot \frac{\rho^i(H \tilde{+}_\phi L, w) \rho(H_{i+1}, w) \cdots \rho(H_n, w) dS(w)}{n \hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)} \\
&\geq G_\phi\left(\frac{\int_{S^{n-1}} \rho^i(H, w) \rho(H_{i+1}, w) \cdots \rho(H_n, w) dS(w)}{n \hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)}, \right. \\
&\quad \left. \frac{\int_{S^{n-1}} \rho^i(L, w) \rho(H_{i+1}, w) \cdots \rho(H_n, w) dS(w)}{n \hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)} \right) \\
&= G_\phi\left(\frac{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)}{\hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)}, \right. \\
&\quad \left. \frac{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n)}{\hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)} \right) \\
&= \phi\left[\left(\frac{\hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)}{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)} \right)^{\frac{1}{i}}, \right. \\
&\quad \left. \left(\frac{\hat{V}(H \tilde{+}_\phi L, \cdots, H \tilde{+}_\phi L, H_{i+1}, \cdots, H_n)}{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n)} \right)^{\frac{1}{i}} \right], \quad (2.6)
\end{aligned}$$

if G_ϕ is strictly convex, equality conditions of Jensen's inequality imply that (2.6) holds with equality if and only if there exist constants c_1 and c_2 such that

$$\frac{\rho(H\tilde{+}_\phi L,w)}{\rho(H,w)}=c_1,\frac{\rho(H\tilde{+}_\phi L,w)}{\rho(L,w)}=c_2,\forall w\in S^{n-1},$$

i.e., $\rho(H,w)/\rho(L,w)=c_2/c_1,\forall w\in S^{n-1}$. This indicates that H and L are dilates, then we get the inequality (2.4).

Similarly, if G_ϕ is strictly concave, by (2.1), Jensen's inequality (2.3) and (2.5), we get

$$\phi\Bigg[\Bigg(\frac{\hat{V}(H\tilde{+}_\phi L,\cdots,H\tilde{+}_\phi L,H_{i+1},\cdots,H_n)}{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}\Bigg)^{\frac{1}{i}},\Bigg(\frac{\hat{V}(H\tilde{+}_\phi L,\cdots,H\tilde{+}_\phi L,H_{i+1},\cdots,H_n)}{\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)}\Bigg)^{\frac{1}{i}}\Bigg]\geq 1, \tag{2.7}$$

the strict concavity of G_ϕ implies that equality holds in (2.7) if and only if H and L are dilates. Consequently, we obtain the Theorm 2.1.

Clearly, if $i=n$ in Theorem 2.1, we derive the dual Orlicz-Brunn-Minkowski inequalities([15]). If $H_{i+1}=\cdots=H_n=B$ in Theorem 2.1, we obtain the dual Orlicz-Brunn-Minkowski inequalities for the dual quermassintegrals([14]).

And moreover, in ([4],[14],[15]), the authors introduced the linear Orlicz radial combination of two star bodies in the following manner:

Definition 2.B. *For $H,L\in\mathcal{S}_o^n,a,b>0,\phi(u,v)=a\phi_1(u)+b\phi_2(v)$, where ϕ_1,ϕ_2 are either both in Φ_1 or both in Ψ_1, the linear Orlicz radial combination, $a\circ H\tilde{+}_{\phi_1,\phi_2}b\circ L$, of H and L is defined by*

$$a\phi_1\Bigg(\frac{\rho(a\circ H\tilde{+}_{\phi_1,\phi_2}b\circ L,w)}{\rho(H,w)}\Bigg)+b\phi_2\Bigg(\frac{\rho(a\circ H\tilde{+}_{\phi_1,\phi_2}b\circ L,w)}{\rho(L,w)}\Bigg)=1,\forall w\in S^{n-1}. \tag{2.8}$$

On the basis of Definition 2.B, the following dual Orlicz-Brunn-Minkowski inequalities are derived:

