Some Eisenstein Identities Involving Borweins' Cubic Theta Functions and Evaluation of Compound Convolution Sum

Smitha Ganesh Bhat, Vidya Harekala Chandrashekara* and Ashwath Rao Badanidiyoor

***Abstract*—The $(p,\ k)$-parametrization technique introduced by Alaca provides an alternative approach for formulating several new Eisenstein series identities involving Borwein's theta functions. As a practical example, we have computed the convolution sum of divisor functions, which exhibit a connection with Eisenstein series.**

***Index Terms*—Cubic Theta Functions, Eisenstein Series, Convolution Sum, Digital Signal Processing.**

I. Introduction

CONVOLUTION is a fundamental operation in signal processing and mathematics, and it plays a crucial role in various fields. It involves the mathematical combination of two signals to create a third signal, typically representing the relationship between the input and output signals in a system. While the description you provided is accurate, I'll rephrase it to offer an alternative explanation:

Convolution serves as a vital mathematical tool employed to generate an output signal by combining two input signals – the input signal and the impulse response. In functional analysis, it stands as a core concept, elucidating the transformation of one signal's shape by another. The convolution of two functions involves integrating the product of these functions, where one function undergoes reflection about the y-axis and subsequent shifting.

In the realm of Digital Signal Processing (DSP), convolution emerges as a potent technique for manipulating and processing signals. Remarkably, it is versatile enough to be applied to both discrete and continuous signals. The impulse response signal, often symbolized as $h(n)$, finds representation as a scaled and shifted unit impulse function denoted as $\delta(n)$. This fundamental mathematical operation plays a pivotal role in comprehending how systems react to diverse inputs, finding widespread applications in areas such as filtering, image processing, and beyond.

Convolution is represented by $*$ symbol. The Fig. 1 shows an input signal $x(n)$ convolved with $h(n)$ to produce $y(n)$ in a linear system.

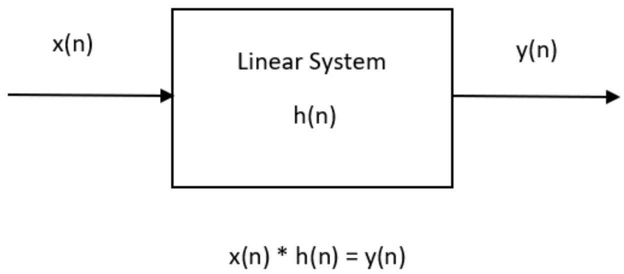

Fig. 1. The input signal $x(n)$ is convolved with impulse function $h(n)$ to produce output signal $y(n)$

In DSP applications [16], it's common to encounter input signals with a large number of data points while impulse signals are typically limited in size. The length of the output signal resulting from convolution is determined by adding the lengths of the input signal and the impulse signal and then subtracting one. Convolution exhibits several important mathematical properties: it is commutative, meaning that $x(n)*h(n) = h(n)*x(n)$; it is associative, so $(x(n)*h(n))*w(n) = x(n)*(h(n)*w(n))$; and it is distributive, which implies that $x(n)*(h(n)+w(n)) = x(n)*h(n)+x(n)*w(n)$.

Convolution finds application in a wide range of mathematical topics, including statistics and probability, as well as in various fields such as acoustics, spectroscopy, signal processing, image processing, geophysics, engineering, physics, computer vision, and the study of differential equations. Some practical applications of convolution include edge detection, correcting out-of-focus photographs, image blurring, applying smoothing filters, simulating reverberation effects, and modeling linear time-invariant systems.

In Section 2, we present preliminary results that align with our core objectives.

Moving on to Sections 3, we introduce and confirm several intriguing mathematical identities, some of which draw inspiration from the work of Earnest Xia but stand as entirely novel contributions. Notable among these discoveries are the Ramanujan-Eisenstein series and Borwein's cubic theta functions.

In Section 4, we analyze a discrete compound convolution sum, expanding on the convolution sum previously introduced in reference [19]. This analysis brings a fresh perspective to the topic.

Manuscript received on 13 Sep 2023, revised on 7 Dec 2023.

