General Function Projective Finite-time Lag Synchronization between Two Coupled Dynamical Networks with Nodes of Different Dimensions

Lihong Yan, Weiyuan Zhang, Yanting Yang, Ning Fu and Tingting Wang

Abstract—**The paper investigated general function projective finite-time lag synchronization(GFPFTLS) between two coupled networks with different dimensions and unknown outer disturbances via Razumikhin approach. Basing on finite-time stabilization theorem of time delay system, the decentralized feedback control strategy and properties of norm for irregular matrices, some useful time-delay independent general function finite-time projective lag synchronization criteria have been obtained. Numerical simulations showed the effectiveness and feasibility of the proposed approaches.**

Index Terms—**general function projective synchronization, finite-time, time delay system, nonidentical nodes, Razumikhin approach.**

I. Introduction

In recent years, complex dynamical networks (CDN) have garnered significant interest due to their applications in natural and social fields[1]. Researchers have focused on problems related to modeling, properties, and synchronization phenomena. Numerous interesting results for synchronization of CDNs have been obtained by using various approaches such as linear feedback[2], adaptive[3,4], pinning[5], and impulsive control to achieve asymptotic, function projective[6], and finite-time[7-10] synchronization.

Two coupled networks may be synchronized with different structures of nodes, even with different dimensions in the real-life world. This can be characterized by ‘function projective synchronization’ generally. Synchronization between two coupled CDNs has been discussed in several works[11-15]. However, more recent studies have focused on CDNs with distributed time-delay, stochastic disturbance, and switching topology[16-19].

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Wu and Lu[20] discussed GFPS (lag, anticipated, and complete) between two complex networks with nonidentical nodes based on Barbalat lemma. The dynamics of the nodes of CDNs are any chaotic systems without the limitation of partial linearity with the same dimensions. In [21], sufficient conditions for synchronization were derived using the Lyapunov asymptotic stability theory with the strategy of adaptive control for a CDN with constant time delay and identical nodes. However, finite-time synchronization was not taken into account, along with constant but not time-varying delay in the meantime. Dai et al. [22] investigated the problem of finite-time generalized function matrix projective lag synchronization(GFMPLS) between two different coupled dynamical networks with different dimensions. They used the double power function nonlinear feedback control method but did not consider the influence of time delay. Furthermore, the nodes in each isolate network had the same dimensions, although the dimensions were different from each other between the drive and response network. Tan[3] constructed a time-delay coupled network model by nonidentical nodes with different state dimensions and realized the stabilization and synchronization of such complex networks. However, general function projective finite-time lag synchronization of this network has not been considered so far.

In addition, many large Internet companies such as Alibaba Amazon and Google, have built data centers worldwide to provide users with high-quality cloud services. Within such a data center, thousands of servers are connected through a data center network with high bandwidth and low latency. There are many delay-sensitive real-time applications running in the data center, such as social networking, retail, e-commerce and more. When the network flow bursts instantaneously, congestion will easily occur at the receiving end of the flow, and inappropriate routing will also lead to unbalanced flow within the network, resulting in internal congestion, which will lead to delay or packet loss. Requests of real-time users need to be responded to as soon as possible, and higher response latency will seriously affect the user experience, thereby reducing the company’s operating revenue.

Also in the field of power systems, as the non-linear load in a power system increases, and with the extensive application of high-frequency switch gear, the network structure becomes more complex, and the number of nodes and branches in a power grid is generally not equal [23]. Achieving synchronization for CDNs in finite time for time-delay systems with external disturbances is crucial in engineering applications for power systems, as it can help to improve the stability and reliability of power grids.

Hence, the existence of state delays in complex dynamical network nodes is inevitable. Time delays are ubiquitous in control systems, and if the stability analysis is time-delay independent, the results are much more conservative. In addition, finite-time stability has much quicker convergence speed and better robustness and interference immunity, considering that external disturbances exist widely in engineering. Therefore, achieving synchronization for complex dynamical networks in finite time for time-delay systems with external disturbances is crucial in engineering applications.

This paper presents a Razumikhin analytical approach to achieving general function projective finite-time lag synchronization (GFPFTLS) between two coupled dynamical networks with nodes of different dimensions based on the finite-time stability theorem of nonlinear time-varying dynamical systems. Which is a widely used method in the analysis of stability and convergence for time-delay systems. The key idea of the Razumikhin approach is to divide the time-delay system into two parts: the delayed part and the non-delayed part. The delayed part is characterized by the time delay, while the non-delayed part is characterized by the current state of the system. The Razumikhin approach then uses the Lyapunov-Krasovskii functional to analyze the stability and convergence properties of the delayed and non-delayed parts separately. The Razumikhin approach has several advantages over other methods for analyzing time-delay systems. First, it is relatively easy to implement and does not require complex mathematical tools. Second, it can handle nonlinear time-delay systems with unknown functions and parameters. Third, it can provide sufficient conditions for stability and convergence that are easy to verify.

There have been few results about finite-time synchronization of coupling CDNs with nonidentical node dynamics by the Razumikhin approach so far. The proposed approach can effectively handle the influence of time delay and external disturbances, which are ubiquitous in engineering applications.

So the main contribution is to derive sufficient conditions for achieving GFPFTLS in coupled dynamical networks with nonidentical node dynamics. The theoretical results are supported by numerical simulations, which demonstrate the validity of the proposed approach. Finally, we show that the results of general function projective finite-time lag synchronization contain a large amount of basic conclusions proposed before.

II. Problem Statement and preliminaries

Consider the following coupled CDNs, the drive network with external disturbances is:

$$\dot{m}_i(t) = f_i(m_i(t)) + \sum_{\substack{j=1,\\ j\neq i}}^{N} c_{ij}(\Gamma_{ij}^{(1)} m_j(t) - \Gamma_{ii}^{(1)} m_i(t)) + \omega_i(t), \quad i = 1, 2, \cdots, N. \tag{1}$$

Where $m_i(t) = (m_{i1}(t), m_{i2}(t), \cdots, m_{in}(t))^T \in R^{n_i}$ is the state variable of the $i-$th node, $f_i : R^{n_i} \to R^{n_i}$ is a smooth nonlinear function, describes the local dynamics of each node of the drive network. $C = (c_{ij})_{N\times N}$ is the outer coupling strength and satisfy $c_{ii} = -\sum_{j=1,j\neq i}^{N} c_{ij}, i = 1, 2, \cdots, N$, and $\Gamma_{ij}^{(1)} \in R^{n_i\times n_j}$ is the inner coupling matrix from node j to i, $\omega_i(t) \in R^{n_i}$ are the unknown external time-varying disturbances.