Corollary 2.1. *Let $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n,i\neq 0,a,b>0$, $\phi(u,v)=a\phi_1(u)+b\phi_2(v)$, $G_1(u)=\phi_1(\frac{1}{u^{\frac{1}{i}}})$, $G_2(v)=\phi_2(\frac{1}{v^{\frac{1}{i}}})$, where $\phi_1,\phi_2\in\Phi_1$ or $\phi_1,\phi_2\in\Psi_1$. And for simplicity, let $\hat{V}(T)=\hat{V}(a\circ H\tilde{+}_{\phi_1,\phi_2}b\circ L,\cdots,a\circ H\tilde{+}_{\phi_1,\phi_2}b\circ L,H_{i+1},\cdots,H_n)$. If both G_1 and G_2 are convex, then*

$$a\phi_1\Bigg[\Bigg(\frac{\hat{V}(T)}{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}\Bigg)^{\frac{1}{i}}\Bigg]+b\phi_2\Bigg[\Bigg(\frac{\hat{V}(T)}{\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)}\Bigg)^{\frac{1}{i}}\Bigg]\leq 1, \tag{2.9}$$

the equality in (2.9) is achieved if and only if both G_1 and G_2 are strictly convex, indicating that H and L must be dilates. Conversely, if both G_1 and G_2 are concave, the inequality is reversed.

Remark. (1) If $\phi_1(u)=u^p(u>0)$ and $\phi_2(v)=v^p(v>0)$ with $p\geq 1$ in Corollary 2.1, when $-\frac{p}{i}>1$(*or* $-\frac{p}{i}<0$), we have $G_1(u)=\phi_1(\frac{1}{u^{\frac{1}{i}}})=u^{\frac{-p}{i}},G_2(v)=\phi_2(\frac{1}{v^{\frac{1}{i}}})=v^{\frac{-p}{i}}$, we get that both G_1 and G_2 are convex; when $0<-\frac{p}{i}<1$, we obtain that both G_1 and G_2 are concave. Combining the Lp radial harmonic combination and Corollary 2.1, the Lp dual Brunn-Minkowski inequality is derived in the following manner([8]):

If $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n$, and $-\frac{p}{n}<0,p\geq 1,a,b>0$,

$$V(a\circ H\tilde{+}_{-p}b\circ L)^{-\frac{p}{n}}\geq aV(H)^{-\frac{p}{n}}+bV(L)^{-\frac{p}{n}},$$

the equality holds if and only if H and L are dilates.

(2) If $\phi_1(u)=u^{-p}(u>0)$ and $\phi_2(v)=v^{-p}(v>0)$ with $p>0$ in Corollary 2.1, when $0<\frac{p}{i}<1$, we have $G_1(u)=\phi_1(\frac{1}{u^{\frac{1}{i}}})=u^{\frac{p}{i}},G_2(v)=\phi_2(\frac{1}{v^{\frac{1}{i}}})=v^{\frac{p}{i}}$, we get that both G_1 and G_2 are concave; when $\frac{p}{i}>1$(or $\frac{p}{i}<0$), we obtain that both G_1 and G_2 are convex. Combining the Lp radial combination and Corollary 2.1, the Lp dual Brunn-Minkowski inequality is derived in the following manner([13]):

If $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n$, and for $0<\frac{1}{i}<1,a,b>0$, we have

$$\hat{V}(a\circ H\tilde{+}b\circ L,\cdots,a\circ H\tilde{+}b\circ L,H_{i+1},\cdots,H_n)^{\frac{1}{i}}\leq a\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)^{\frac{1}{i}}+b\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)^{\frac{1}{i}},$$

equality holds true under the condition that H and L are dilates; for $\frac{1}{i}>1,a,b>0$, the inequality is reversed.

If $\phi_1=\phi_2=\phi$, we write $a\circ H\tilde{+}_{\phi_1,\phi_2}b\circ L=b\circ H\tilde{+}_\phi(1-b)\circ L$, from Corollary 2.1, the following Corollary 2.2 is obtained.:

Corollary 2.2. *Let $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n$, real $i>0,0<a<1$, ϕ (with second derivative): $(0,\infty)\to(0,\infty)$ and $\phi_1=\phi_2=\phi$, $\phi(1)=1$. If $\phi\in\Phi_1$ is a convex function, or $\phi\in\Psi_1$ is a concave function, then*

$$\hat{V}(b\circ H\tilde{+}_\phi(1-b)\circ L,\cdots,b\circ H\tilde{+}_\phi(1-b)\circ L,H_{i+1},\cdots,H_n)^{\frac{1}{i}}\leq\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)^{\frac{b}{i}}\cdot\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)^{\frac{1-b}{i}}, \tag{2.10}$$

the equality holds if and only if H and L are dilates when ϕ is strictly convex (or concave).