Smitha Ganesh Bhat is Assistant Professor-Senior Scale at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (e-mail:smitha.holla@manipal.edu)

Vidya Harekala Chandrashekara is Assistant Professor-Senior Scale at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (corresponding author to provide e-mail:vidyaashwath@gmail.com)

Ashwath Rao Badanidiyoor is Assistant Professor-Selection Grade at Department of Computer Science, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (e-mail:ashwath.rao.b@gmail.com)

II. Preliminaries

The origins of the arithmetic-geometric mean iteration can be traced back to the theories of elliptic functions and theta functions. The Borwein brothers [8], [9] derived the following multidimensional theta functions:

$$a(q) := \sum_{r,s=-\infty}^{\infty} q^{r^2+rs+s^2}.$$
$$b(q) := \sum_{r,s=-\infty}^{\infty} \omega^{r-s} q^{r^2+rs+s^2}.$$
$$c(q) := \sum_{r,s=-\infty}^{\infty} q^{\left(r+\frac{1}{3}\right)^2+\left(r+\frac{1}{3}\right)\left(s+\frac{1}{3}\right)+\left(s+\frac{1}{3}\right)^2}.$$

for $|q| < 1$, where q represents complex numbers, and$\omega = exp(2\pi i/3)$ is the principal cube root of unity, the given expressions for two-dimensional theta functions reveal that when $q = 0$, the values become $a(q) = 1$, $b(q) = 1$, and $c(q) = 0$.

From Euler's binomial theorem, Borwein brothers have devised expressions for $b(q)$ and $c(q)$ in terms of infinite products, which are given by,

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty},$$
$$c(q) = \frac{3q^{\frac{1}{3}}(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

The function $a(q)$ can be expressed as follows (J.M. Borwein & P. B. Borwein [8]):

$$a(q) = (-q;q^2)_\infty^2(q^2;q^2)_\infty(q^3;q^6)_\infty^2(q^6;q^6)_\infty + 4q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}.$$

The proofs for Equations (1), (1) and (1) are quintessentially classic and can be found in the works of Borwein et al. [9].

Besides that, they have established the fundamental relationship between $a(q)$, $b(q)$ and $c(q)$ which is a basic cubic identity given by,

$$a^3(q) = b^3(q) + c^3(q).$$

Definition II.1. *Srinivasa Ramanujan in his second notebook [14] has provided the definitions for the Eisenstein Series $L(q)$, $M(q)$ and $N(q)$ as follows:*

$$L(q) := 1 - 24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} := 1 - 24\sum_{r=1}^{\infty}\delta_1(r)q^r,$$
$$M(q) := 1 + 240\sum_{r=1}^{\infty}\frac{r^3q^r}{1-q^r} := 1 + 240\sum_{r=1}^{\infty}\delta_3(r)q^r,$$
$$N(q) := 1 - 504\sum_{r=1}^{\infty}\frac{r^5q^r}{1-q^r} := 1 - 504\sum_{r=1}^{\infty}\delta_5(r)q^r.$$

Definition II.2. *For any complex c and d, Ramanujan[7, p.35] documented a general theta function,*

$$f(c,d) := \sum_{m=-\infty}^{\infty} c^{m(m+1)/2}d^{m(m-1)/2} := (-c;cd)_\infty(-d;cd)_\infty(cd;cd)_\infty,$$

where

$$(c;q)_\infty := \prod_{m=0}^{\infty}(1-cq^m), \qquad |q|<1.$$

The special case of theta function defined by Ramanujan[7, p.35],

$$\varphi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty.$$

In their remarkable article, Alaca et al. [1] have defined the $(p,\ k)$ parametrization of theta functions. These are highly significant in designing the duplication and triplication principle and further obtaining certain sum to product identities. The parameters p and k are defined as:

$$p := p(q) = \frac{\varphi^2(q)-\varphi^2(q^3)}{2\phi^2(q^3)}.$$
$$k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}.$$

Since $\varphi(0) = 1$, it clear that $p(0) = 0$ and $k(0) = 1$.