And the response network with different dimensions and numbers of nodes is:

$$\dot{s}_i(t) = g_i(s_i(t)) + \sum_{\substack{j=1,\\ j\neq i}}^{N} d_{ij}(\Gamma_{ij}^{(2)} s_j(t) - \Gamma_{ii}^{(2)} s_i(t)) + \varpi_i(t) + u_i(t), \quad i = 1, 2, \cdots, N. \tag{2}$$

Where $s_i(t) = (s_{i1}(t), s_{i2}(t), ., s_{iq_i}(t))^T \in R^{q_i}$ is the state variable of node i, $g_i : R^{q_i} \to R^{q_i}$ is a smooth nonlinear function which describes the local dynamics of each node of the response network. $D = (d_{ij})_{N\times N}$ is the outer coupling strength and satisfy $d_{ii} = -\sum_{j=1,j\neq i}^{N} d_{ij}, i = 1, 2, \cdots, N$ is the inner coupling matrix from node j to node i. $\varpi_i(t) \in R^{q_i}$ are the unknown external time-varying disturbances and $u_i(t), i = 1, 2, \cdots, N$ is the outer input controller.

Let's denote the time-delay error $\varrho_i(t)$ of the drive-response dynamical network (1) and (2) as following,

$$\varrho_i(t) = s_i(t) - \Lambda_i(t) m_i(t - d_\tau(t)),$$

So the general function projective time-delay error dynamical system between the networks (1) and (2) is as following:

$$\begin{cases} \dot{\varrho}_i(t) = \dot{s}_i(t) - \dot{\Lambda}_i(t) m_i(t - d_\tau(t)) \\ \qquad -\Lambda_i(t)(1 - \dot{d}_\tau(t))\dot{m}_i(t - d_\tau(t)), \\ \varrho_i(\vartheta) = \varphi(\vartheta), \forall \vartheta \in [-d_\tau, 0]. \end{cases} \tag{3}$$

Where $\Lambda_i(t) = (a_{kj}^i(t)) \in R^{q_i\times n_i}, i = 1, 2, \cdots, N$ are some bounded continuously differentiable function matrices. And $\varphi : [-d_\tau, 0] \to R^n$ is the continuously function with finite norm $||\varphi||_{d_\tau} = \sup_{-d_\tau\leq t\leq 0}||\varphi(t)||$. $d_\tau(t)$ is time-varying delay and satisfies the following assumption 1.

Assumption 1 The derivation of time delay $d_\tau(t)$ is bounded. There exist a positive constant τ, such that $\dot{d}_\tau(t) \leq d_\tau$.

It is worth noting that the upper bound need not to be less than 1, without the constraint, the conclusions presented in the paper is much more general. Also the two external disturbances satisfy assumption 2, the unknown external time-varying disturbances $\omega_i(t)$ and $\varpi_i(t)$ are bounded.

Assumption 2 There exist non-negative constants γ_1 and γ_2, such that,

$$\begin{cases} \|\omega_i(t)\| \leq \gamma_1 \\ \|\varpi_i(t)\| \leq \gamma_2 \end{cases} \quad i = 1, 2, \cdots, N.$$

Where $||\cdot||$ stands for the 1- norm.

Assumption 3[24] There exists a positive constant λ, such that $||\Lambda_i(t)|| \leq \lambda$.

Where $||\Lambda_i(t)||, i = 1, 2, \cdots, N$ are the spectral norm of matrix $\Lambda_i(t)$.

Furthermore, lets denote L_n^τ be the space of continuously function with $\tau > 0$, and $L_\delta := \{\psi \in L_n : \|\psi\|_{L_n^\tau} < \delta\}$ with $\|\psi\|_{L_n^\tau} = \sup_{-\tau\leq s\leq 0}\|\psi(s)\|_n$ and $\|\psi(s)\|_n$ is the Eulidean norm in R^n.

We will design proper controllers $u_i(t), i = 1, 2, \cdots, N$ to realize GFPFTLS between the drive and response networks (1) and (2) soon follow. As we can see, general function projective finite-time lag synchronization between the networks

(1) and (2) is equal to the time-delay error system (3) is stable at the origin in a finite time.

The definition of GFPFTLS between drive-response networks (1) and (2) is as following.

Definition 1 The drive-response networks (1) and (2) can realize general function projective finite-time lag synchronization, if there exists $\delta > 0$ such that, for any $\psi \in L_\delta$, there exists $0 \leq T(\psi) < +\infty$ which satisfies $\lim\limits_{t\to T(\psi)} \|s_i(t) - \Lambda_i(t)m_i(t-d_\tau(t))\| = 0$, and when $t \geq T(\psi)$, $\|s_i(t) - \Lambda_i(t)m_i(t-d_\tau(t))\| = 0$.

There followed a designed proper controller, add it to the error dynamical equation (3) and investigate the GFPFTLS of drive-response network (1) and (2). For further discussion, the following lemmas about finite-time stability of time-delay system will be introduced.

Lemma 1[24, 25] For the system

$$\begin{cases} \dot m(t) = f(m(t), m(t-\tau_h)), \\ m(\vartheta) = \psi(\vartheta), \forall \vartheta \in [-\tau_h, 0], \end{cases} \tag{4}$$

which has unique solution in forward time. If there exist a Class-$\mathcal{K}$ function σ, real numbers $\ell > 0, \alpha > 1$ and a C^1 Lyapunov function $V(m)$ of system (4) such that

$$\begin{aligned} &(1)\sigma(\|m\|) \leq V(m), \\ &(2)\dot V(m) \leq -\ell V^{\frac{1}{\alpha}}(m), m \in \Omega \end{aligned}$$

hold, then the above system (4) is finite-time stable.