Proof. *Firstly, if $\phi\in\Phi_1$ is convex, we will prove that $G(t)=\phi(\frac{1}{t^{\frac{1}{i}}})$ is a convex function for $i>0$.*

In fact, let $u=\frac{1}{t^{\frac{1}{i}}}$ in $G(t)=\phi(\frac{1}{t^{\frac{1}{i}}})$, then

$$\frac{dG(t)}{dt}=\frac{d\phi(u)}{du}\frac{du}{dt},$$

and

$$\frac{d^2G(t)}{dt^2}=\frac{d^2\phi(u)}{du^2}(\frac{du}{dt})^2+\frac{d\phi(u)}{du}\frac{d^2u}{dt^2}, \tag{2.11}$$

where

$$\frac{du}{dt}=-\frac{1}{i}t^{-\frac{1+i}{i}},$$

$$\frac{d^2u}{dt^2}=\frac{1}{i}\frac{1+i}{i}t^{-\frac{1+2i}{i}}.$$

Thus, if function $\phi\in\Phi_1$ is strictly increasing and convex, then $\frac{d\phi(u)}{dt}>0$ and $\frac{d^2\phi(u)}{du^2}\geq 0$. And for $i>0$, then $\frac{d^2u}{dt^2}>0$, we get $\frac{d^2G(t)}{dt^2}\geq 0$. So G is convex.

Next, let $\phi_1=\phi_2=\phi$, and for simplicity, let $\hat{V}(S)=\hat{V}(b\circ H\tilde{+}_\phi(1-b)\circ L,\cdots,b\circ H\tilde{+}_\phi(1-b)\circ L,H_{i+1},\cdots,H_n)$, $\hat{V}(S_1)=\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)$, $\hat{V}(S_2)=$

$\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)$, according to (2.9), the arithmetic-geometric mean inequality holds true and $\phi(1)=1$, we have

$$
\begin{aligned}
1&=\phi(1)\\
&\geq b\phi\left[\left(\frac{\hat{V}(S)}{\hat{V}(S_1)}\right)^{\frac{1}{i}}\right]+(1-b)\phi\left[\left(\frac{\hat{V}(S)}{\hat{V}(S_2)}\right)^{\frac{1}{i}}\right]\\
&\geq \phi\left[b\left(\frac{\hat{V}(S)}{\hat{V}(S_1)}\right)^{\frac{1}{i}}+(1-b)\left(\frac{\hat{V}(S)}{\hat{V}(S_2)}\right)^{\frac{1}{i}}\right]\\
&\geq \phi\left(\frac{\hat{V}(S)^{\frac{1}{i}}}{\hat{V}(S_1)^{\frac{b}{i}}\hat{V}(S_2)^{\frac{1-b}{i}}}\right),
\end{aligned}
\tag{2.12}
$$

for ϕ is strictly increasing in (2.12), we obtain

$$\frac{\hat{V}(S)^{\frac{1}{i}}}{\hat{V}(S_1)^{\frac{b}{i}}\hat{V}(S_2)^{\frac{1-b}{i}}}\leq 1, \tag{2.13}$$

and for $i>0$ in (2.13), we get

$$\hat{V}(S)^{\frac{1}{i}}\leq \hat{V}(S_1)^{\frac{b}{i}}\hat{V}(S_2)^{\frac{1-b}{i}}.$$

i.e.

$$
\begin{aligned}
&\hat{V}(b\circ H\tilde{+}_{\phi}(1-b)\circ L,\cdots,b\circ H\tilde{+}_{\phi}(1-b)\circ L,H_{i+1},\cdots,H_n)^{\frac{1}{i}}\\
&\leq \hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)^{\frac{b}{i}}\\
&\cdot\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)^{\frac{1-b}{i}},
\end{aligned}
\tag{2.14}
$$

when ϕ is strictly convex, the conditions for equality to hold in (2.14) imply that H and L are dilates.

Similarly, if $\phi\in\Psi_1$ is strictly decreasing and concave in (2.11), then $\frac{d\phi(u)}{dt}<0$ and $\frac{d^2\phi(u)}{du^2}\leq 0$. And for $i>0$, then $\frac{d^2u}{dt^2}>0$, we get $\frac{d^2G(t)}{dt^2}\leq 0$. So G is concave.