Lemma II.3. *[1] For aforementioned Eisenstein series [8], [9], the representations of $M(q)$, $M(q^l)$, $L(q) - lL(q^l)$, $(l = 2,3,4,6,12)$ and also $L(-q^l) - rL(q^r)$, $l \in \{1,3\}$ and $r \in \{1,2,3\}$, in terms of the parameters p and k are given by,*

$$M(q) = (1 + 124p(1+p^6) + 964p^2(1+p^4) + 2788p^3(1 + p^2) + 3910p^4 + p^8)k^4,$$
$$M(q^2) = (1 + 4p(1+p^6) + 64p^2(1+p^4) + 178p^3(1+p^2) + 235p^4 + p^8)k^4,$$
$$M(q^3) = (1 + 4p(1+p^6) + 4p^2(1+p^4) + 28p^3(1+p^2) + 70p^4 + p^8)k^4,$$
$$M(q^6) = (1 + 4p(1+p^6) + 4p^2(1+p^4) - 2p^3(1+p^2) - 5p^4 + p^8)k^4,$$
$$M(q^{12}) = (1 + 4p(1+p) - 2p^3(1+p^2) - 5p^4 + p^6(1+p)/4 + p^8/16)k^4,$$
$$L(-q) - L(q) = 3(8p + 12p^2 + 6p^3 + p^4)k^2,$$
$$L_{1,2}(q) = (L(-q) - L(q))/48 = (p/2 + 3p^2/4 + 3p^3/8 + p^4/16)k^2,$$
$$L_{1,2}(q^3) = (L(-q^3) - L(q^3))/48 = p^3(2+p)k^2/16,$$
$$L(-q) - 2L(q^2) = -(1 - 10p - 12p^2 - 4p^3 - 2p^4)k^2,$$
$$L(q) - 2L(q^2) = -(1 + 14p(1+p^2) + 24p^2 + p^4)k^2,$$
$$L(q) - 3L(q^3) = -(1 + 8p(1+p^2) + 18p^2 + p^4)k^2,$$
$$L(q) - 6L(q^6) = -(5 + 22p(1+p^2) + 36p^2 + 5p^4)k^2,$$
$$L(q^2) - 3L(q^6) = -2(1 + 2p(1+p^2) + 3p^2 + p^4)k^2,$$
$$L(q^3) - 2L(q^6) = -(1 + 2p(1+p^2) + p^4)k^2,$$
$$L(q) - 4L(q^4) = -3(1 + 6p + 12p^2 + 8p^3)k^2,$$
$$L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2.$$

Lemma II.4. *The parametric representations of $a(q^r), b(q^r), c(q^r) (r \in 1,2,4,6)$ and $a(-q)$, $b(-q)$, $c(-q)$ in terms of the parameters p and k deduced by Alaca et al. [1] are as follows.*

$$a(-q) = (1-2p-2p^2)k,$$
$$a(q) = (1+4p+p^2)k,$$
$$a(q^2) = (1+p+p^2)k,$$
$$a(q^4) = (1+p-\frac{1}{2}p^2)k,$$
$$a(q^6) = \frac{(p^2+p+1+2^{1/3}((1-p)(2+p)(1+2p))^{2/3})k}{3},$$
$$b(-q) = 2^{-\frac{1}{3}}((1-p)(1+2p)^4(2+p))^{\frac{1}{3}}k,$$
$$b(q) = 2^{-\frac{1}{3}}((1-p)^4(1+2p)(2+p))^{\frac{1}{3}}k,$$
$$b(q^2) = 2^{-2/3}((1-p)(1+2p)(2+p))^{\frac{2}{3}}k,$$
$$b(q^4) = 2^{-\frac{4}{3}}((1-p)(1+2p)(2+p)^4)^{\frac{1}{3}}k,$$
$$c(-q) = -2^{\frac{1}{3}}3(p(1+p))^{\frac{1}{3}}k,$$
$$c(q) = 2^{-\frac{1}{3}}3(p(1+p)^4)^{\frac{1}{3}}k,$$
$$c(q^2) = 2^{-\frac{2}{3}}3(p(1+p))^{\frac{2}{3}}k.$$
$$c(q^4) = 2^{-\frac{4}{3}}3(p^4(1+p))^{\frac{1}{3}}k,$$
$$c(q^6) = \frac{\left(p^2+p+1-2^{-2/3}((1-p)(2+p)(1+2p))^{2/3}\right)k}{3}.$$