If $\dot V(m) \leq -\ell V^{\frac{1}{\alpha}}(m), m \in \Omega$ and σ is a Class-$\mathcal{K}_\infty$ function, then the origin is a globally finite-time stable equilibrium of the system (5). What's more, the settling time of the system (4) with respect to the initial condition $\phi \in L_\alpha$ satisfies $T_0(\phi) \leq \frac{\alpha}{\ell(1-\alpha)} \cdot V^{\frac{\alpha-1}{\alpha}}(\phi)$ for all $t > 0$. Where settling time $T_0(\phi) = \inf\{T(\phi) \geq 0 : m(t,\phi) = 0, \forall t \geq T(\phi)\}$.

Remark 1 In reference[25], the author pointed out that lemma 1 in paper [24] is incorrect, and showed that if we delete the Razumikhin condition whenever $V(m(t+\vartheta)) \leq V(m(t))$, for $\vartheta \in [-\tau_h, 0]$, the corrected conclusion is presented in Lemma 1. Also the author put forward a another necessary condition to realize finite-time stability for time-delay systems as following.

Lemma 2[25] Let $\dot m(t) = f(m_t), t \geq 0, m \in R^n, m_t \in C[-\tilde\tau, 0]$ be finite-time convergent in Ω, then

$$\forall t \in \Gamma_{m_0} : F[0, m(t, m_0)] = 0 \tag{5}$$

for any initial value $m_0 \in \Omega$.

Where $m_t = m(t - d_\tau(t))$ is a continuous function, and there is a non-empty set of time instants Γ_{m_0}=$\{t \in [T_0(m_0) - \tilde\tau, T_0(m_0)] : m(t, m_0) \neq 0\}$ and $f(\phi) = F(\phi(0), \phi(-\tilde\tau))$with the help of the definition of $T_0(m_0)$, where $F : R^{2n} \to R^n$ is continuous in R^n .

Notations: For convenience, lets denote I_n be $n-$dimension identity matrix. And the matrices $\bar\Gamma^{(1)}, \Lambda(t), \bar\Gamma^{(2)}, L$ separately are:

$$\bar\Gamma^{(1)} = \begin{pmatrix} c_{11}\Gamma^{(1)}_{11} & c_{12}\Gamma^{(1)}_{12} & \cdots & c_{1N}\Gamma^{(1)}_{1N} \\ c_{21}\Gamma^{(1)}_{21} & c_{22}\Gamma^{(1)}_{22} & \cdots & c_{2N}\Gamma^{(1)}_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N1}\Gamma^{(1)}_{N1} & c_{N2}\Gamma^{(1)}_{N2} & \cdots & c_{NN}\Gamma^{(1)}_{NN} \end{pmatrix},$$

$$\Lambda(t) = \begin{pmatrix} \Lambda_1(t) & & & \\ & \Lambda_2(t) & & \\ & & \ddots & \\ & & & \Lambda_N(t) \end{pmatrix},$$

$$\bar\Gamma^{(2)} = \begin{pmatrix} d_{11}\Gamma^{(2)}_{11} & d_{12}\Gamma^{(2)}_{12} & \cdots & d_{1N}\Gamma^{(2)}_{1N} \\ d_{21}\Gamma^{(2)}_{21} & d_{22}\Gamma^{(2)}_{22} & \cdots & d_{2N}\Gamma^{(2)}_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ d_{N1}\Gamma^{(2)}_{N1} & d_{N2}\Gamma^{(2)}_{N2} & \cdots & d_{NN}\Gamma^{(2)}_{NN} \end{pmatrix},$$

$$L = \begin{pmatrix} l_1 I_{q_1} & 0 & \cdots & 0 \\ 0 & l_2 I_{q_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & l_N I_{q_N} \end{pmatrix}.$$

III. General function projective finite-time lag synchronization between two coupled networks

In this part, we consider GFPFTLS between coupled complex dynamical network (1) and (2) with Razumikhin approach. In order to drive the achieve the objective (3) to zero in a finite time, the injected adaptive controller $u_i(t)$ for the $i-$th node are designed as follows,

$$u_i(t) = \begin{cases} (1-\dot d_\tau(t))\Lambda_i(t) f_i(m_i(t-d_\tau(t))) - g_i(s_i(t)) \\ -\frac{1}{4\varepsilon}\frac{\|m(t-d_\tau(t))\|^2}{\|\varrho_i(t)\|^2}\varrho_i(t) - l_i\varrho_i(t) \\ -\frac{\eta}{\sqrt{2^{1+\kappa}}}\mathrm{sign}(\varrho_i(t))|\varrho_i(t)|^\kappa + \dot\Lambda_i(t) m_i(t-d_\tau(t)) \\ -(\gamma_2 - \lambda\gamma_1(1+d_\tau))\mathrm{sign}(\varrho_i(t)), \\ \qquad\qquad if\, \varrho_i(t) \neq 0, \\ 0, \qquad\qquad if\, \varrho_i(t) = 0. \end{cases} \tag{6}$$

Where $|\varrho_i(t)|^\kappa = (|\varrho_{i1}(t)|^\kappa, |\varrho_{i2}(t)|^\kappa, \cdots, |\varrho_{in_i}(t)|^\kappa)^T$, $\mathrm{sign}(\varrho_i(t)) = (\mathrm{sign}(\varrho_{i1}(t)), \mathrm{sign}(\varrho_{i2}(t)), \cdots, \mathrm{sign}(\varrho_{in_i}(t)))^T$, $i = 1, 2, \cdots, N$. $m(t-d_\tau(t))$= $(m_1^T(t-d_\tau(t)), m_2^T(t-d_\tau(t)), \cdots, m_N^T(t-d_\tau(t)))^T$. And κ $(0 < \kappa < 1)$ is the adjusted controller parameter. With the controller (6), a sufficient condition for controlled complex network can realize general function projective finite-time lag synchronization between coupled complex dynamical network (1) and (2).

Remark 2 The controller (6) is determined by the non-identical dimensions of the heterogeneity of complex dynamical network modelling, similar literature [3] and [22] can be referred.