Let $\phi_1=\phi_2=\phi$, where ϕ is a strictly decreasing function. By applying Corollary 2.1 and the arithmetic-geometric mean inequality, while considering that $\phi(1)=1$, we can conclude

$$\frac{\hat{V}(S)^{\frac{1}{i}}}{\hat{V}(S_1)^{\frac{b}{i}}\hat{V}(S_2)^{\frac{1-b}{i}}}\leq 1, \tag{2.15}$$

for $i>0$ in (2.15), then

$$
\begin{aligned}
&\hat{V}(b\circ H\tilde{+}_{\phi}(1-b)\circ L,\cdots,b\circ H\tilde{+}_{\phi}(1-b)\circ L,H_{i+1},\cdots,H_n)^{\frac{1}{i}}\\
&\leq \hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)^{\frac{b}{i}}\\
&\cdot\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)^{\frac{1-b}{i}},
\end{aligned}
\tag{2.16}
$$

the equality holds in (2.16) if and only if H and L are dilates when ϕ is strictly concave.

III. The General Dual Orlicz Mixed Volume and The Dual Orlicz-Minkowski Inequality

In order to prove Theorem 3.1, we need the following two Lemmas([15]).

Lemma 3.1. *Let $H,L\in\mathcal{S}_o^n,\varepsilon>0,\phi_1,\phi_2$ be either both in Φ_1 or both in Ψ_1 and $\phi_1(1)=\phi_2(1)=1$. If $\varepsilon\to 0^+$, then $\rho(H\widetilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)\to\rho(H,w)$ uniformly on S^{n-1}.*

Lemma 3.2. *Let $H,L\in\mathcal{S}_o^n,0\leq\varepsilon<1,\phi_1,\phi_2$ be either both in Φ_1 or both in Ψ_1 and $\phi_1(1)=\phi_2(1)=1$.*

If $\phi_1,\phi_2\in\Phi_1$, then

$$H\widetilde{+}_{\phi_1,\phi_2}L\subset H\widetilde{+}_{\phi_1,\phi_2}\varepsilon\circ L\subset H.$$

If $\phi_1,\phi_2\in\Psi_1$, then

$$H\widetilde{+}_{\phi_1,\phi_2}L\supset H\widetilde{+}_{\phi_1,\phi_2}\varepsilon\circ L\supset H.$$

Theorem 3.1. *Let $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n$, real $i\neq 0$, $\varepsilon\to 0^+,\phi_1,\phi_2\in\Phi_1$ or $\phi_1,\phi_2\in\Psi_1$. Moreover, let $H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L$ be the linear Orlicz radial combination of H and L, $\phi_1(1)=\phi_2(1)=1$. And for simplicity, let $\hat{V}(T)=\hat{V}(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,\cdots,H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,H_{i+1},\cdots,H_n)$.*

(1)The left-derivative of $\phi_1(u)$ at $u=1$, denoted as $\phi'_{1^-}(1)$, exists and is finite for any $\phi_1,\phi_2\in\Phi_1$, then

$$
\begin{aligned}
&-\frac{\phi'_{1^-}(1)}{i}\lim_{\varepsilon\to 0^+}\left[\frac{\hat{V}(T)}{\varepsilon}-\frac{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}{\varepsilon}\right]\\
&=\hat{V}_{\phi_2}(H,\cdots,H,L,H_{i+1},\cdots,H_n).
\end{aligned}
\tag{3.1}
$$

(2)The right-derivative of $\phi_1(u)$ at $u=1$, denoted as $\phi'_{1^+}(1)$, exists and is finite for any $\phi_1,\phi_2\in\Phi_1$, then

$$
\begin{aligned}
&-\frac{\phi'_{1^+}(1)}{i}\lim_{\varepsilon\to 0^+}\left[\frac{\hat{V}(T)}{\varepsilon}-\frac{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}{\varepsilon}\right]\\
&=\hat{V}_{\phi_2}(H,\cdots,H,L,H_{i+1},\cdots,H_n).
\end{aligned}
\tag{3.2}
$$