III. Relations comprising Ramanujan's Eisenstein series and Cubic Theta Functions

Ramanujan in his notebook [14] recorded some interesting series that involves L, M & N. Consequently, he furnished numerous notable identities for infinite series that encompass theta functions. Xia et. al [22], as described in their work, employed computational methods to discover elegant mathematical identities involving Eisenstein series and cubic theta functions. These identities revolved around expressions of the form $L(q) - rL(q^r)$, where 'r' takes on values from the set $\{2, 3, 4, 6, 12\}$. Some additional identities have been deduced by Shruti and Srivatsakumar B.R. [15] and the convolution sum has been evaluated. Recently, Vidya H. C. and Ashwath Rao B. [18], Vidya H. C. and Smitha G. Bhat [19] have formulated few identities that includes $L(-q^l) - L(q^l)$, for $l \in \{1,3\}$ as well. Likewise, Vidya H. C. and Ashwath Rao B. [11], have deduced relationships among theta functions. In our paper, we established specific identities connecting Ramanujan-type Eisenstein series and cubic theta functions, with a particular emphasis on Eisenstein series $M(q^n)$, for $n = 1, 2, 3, 6, 12$, accomplished without computer assistance. Additionally, we applied these findings to evaluate compound convolution sums.

Theorem III.1. *The connection between an infinite series and theta functions is as follows:*

$$(i)\ 1-24(1+u)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}\right]+(132 + 144u)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right]-96u\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right] + 108\sum_{r=1}^{\infty}\left[\frac{rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}. \quad (1)$$

$$(ii)\ 1+(6-24u)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right]-24u\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] -(3+144u)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right]-96u\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right] -27\sum_{r=1}^{\infty}\left[\frac{rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(q^2)b(q^4)}{b(-q)}. \quad (2)$$

$$(iii)\ 6+(48-24u)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right]-24u\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] -(48-144u)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right]-96u\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right], -144\sum_{r=1}^{\infty}\left[\frac{rq^{6r}}{1-q^{6r}}\right]=\frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}. \quad (3)$$

$$(iv)\ 3-24u\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right]+(24-24u)\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] -(24-144u)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right]-96u\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right] -72\sum_{r=1}^{\infty}\left[\frac{rq^{6r}}{1-q^{6r}}\right]=\frac{a(q)c^2(q)}{c(q^2)}. \quad (4)$$

$$(v)\ (8-24u)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right]+(8-24u)\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] +(4+144u)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right]-48\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}\right] -96u\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right]-12\sum_{r=1}^{\infty}\left[\frac{rq^{6r}}{1-q^{6r}}\right]+\frac{5}{3} =\frac{b^2(q)c^2(-q)}{2^{\frac{4}{3}}b(q^2)c(q^2)}. \quad (5)$$

Proof: Let us persume that,

$$C_1L_{1,2}(q)+C_2[2L(q^2)-L(q)]+C_3[3L(q^3)-L(q)] +C_4[4L(q^4)-L(q)]+C_5[6L(q^6)-L(q)] =\frac{b(q)b(-q)b(q^2)}{b(q^4)}. \quad (6)$$

The equation above undergoes transformation through (p,k) parametrization using Lemma II.3. This leads to the derivation of a system of non-homogeneous linear equations, where the coefficients of terms involving k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 in the left-hand side are equated with their corresponding terms in the right-hand side. These equations must then be solved to find the unknown values.

$$\begin{pmatrix} 0 & 1 & 1 & 3 & 5 \\ \frac{1}{2} & 14 & 8 & 18 & 22 \\ \frac{3}{4} & 24 & 18 & 36 & 36 \\ \frac{3}{8} & 14 & 8 & 24 & 22 \\ \frac{1}{16} & 1 & 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ -4 \\ 4 \end{pmatrix}.$$