Substitute the dynamical network equations (1) and (2) which satisfying the dissipative coupling conditions to the error dynamical equation (3), the error equation can be derived,

$$\begin{aligned}
&\dot{\varrho}_i(t)=\dot{y}_i(t)-\dot{\Lambda}_i(t)m_i(t-d_\tau(t))\\
&-\Lambda_i(t)(1-\dot{d}_\tau(t))\dot{m}_i(t-d_\tau(t))\\
&=g_i\left(s_i(t)\right)+\sum_{j=1,j\neq i}^{N}d_{ij}(\Gamma_{ij}^{(2)}s_j(t)-\Gamma_{ii}^{(2)}s_i(t))\\
&+\varpi_i(t)+u_i(t)-\Lambda_i(t)(1-\dot{d}_\tau(t))f_i(m_i(t-d_\tau(t)))\\
&-\Lambda_i(t)(1-\dot{d}_\tau(t))\sum_{j=1}^{N}c_{ij}\Gamma_{ij}^{(1)}m_j(t-d_\tau(t))\\
&-\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t)))-\dot{\Lambda}_i(t)m_i(t-d_\tau(t))\\
&=g_i\left(s_i(t)\right)-\dot{\Lambda}_i(t)m_i\left(t-d_\tau(t)\right)+\varpi_i(t)+u_i(t)\\
&-\Lambda_i(t)(1-\dot{d}_\tau(t))\left(\omega_i(t-d_\tau(t))+f_i(m_i(t-d_\tau(t)))\right)\\
&+\sum_{j=1,j\neq i}^{N}d_{ij}\left(\Gamma_{ij}^{(2)}\varrho_j(t)-\Gamma_{ii}^{(2)}\varrho_i(t)\right)\\
&+\sum_{j=1}^{N}\left(d_{ij}\Gamma_{ij}^{(2)}\Lambda_j(t)-(1-\dot{d}_\tau(t))c_{ij}\Lambda_i(t)\Gamma_{ij}^{(1)}\right)m_j(t-d_\tau(t))
\end{aligned}\tag{7}$$

Theorem 1 Suppose that Assumptions 1 holds, if there exist positive constants κ,η,ε and proper constants $l_i, i=1,2,\cdots,N$, such that $\Omega=\bar{\Gamma}^{(2)}+\varepsilon Q^TQ-L<0$, then the two coupled networks (1) and (2) with controllers (6) can realize general function projective finite-time lag synchronization. Furthermore, the synchronization settling time between the drive-response networks (1) and (2) with respect to the initial condition $\phi\in L_\delta$ satisfies: $T(\phi)\le\frac{2}{\eta(1-\kappa)}V_\varrho^{\frac{1-\kappa}{2}}(\phi)$.

Proof. Choose the Lyapunov function $V_\varrho(t)$ as following,

$$V_\varrho(t)=\frac{1}{2}\sum_{i=1}^{N}\varrho_i^T(t)\varrho_i(t)$$

Then one gets

$$\dot{V}_\varrho(t)=\sum_{i=1}^{N}\varrho_i^T(t)\dot{\varrho}_i(t)$$

If $||\varrho_i(t)||\neq 0$, substituting the first part of controller (6) to the error system (7), one can have

$$\begin{aligned}
&\dot{\varrho}_i(t)=g_i\left(s_i(t)\right)-\dot{\Lambda}_i(t)m_i\left(t-d_\tau(t)\right)+\varpi_i(t)\\
&-\Lambda_i(t)(1-\dot{d}_\tau(t))(\omega_i(t-d_\tau(t))+f_i(m_i(t-d_\tau(t))))\\
&+\sum_{j=1,j\neq i}^{N}d_{ij}\left(\Gamma_{ij}^{(2)}\varrho_j(t)-\Gamma_{ii}^{(2)}\varrho_i(t)\right)\\
&+\sum_{j=1}^{N}\left(d_{ij}\Gamma_{ij}^{(2)}\Lambda_j(t)-\left(1-\dot{d}_\tau(t)\right)c_{ij}\Lambda_i(t)\Gamma_{ij}^{(1)}\right)m_j\left(t-d_\tau(t)\right)\\
&+(1-\dot{d}_\tau(t))\Lambda_i(t)f_i(m_i(t-d_\tau(t)))-g_i(s_i(t))-l_i\varrho_i(t)\\
&-\tfrac{1}{4\varepsilon}\tfrac{\|m(t-d_\tau(t))\|^2}{||\varrho_i(t)||^2}\varrho_i(t)-(\gamma_2-\lambda\gamma_1(1+\tau))\text{sign}(\varrho_i(t))\\
&+\dot{\Lambda}_i(t)m_i(t-d_\tau(t))-\tfrac{\eta}{\sqrt{2^{1+\kappa}}}\text{sign}(\varrho_i(t))|\varrho_i(t)|^\kappa\\
&=-\tfrac{\eta}{\sqrt{2^{1+\kappa}}}sign(\varrho_i(t))|\varrho_i(t)|^\kappa-l_i\varrho_i(t)+\varpi_i(t)\\
&+\sum_{j=1}^{N}\left(d_{ij}\Gamma_{ij}^{(2)}\Lambda_j(t)-\left(1-\dot{d}_\tau(t)\right)c_{ij}\Lambda_i(t)\Gamma_{ij}^{(1)}\right)\varrho_j\left(t-d_\tau(t)\right)\\
&-\tfrac{1}{4\varepsilon}\tfrac{\|m(t-d_\tau(t))\|^2}{||\varrho_i(t)||^2}\varrho_i(t)-\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t))\\
&+\sum_{j=1,j\neq i}^{N}d_{ij}\left(\Gamma_{ij}^{(2)}\varrho_j(t)-\Gamma_{ii}^{(2)}\varrho_i(t)\right)\\
&-(\gamma_2-\lambda\gamma_1(1+d_\tau))\text{sign}(\varrho_i(t)).
\end{aligned}\tag{8}$$