Proof. *Let $H,L\in\mathcal{S}_o^n$, according to (2.8), the radial function, $H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L$, of H and L satisfies*

$$
\begin{aligned}
&\phi_1\left(\frac{\rho(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)}{\rho(H,w)}\right)\\
&+\varepsilon\phi_2\left(\frac{\rho(H\tilde{+}_{\varphi_1,\phi_2}\varepsilon\circ L,w)}{\rho(L,w)}\right)=1,\forall w\in S^{n-1}.
\end{aligned}
\tag{3.3}
$$

Let $g(\varepsilon)=\frac{\rho(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)}{\rho(H,w)},\forall w\in S^{n-1}$, we have

$$\frac{1}{\varepsilon}=\frac{\phi_2\left(\frac{\rho(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)}{\rho(L,w)}\right)}{1-\phi_1(g(\varepsilon))},\forall w\in S^{n-1}. \tag{3.4}$$

(1) If $\phi_1,\phi_2\in\Phi_1$, and $\varepsilon\to 0^+$, by Lemma 3.1 and Lemma 3.2, we get $g(\varepsilon)\to 1^-$. Noting that $g(0)=1$, according to $\phi_1(g(0))=\phi_1(1)=1$ and the definition of derivative, we have

$$
\begin{aligned}
&-\frac{\phi'_{1^-}(1)}{i}\lim_{\varepsilon\to 0^+}\left[\frac{\hat{V}(T)}{\varepsilon}-\frac{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}{\varepsilon}\right]\\
&=-\frac{\phi'_{1^-}(1)}{i}\lim_{\varepsilon\to 0^+}\frac{1}{n}\int_{S^{n-1}}\frac{\rho^i(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)-\rho^i(H,w)}{\varepsilon}\\
&\quad\cdot\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w)\\
&=-\frac{\phi'_{1^-}(1)}{i}\lim_{\varepsilon\to 0^+}\frac{1}{n}\int_{S^{n-1}}\frac{1}{\varepsilon}\left(\left(\frac{\rho(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)}{\rho(H,w)}\right)^i-1\right)\\
&\quad\cdot\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w)\\
&=-\frac{\phi'_{1^-}(1)}{i}\frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left(\left(\frac{\rho(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)}{\rho(H,w)}\right)^i-1\right)\\
&\cdot\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w)\\
&=-\frac{\phi'_{1^-}(1)}{i}\frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\left(\frac{g(\varepsilon)^i-1}{1-\phi_1(g(\varepsilon))}\right)\\
&\quad\cdot\phi_2\left(\frac{\rho(H\tilde{+}_{\phi_1,\phi_2}\varepsilon\circ L,w)}{\rho(L,w)}\right)
\end{aligned}
$$

$$\cdot\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w)$$

$$=\frac{1}{n}\int_{S^{n-1}}\phi_2\left(\frac{\rho(H,w)}{\rho(L,w)}\right)$$

$$\cdot\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w)$$

$$=\hat{V}_{\phi_2}(H,\cdots,H,L,H_{i+1},\cdots,H_n).$$

The desired relation (3.1) is thus obtained.

(2) If $\phi_1,\phi_2\in\Psi_1$ and as $\varepsilon\to 0^+$, according to Lemma 3.1 and Lemma 3.2, we observe that $g(\varepsilon)\to 1^+$. It is worth noting that $g(0)=1$, and since $\phi_1(g(0))=\phi_1(1)=1$, utilizing equation (3.4), similar to the above discussion, we have

$$-\frac{\phi'_{1+}(1)}{i}\lim_{\varepsilon\to 0^+}\left[\frac{\hat{V}(T)}{\varepsilon}-\frac{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}{\varepsilon}\right]$$

$$=\hat{V}_{\phi_2}(H,\cdots,H,L,H_{i+1},\cdots,H_n).$$

Hence, we obtain the desired relation (3.2).