Obviously, the necessary condition Lemma 2 is satisfied. Then combining the above equation with the derivative of $V_\varrho(t)$, we get

$$\begin{aligned}
&\dot{V}_\varrho(t)=\sum_{i=1}^{N}\varrho_i^T(t)\left(\sum_{j=1,j\neq i}^{N}d_{ij}\left(\Gamma_{ij}^{(2)}\varrho_j(t)-\Gamma_{ii}^{(2)}\varrho_i(t)\right)+\varpi_i(t)\right)\\
&+\sum_{i=1}^{N}\varrho_i^T(t)\sum_{j=1}^{N}d_{ij}\Gamma_{ij}^{(2)}\Lambda_j(t)m_j\left(t-d_\tau(t)\right)\\
&-\sum_{i=1}^{N}\varrho_i^T(t)\sum_{j=1}^{N}\left(1-\dot{d}_\tau(t)\right)c_{ij}\Lambda_i(t)\Gamma_{ij}^{(1)}m_j\left(t-d_\tau(t)\right)\\
&-\sum_{i=1}^{N}\varrho_i^T(t)\left(-\tfrac{1}{4\varepsilon}\tfrac{\|m(t-d_\tau(t))\|^2}{||\varrho_i(t)||^2}\varrho_i(t)-l_i\varrho_i(t)\right)\\
&-\sum_{i=1}^{N}\varrho_i^T(t)\tfrac{\eta}{\sqrt{2^{1+\kappa}}}sign(\varrho_i(t))|\varrho_i(t)|^\kappa\\
&-\sum_{i=1}^{N}\varrho_i^T(t)\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t))\\
&-\sum_{i=1}^{N}\varrho_i^T(t)(\gamma_2-\lambda\gamma_1(1+d_\tau))\cdot\text{sign}(\varrho_i(t))
\end{aligned}\tag{9}$$

With the bound of outer disturbances Assumption 2, one has

$$\begin{aligned}
&\left\|\varpi_i(t)-\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t))\right\|\\
&\le\|\varpi_i(t)\|+\left\|-\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t))\right\|\\
&\le\|\varpi_i(t)\|+\|\Lambda_i(t)\|\cdot\left|-(1-\dot{d}_\tau(t))\right|\cdot\|\omega_i(t-d_\tau(t))\|\\
&\le\gamma_2+\lambda(1+d_\tau)\gamma_1.
\end{aligned}\tag{10}$$

So

$$\begin{aligned}
&\sum_{i=1}^{N}\varrho_i^T(t)\left(\omega_i(t)-\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t))\right)\\
&-\sum_{i=1}^{N}\varrho_i^T(t)(\gamma_2-\lambda\gamma_1(1+d_\tau))\text{sign}(\varrho_i(t))\\
&\le\sum_{i=1}^{N}\left\|\varrho_i^T(t)\right\|\cdot\left\|v_i(t)-\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t))\right\|\\
&-\sum_{i=1}^{N}\varrho_i^T(t)(\gamma_2-\lambda\gamma_1(1+d_\tau))\text{sign}(\varrho_i(t))\\
&\le\left\|\varpi_i(t)-\Lambda_i(t)(1-\dot{d}_\tau(t))\omega_i(t-d_\tau(t))\right\|\cdot\|\varrho(t)\|\\
&-(\gamma_2-\lambda\gamma_1(1+d_\tau))\|\varrho(t)\|\\
&\le 0.
\end{aligned}\tag{11}$$

With the denotations of matrices $\bar{\Gamma}^{(1)},\bar{\Gamma}^{(2)},\Lambda(t)$, let $P(t)=(\varrho_1(t),\varrho_2(t),\cdots,\varrho_N(t))$, rewriting the second part of (9) to the vector form, the following results holds,

$$\begin{aligned}
&\sum_{i=1}^{N}\varrho_i^T(t)\sum_{j=1}^{N}d_{ij}\Gamma_{ij}^{(2)}\Lambda_j(t)m_j\left(t-d_\tau(t)\right)\\
&-\sum_{i=1}^{N}\varrho_i^T(t)\sum_{j=1}^{N}\left(1-\dot{d}_\tau(t)\right)c_{ij}\Lambda_i(t)\Gamma_{ij}^{(1)}m_j\left(t-d_\tau(t)\right)\\
&=\varrho^T(t)(\bar{\Gamma}^{(2)}\Lambda(t)-\left(1-\dot{d}_\tau(t)\right)\Lambda(t)\bar{\Gamma}^{(1)})m\left(t-d_\tau(t)\right),
\end{aligned}\tag{12}$$

With the help of young's inequality, one can see there exist a positive constant ε such that

$$\begin{aligned}
&\varrho^T(t)(\bar{\Gamma}^{(2)}\Lambda(t)-\left(1-\dot{d}_\tau(t)\right)\Lambda(t)\bar{\Gamma}^{(1)})m\left(t-d_\tau(t)\right)\\
&\le\varepsilon\varrho^T(t)Q^TQ\varrho(t)+\tfrac{1}{4\varepsilon}m^T\left(t-d_\tau(t)\right)m\left(t-d_\tau(t)\right),
\end{aligned}\tag{13}$$

Where $Q^T=\bar{\Gamma}^{(2)}\Lambda(t)-\left(1-\dot{d}_\tau(t)\right)\Lambda(t)\bar{\Gamma}^{(1)}$.

Substitute (11) and (13) to (9), one get

$$\begin{aligned}\dot{V}_{\varrho}(t)=&\sum_{i=1}^{N}\varrho_i^T(t)\left(\sum_{j=1,j\neq i}^{N}d_{ij}\left(\Gamma_{ij}^{(2)}\varrho_j(t)-\Gamma_{ii}^{(2)}\varrho_i(t)\right)\right)\\&-\sum_{i=1}^{N}l_i\varrho_i^T(t)\varrho_i(t)-\tfrac{\eta}{\sqrt{2^{1+\kappa}}}|\varrho_i(t)|^{\kappa+1}\\&+\varepsilon\varrho^T(t)Q^TQ\varrho(t)+\tfrac{1}{4\varepsilon}\|m\left(t-d_\tau(t)\right)\|^2\\&-\sum_{i=1}^{N}\varrho_i^T(t)\left(\tfrac{1}{4\varepsilon}\sum_{i=1}^{N}m_i^T(t-d_\tau(t))m_i(t-d_\tau(t))\tfrac{\varrho_i(t)}{\|\varrho_i(t)\|^2}\right)\\&=\sum_{i=1}^{N}\varrho_i^T(t)\left(\sum_{j=1,j\neq i}^{N}d_{ij}\left(\Gamma_{ij}^{(2)}\varrho_j(t)-\Gamma_{ii}^{(2)}\varrho_i(t)\right)\right)\\&+\varepsilon\varrho^T(t)Q^TQ\varrho(t)-\sum_{i=1}^{N}l_i\varrho_i^T(t)\varrho_i(t)-\tfrac{\eta}{\sqrt{2^{1+\kappa}}}|\varrho_i(t)|^{\kappa+1}.\end{aligned}$$