In light of Theorem 3.1, we give the definition of the general dual Orlicz mixed volumes as follows:

Definition 3.1. *For $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n$, i is real, $\phi:(0,\infty)\to\mathbb{R}$, the general dual Orlicz mixed volume, $\hat{V}_{\phi,i}(H,\cdots,H,L,H_{i+1},\cdots,H_n)$, of $H,L,H_{i+1},\cdots,H_n$ is defined by*

$$\hat{V}_{\phi,i}(H,\cdots,H,L,H_{i+1},\cdots,H_n)$$

$$=\frac{1}{n}\int_{S^{n-1}}\phi(\frac{\rho(H,w)}{\rho(L,w)})$$

$$\cdot\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w). \tag{3.5}$$

It is evident that when $i=n$ in equation (3.5), $\hat{V}_{\phi,n}(H,\ldots,H,L,H_{i+1},\ldots,H_n)=\hat{V}_\phi(H,L)$, which demonstrates the extension of the dual Orlicz mixed volume to a general case ([15]). Similarly, if $H_{i+1}=\ldots=H_n=B$ in equation (3.5), we obtain the definition of dual Orlicz mixed quermassintegrals([14]).

The subsequent step involves establishing the dual Orlicz-Minkowski inequalities for the general dual Orlicz mixed volume:

Theorem 3.2. *Let $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n, i\neq 0$, $\phi:(0,+\infty)\to\mathbb{R}$, and $G(u)=\phi(\frac{1}{u^{\frac{1}{i}}}), u>0$. If G is convex, then*

$$\hat{V}_{\phi,i}(H,\cdots,H,L,H_{i+1},\cdots,H_n)$$

$$\geq\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)$$

$$\cdot\phi\left[\left(\frac{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}{\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)}\right)^{\frac{1}{i}}\right], \tag{3.6}$$

the equality holds in (3.6) if and only if H and L are dilates when ϕ is strictly convex. Conversely, if G is concave, the inequality is reversed.

Proof. *For $H,L,H_{i+1},\cdots,H_n\in\mathcal{S}_o^n$, if $G(u)=\phi(\frac{1}{u^{\frac{1}{i}}})$ is convex, according to (2.2), (3.5), we have*

$$\hat{V}_{\phi,i}(H,\cdots,H,L,H_{i+1},\cdots,H_n)$$

$$=\frac{1}{n}\int_{S^{n-1}}\phi(\frac{\rho(H,w)}{\rho(L,w)})$$

$$\cdot\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w)$$

$$=\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)\int_{S^{n-1}}\phi\left(\frac{\rho(H,w)}{\rho(L,w)}\right)$$

$$\cdot\frac{\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)}{n\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}dS(w)$$

$$=\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)\int_{S^{n-1}}G\left[\left(\frac{\rho(L,w)}{\rho(H,w)}\right)^i\right]$$

$$\cdot\frac{\rho^i(H,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)}{n\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}dS(w)$$

$$\geq\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)$$

$$\cdot G\left(\frac{\frac{1}{n}\int_{S^{n-1}}\rho^i(L,w)\rho(H_{i+1},w)\cdots\rho(H_n,w)dS(w)}{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}\right)$$

$$=\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)$$

$$\cdot G\left(\frac{\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)}{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}\right)$$

$$=\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)$$

$$\cdot\phi\left[\left(\frac{\hat{V}(H,\cdots,H,H_{i+1},\cdots,H_n)}{\hat{V}(L,\cdots,L,H_{i+1},\cdots,H_n)}\right)^{\frac{1}{i}}\right], \tag{3.7}$$

when G is strictly convex, the conditions for equality to hold in (3.7) imply the existence of a constant $c>0$, such that $\frac{\rho(L,w)}{\rho(H,w)}=c$ for all $w\in S^{n-1}$, indicating that both H and L are dilates.

Similarly, in the case of strict concavity for function G, as indicated by equation (2.3), Theorem 3.2 can be derived, with the conditions for equality being satisfied exclusively when both H and L are dilates.

Clearly, when $i=n$ in Theorem 3.2, we obtain the dual Orlicz-Minkowski inequalities for the dual Orlicz mixed volume ([15]). Similarly, if $H_{i+1}=\cdots=H_n=B$ in Theorem 3.2, we derive the dual Orlicz-Minkowski inequalities for the dual Orlicz mixed quermassintegrals([14]).

Remark. (1) When $\phi(u)=u^p(u>0,p\geq 1,i=n)$ in (3.5), which becomes the following Lp dual mixed volume (see [8])

$$\hat{V}_{-p}(H,L)=\frac{1}{n}\int_{S^{n-1}}\rho^{n+p}(H,w)\rho^{-p}(L,w)dS(w).$$

And according to Theorem 3.2, we get the Lp dual mixed Minkowski inequality(see [8]): If $H,L\in\mathcal{S}_o^n, p\geq 1$, then

$$\hat{V}_{-p}(H,L)\geq V(H)^{\frac{n+p}{n}}V(L)^{\frac{-p}{n}},$$

with equality if and only if H and L are dilates.

(2) When $\phi(u)=u^{-p}(u>0,p>0,i=n)$ in (3.5), which becomes the following Lp dual mixed volume(see [16])

$$\hat{V}_p(H,L)=\frac{1}{n}\int_{S^{n-1}}\rho^{n-p}(H,w)\rho^p(L,w)dS(w).$$

And according to Theorem 3.2, we get the Lp dual mixed Minkowski inequality(see [16]): If $H,L\in\mathcal{S}_o^n$, $\frac{p}{n}>1$, then

$$\hat{V}_p(H,L)\geq V(H)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{3.8}$$

the inequality (3.8) holds with equality if and only if H and L are dilates; the reverse occurs when $\frac{p}{n}<1$; when $n=p$, inequality (3.8) becomes an equality.

Let $\phi(u) = log^u$ in (3.5) and write

$$\overline{d\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n, w)} = \frac{\rho^i(H, w)\rho(H_{i+1}, w) \cdots \rho(H_n, w)dS(w)}{n\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)}, \quad (3.9)$$

for all $w \in S^{n-1}$.

The subsequent step involves the establishment of the general dual mixed Log-Minkowski inequalities as presented below:

Theorem 3.3. *For $H, L, H_{i+1}, \cdots, H_n \in \mathcal{S}_o^n$, real $i \neq 0$. If $i > 0$, then*

$$\int_{S^{n-1}} log \frac{\rho(H, w)}{\rho(L, w)} \overline{d\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n, w)} \geq \frac{1}{i} log \frac{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n, w)}{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n, w)}, \quad (3.10)$$

the inequality is reversed if $i < 0$. In each case, the equality holds if and only if H and L undergo dilations.

Proof. *Let $\phi(u) = log^u$, then $G(u) = \phi(\frac{1}{u^{\frac{1}{i}}}) = -\frac{1}{i} log^u$. Obviously, if $i > 0$, G is convex; if $i < 0$, G is concave.*

For $H, L, H_{i+1}, \cdots, H_n \in \mathcal{S}_o^n$, if $i > 0$, according to (2.2), (3.5), and (3.9), we have

$$\begin{aligned}
&\hat{V}_{\phi,i}(H, \cdots, H, L, H_{i+1}, \cdots, H_n) \\
&= \frac{1}{n} \int_{S^{n-1}} \phi(\frac{\rho(H, w)}{\rho(L, w)}) \\
&\cdot \rho^i(H, w)\rho(H_{i+1}, w) \cdots \rho(H_n, w)dS(w) \\
&= \hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n) \frac{1}{n} \int_{S^{n-1}} G(\frac{\rho^i(L, w)}{\rho^i(H, w)}) \\
&\cdot \frac{\rho^i(H, w)\rho(H_{i+1}, w) \cdots \rho(H_n, w)dS(w)}{n\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)} \\
&\geq \hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n) \\
&\cdot G\left(\int_{S^{n-1}} \frac{\rho^i(L, w)\rho(H_{i+1}, w) \cdots \rho(H_n, w)dS(w)}{n\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)} \right) \\
&= \hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n) \\
&\cdot G\left(\frac{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n)}{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)} \right) \\
&= \hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n) \\
&\cdot \frac{-1}{i} log \frac{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n)}{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)} \\
&= \frac{1}{i} \hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n) \\
&\cdot log \frac{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)}{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n)}.
\end{aligned}$$

So we get

$$\int_{S^{n-1}} \phi(\frac{\rho(H, w)}{\rho(L, w)}) \overline{d\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n, w)} \geq \frac{1}{i} log \frac{\hat{V}(H, \cdots, H, H_{i+1}, \cdots, H_n)}{\hat{V}(L, \cdots, L, H_{i+1}, \cdots, H_n)}.$$

If $i > 0$, the function G exhibits strict convexity. According to the conditions for equality stated in (3.10), equality holds if and only if both H and L are dilates. Similarly, when $i < 0$, Theorem 3.3 guarantees that equality is achieved only when both H and L are dilates.

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