By simplification, the following results are obtained,

$$\dot{V}_{\varrho}(t)\leq P^T(t)(\bar{\Gamma}^{(2)}+\varepsilon Q^TQ-L)P(t)-\frac{\eta}{\sqrt{2^{1+\kappa}}}\sum_{i=1}^{N}|\varrho_i(t)|^{\kappa+1}. \tag{14}$$

Let's denote $\Omega=\bar{\Gamma}^{(2)}+\varepsilon Q^TQ-L$, then one get

$$\dot{V}_{\varrho}(t)\leq P^T(t)\Omega P(t)-\frac{\eta}{\sqrt{2^{1+\kappa}}}\sum_{i=1}^{N}|\varrho_i(t)|^{\kappa+1}.$$

In addition, basing on the property of inequality $\sum_{i=1}^{n}\|m_i\|^{\theta}\leq\left(\sum_{i=1}^{n}\|m_i\|^2\right)^{\theta/2}(0<\theta<2)$, the following result can be achieved,

$$\begin{aligned}\dot{V}_{\varrho}(t)&\leq P^T(t)\Omega P(t)-\eta\left(\sum_{i=1}^{N}\tfrac{1}{2}\|\varrho_i(t)\|^2\right)^{\frac{\kappa+1}{2}}\\&=P^T(t)\Omega P(t)-\eta V_{\varrho}^{\frac{1+\kappa}{2}}(t).\end{aligned} \tag{15}$$

Combining the conclusion (15), if there exist large enough feedback gains $l_i, i=1,2\cdots,N$ such that $\Omega<0$, then we have

$$\dot{V}_{\varrho}(t)\leq-\eta V_{\varrho}^{\frac{1+\kappa}{2}}(t). \tag{16}$$

Obviously, the constructed Lyapunov function $V_{\varrho}(t)$ satisfies the conditions (1) and (2) of Lemma 1, then in the light of Lemma 1, one can obtain the error system (3) will be stabilized in a finite time, that is, the drive and response complex dynamical networks (1) and (2) could realize function projective finite-time lag synchronization in a finite time. At the same time, the synchronization settling time between drive-response networks (1) and (2) with respect to the initial condition $\phi\in L_\delta$ satisfies:

$$T(\phi)\leq\frac{2}{\eta(1-\kappa)}V_{\varrho}^{\frac{1-\kappa}{2}}(\phi). \tag{17}$$

One can see that the derivative item $\dot{d_\tau}(t)$ is used in Theorem 1, but time delay $d_\tau(t)$ is not always known previously, and it is not easy to obtain precise information of time delay in practical applications, in addition, the calculation of matrix inequality $\bar{\Gamma}^{(2)}+\varepsilon Q^TQ-L<0$ will increase control cost, although it can be calculated by Schur complement theorem via Matlab software. So next we will investigate the synchronization condition basing on the definition and properties of norm for irregular matrices, making it easy to be verifiable and applied for engineering and scientific research.

Next we will give a corollary which is an simplified representation of Theorem 1 according to the spectral norm and its properties of irregular matrices.

For the matrix inequality $\Omega=\bar{\Gamma}^{(2)}+\varepsilon Q^TQ-L$, we can see that when the matrix $\Omega<0$, then $L>\bar{\Gamma}^{(2)}+\varepsilon Q^TQ$. furthermore, it means $l_{\min}\geq\lambda_{\max}\{\bar{\Gamma}^{(2)}\}+\varepsilon\lambda_{\max}\{Q^TQ\}$, and the highest eigenvalue for the matrix $\lambda_{\max}\{Q^TQ\}=\|Q\|^2$, where $\|Q\|$ is the spectral norm of matrix Q. While according to the deduced inequality (13), the norm of Q equals to $||\bar{\Gamma}^{(2)}\Lambda(t)-\left(1-\dot{d_\tau}(t)\right)\Lambda(t)\bar{\Gamma}^{(1)}||$, basing on compatibility conditions of norm for irregular matrix and Assumption 3, one can obtain

$$\begin{aligned}\|Q\|&=||\bar{\Gamma}^{(2)}\Lambda(t)-\left(1-\dot{d_\tau}(t)\right)\Lambda(t)\bar{\Gamma}^{(1)}||\\&\leq||\bar{\Gamma}^{(2)}\Lambda(t)||+||-\left(1-\dot{d_\tau}(t)\right)\Lambda(t)\bar{\Gamma}^{(1)}||\\&\leq||\bar{\Gamma}^{(2)}||\cdot||\Lambda(t)||+(1+d_\tau)||\Lambda(t)||\cdot||\bar{\Gamma}^{(1)}||\\&=\lambda(\mu_2+(1+d_\tau)\mu_1)\end{aligned} \tag{18}$$

Where $\mu_i, i=1,2$ are the upper bounds of $||\bar{\Gamma}^{(i)}||, i=1,2$. Thus, another GFPFTLS results for drive-response dynamical networks (1) and (2) will be presented.

Corollary 1 Suppose that Assumptions 1 holds, if there exist positive constants κ,η,ε and proper constants $l_i, i=1,2,\cdots,N$, such that

$$l_{\min}\geq\lambda_{\max}\{\bar{\Gamma}^{(2)}\}+\varepsilon\lambda^2(\mu_2+(1+d_\tau)\mu_1)^2 \tag{19}$$

Where $l_{\min}=\min\{l_i\}, i=1,2,\cdots,N$. Then the two coupled networks (1) and (2) with controllers (6) can realize general function projective finite-time lag synchronization. Furthermore, the synchronization settling time between drive-response networks (1) and (2) with respect to the initial condition $\phi\in L_\delta$ satisfies: $T(\phi)\leq\frac{2}{\eta(1-\kappa)}V^{\frac{1-\kappa}{2}}(\phi)$.

Remark 3 The above results about GFPFTLS between the drive-response networks contain a lot of projective synchronization cases in papers[3,6,9,10]. Such as:

1. If $d_\tau(t)=0$, the drive-response network (1) and (2) realize general function finite-time projective synchronization(GFFTPS) with different dimensions of nodes.
2. If $d_\tau(t)=0, q_i=n_i, M=N, \Lambda(t)=I$, the network (1) and (2) can realize function projective finite-time synchronization(FPFTS).
3. If $d_\tau(t)\neq0, M=N$, the drive-response network (1) and (2) can realize general function projective finite-time lag synchronization(GFPFTS) with different dimensions of nodes.

In all, Theorem 1 contains a large amount of synchronization cases as to the presented drive-response dynamical networks (1) and (2).

IV. Numerical simulations

Without loss of generality, in this section, we will design some numerical simulation examples with MATLAB software to illustrate the proposed approaches to realize GFPFTLS for complex networks (1) and (2) with 4 heterogeneous chaos as the node dynamics respectively. The four nodes of driven network (1) separately are sprott-O, hyperchaotic Lv, Lorenz and Duffing systems. The dynamical equations of the drive network are as following,

$$\begin{pmatrix} \dot{m}_{11} \\ \dot{m}_{12} \\ \dot{m}_{13} \end{pmatrix} = \begin{pmatrix} m_{12} \\ m_{11} - m_{13} \\ m_{11} + m_{11}m_{13} + 2.7m_{12} \end{pmatrix},$$
$$\begin{pmatrix} \dot{m}_{21} \\ \dot{m}_{22} \\ \dot{m}_{23} \\ \dot{m}_{24} \end{pmatrix} = \begin{pmatrix} 36(m_{22} - m_{21}) + m_{24} \\ -m_{21}m_{23} + 20m_{22} \\ m_{21}m_{22} - 3m_{23} \\ m_{21}m_{23} + m_{24} \end{pmatrix}, \quad (20)$$
$$\begin{pmatrix} \dot{m}_{31} \\ \dot{m}_{32} \\ \dot{m}_{33} \end{pmatrix} = \begin{pmatrix} 10(m_{32} - m_{31}) \\ 28m_{31} - m_{31}m_{33} - m_{32} \\ m_{31}m_{32} - 8/3m_{33} \end{pmatrix},$$
$$\begin{pmatrix} \dot{m}_{41} \\ \dot{m}_{42} \end{pmatrix} = \begin{pmatrix} m_{42} \\ -0.25m_{41} - m_{41}^3 + 11cos(t) \end{pmatrix}.$$

The response network with 4 nodes are chosen as Sprott-O, duffing and rosslor and Chen chaotic attractor as follows,

$$\begin{pmatrix} \dot{s}_{11} \\ \dot{s}_{12} \\ \dot{s}_{13} \end{pmatrix} = \begin{pmatrix} s_{12} \\ s_{11} - s_{13} \\ s_{11} + s_{11}s_{13} + 2.7s_{12} \end{pmatrix},$$
$$\begin{pmatrix} \dot{s}_{21} \\ \dot{s}_{22} \end{pmatrix} = \begin{pmatrix} s_{22} \\ -0.25s_{21} - s_{21}^3 + 10\cos(t) \end{pmatrix},$$
$$\begin{pmatrix} \dot{s}_{31} \\ \dot{s}_{32} \\ \dot{s}_{33} \end{pmatrix} = \begin{pmatrix} s_{32} - s_{33} \\ 0.2s_{32} - s_{31} \\ 0.2 + s_{33}(s_{31} - 5.7) \end{pmatrix}, \quad (21)$$
$$\begin{pmatrix} \dot{s}_{41} \\ \dot{s}_{42} \\ \dot{s}_{43} \end{pmatrix} = \begin{pmatrix} -8/3s_{41} + s_{42}s_{43} \\ 10(s_{43} - s_{42}) \\ -15s_{42} - s_{42}s_{41} - s_{43} \end{pmatrix}.$$

We can see that the parameters $m = n = 4$. The two outer coupling matrices are chosen as follows,

$$C = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 2 & -3 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$
$$D = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -3 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0.5 & 1 & -1.5 \end{pmatrix}.$$

The inner coupling matrices of network (1) and (2) are respectively designed as:

$$\Gamma_{12}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$
$$\Gamma_{14}^{(1)} = \Gamma_{12}^{(2)} = \Gamma_{32}^{(2)} = \Gamma_{42}^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix},$$
$$\Gamma_{24}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}, \Gamma_{22}^{(1)} = I_4, \Gamma_{32}^{(1)} = \Gamma_{12}^{(1)},$$
$$\Gamma_{21}^{(1)} = \Gamma_{23}^{(1)} = \left(\Gamma_{12}^{(1)}\right)^T, \Gamma_{34}^{(1)} = \Gamma_{32}^{(2)} = \Gamma_{14}^{(1)},$$
$$\Gamma_{41}^{(1)} = \Gamma_{43}^{(1)} = \Gamma_{21}^{(2)} = \Gamma_{23}^{(2)} = \Gamma_{24}^{(2)} = \left(\Gamma_{14}^{(1)}\right)^T,$$
$$\Gamma_{44}^{(1)} = \Gamma_{22}^{(2)} = \Gamma_2, \Gamma_{ij}^{(1)} = \Gamma_{ji}^{(1)} = I_3, i,j = 1,3$$
$$\Gamma_{42}^{(1)} = \left(\Gamma_{24}^{(1)}\right)^T, \Gamma_{11}^{(1)} = \Gamma_{33}^{(1)} = \Gamma_{11}^{(2)} = \Gamma_{31}^{(2)} =$$
$$\Gamma_{33}^{(2)} = \Gamma_{34}^{(2)} = \Gamma_{41}^{(2)} = \Gamma_{43}^{(2)} = \Gamma_{44}^{(2)} = I_3.$$

By calculation, we can see that $||\bar{\Gamma}^{(1)}||$ =2.099, $||\bar{\Gamma}^{(2)}||$=0.839, so the upper bounds are $\mu_1 = 2.1$ $\mu_2 = 0.9$.

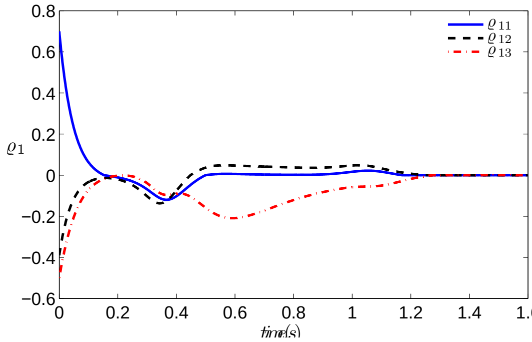

Fig. 1. Error evolution of the first node between the drive-response network (20) and (21)

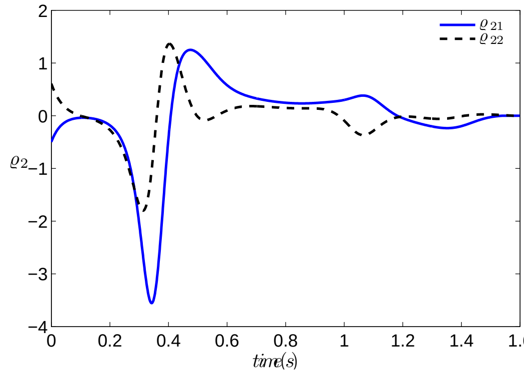

Fig. 2. Error evolution of the second node between the drive-response network (20) and (21)

For convenience, the function projective matrices are chosen as $\Lambda_1(t) = \Lambda_3(t) = I_3$ respectively, while $\Lambda_2(t) = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}^T$, $\Lambda_4(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0.5 & 1 \end{pmatrix}^T$. Moreover, by calculation of the matrix norm, we can get $\lambda = 1.91$. The time delay $d_\tau(t) = 0.1sin(t)$, so the bound of time delay are $d_\tau = \tilde{\tau} = 0.1$. Matrices Q_i are identity matrices, i.e.$Q_3 = Q_4 = I_3, Q_2 = I_2$. The external disturbances of drive and response network are respectively:$\omega_1 = 0.1\sin(10t); \omega_2 = 0.1; \omega_3 = 0.2\sin(4t); \omega_4 = \sin(4t); \omega_4 = 0.15\cos(5t); \varpi_1 = 0.1\sin(2t); \varpi_1 = 0.05\cos(10t); \varpi_3 = 0; \varpi_4 = 0.04\sin(5t)$. Basing on the condition of Corollary 1, one can have that $\eta = 0.4, \kappa = 0.6, \varepsilon$=0.5, the feedback gains are chosen as l_1=20, $l_2 = 25, l_3 = 30, l_4$= 25. Choosing initial values randomly in $[-10, 10]$, since $t_0 = 0$, the GFPFTLS of the drive-response network (1) and (2)with controller (6) is achieved, and the time needed is consistent with theoretical results (18) . The states error curves between the complex network (1) and (2) are showed in Figure 1-4. From the simulation results, we can found that the error will tend to 0 in a finite time when the controller (6) is added to the coupled dynamical network (2). Also if we choose larger feedback gains, $l_i, i = 1, 2, 3, 4$, the less time needed to realize general function projective finite-time lag synchronization.

V. Conclusions

We investigated GFPFTLS problems of CDNs with d-ifferent dimensions of nodes here. Based on finite-time

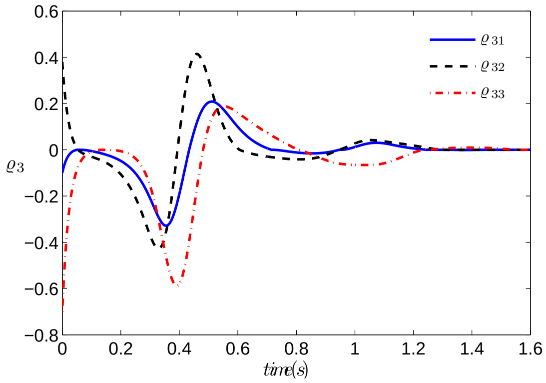

Fig. 3. Error evolution of the third node between the drive-response network (20) and (21)

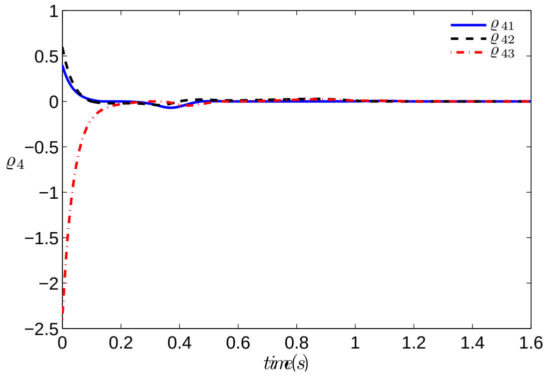

Fig. 4. Error evolution of the fourth node between the drive-response network (20) and (21)

stabilization theorem of time-delay system, some useful time-delay independent GFPFTLS criteria have been obtained with Razumikhin analytical approach.

Also an illustrative example with MATLAB numerical simulation basing on the different dimensions of nodes was given to demonstrate the feasibility of the proposed synchronization methods. In the simulations setction, the author took heterogeneity of nodes and time-varying delay caused by information transmission into account. As the whole systems' error needed for the controller designed rather than the isolate node error of each network, much larger feedback gains are needed in some cases.

With the help of the Razumikhin approach, considering the systems' general function prescribed/fixed-time projective lag synchronization with distributed time delay, stochastic noise which is agreement with the branches of science and industry will be the author's future work. Also how to expand the theoretical results to the practical industrial and medical applications will be taken into account next.

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