On solving the above system, we get,

$$C_1 = 48 + 48u,\ C_2 = -\frac{11}{4} - 3u,\ C_3 = 0,\ C_4 = u \text{ and } C_5 = \frac{3}{4}.$$

Upon replacing the previously mentioned statistics into (6) and subsequently streamlining the process with the help of Definition II.1, we arrive at equation (1). Likewise, employing the same approach, we deduce the following identities. Altering the right-hand side of (1) and subsequently using (6), results in equations (i) to (iv).

$$(i)\ (-12 + 48u)L_{1,2}(q) + \left(\frac{1}{16} - 3u\right)[2L(q^2) - L(q)] + u[4L(q^4) - L(q)] + \frac{3}{16}[6L(q^6) - L(q)] = \frac{b(q)b(q^2)b(q^4)}{b(-q)}.$$

$$(ii)\ (-96 + 48u)L_{1,2}(q) + (1 - 3u)[2L(q^2) - L(q)] + u[4L(q^4) - L(q)] + [6L(q^6) - L(q)] = \frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$(iii)\ 48uL_{1,2}(q) + (\frac{1}{2} - 3u)[2L(q^2) - L(q)] + u[4L(q^4) - L(q)] + \frac{1}{2}[6L(q^6) - L(q)] = \frac{a(q)c^2(q)}{c(q^2)}.$$

$$(iv)\ (-16 + 48u)L_{1,2}(q) - (\frac{1}{12} + 3u)[2L(q^2) - L(q)] + \frac{2}{3}[3L(q^3) - L(q)] + u[4L(q^4) - L(q)] + \frac{1}{12}[6L(q^6) - L(q)] = \frac{b^2(q)c^2(-q)}{2^{\frac{4}{3}}b(q^2)c(q^2)}.$$

On simplifying the above equations we get equations (2) to (5). ■

Theorem III.2. *The relation amongst an infinite series and theta functions holds:*

$$(i)\ \left(2u - \frac{7}{10}\right) - \left(\frac{18}{4} - 48u\right)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right] - \frac{84}{5}\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] + \left(\frac{852}{5} - 96u\right)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right] - 378\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}\right] + \frac{1224}{5}\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right] = \frac{b(q)b(-q)b(q^2)}{b(q^4)}. \quad (7)$$

$$(ii)\ \left(2u - \frac{37}{40}\right) - \left(\frac{249}{10} - 48u\right)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right] + \frac{9}{5}\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] + \left(\frac{393}{5} - 96u\right)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right] - \frac{189}{2}\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}\right] + \frac{306}{5}\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right] = \frac{b(q)b(q^2)b(q^4)}{b(-q)}. \quad (8)$$

$$(iii)\ \left(2u - \frac{28}{5}\right) - \left(\frac{744}{5} - 48u\right)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right] + \frac{48}{5}\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] + \left(\frac{2256}{5} - 96u\right)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right] - 504\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}\right] + \frac{1632}{5}\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right] = \frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}. \quad (9)$$

$$(iv)\ \left(2u - 4\right) - \left(\frac{492}{5} - 48u\right)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right] + \frac{144}{5}\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right] + \left(168 - 96u\right)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right] - \frac{828}{5}\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}\right] + \frac{816}{5}\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right] = \frac{a(q)c^2(q)}{c(q^2)}. \quad (10)$$

$$(v)\ \left(2u - \frac{49}{30}\right) - \left(\frac{202}{5} - 48u\right)\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}\right]$$

$$+\frac{44}{5}\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}\right]+\left(\frac{388}{5}-96u\right)\sum_{r=1}^{\infty}\left[\frac{rq^{2r}}{1-q^{2r}}\right]$$
$$-66\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}\right]+\frac{296}{5}\sum_{r=1}^{\infty}\left[\frac{rq^{4r}}{1-q^{4r}}\right]$$
$$=\frac{b^2(q)c^2(-q)}{2^{\frac{4}{3}}b(q^2)c(q^2)}. \quad (11)$$

Proof: Let us persume that,

$$C_1[L(-q)-L(q)]+C_2[2L(q^2)-L(-q)]+C_3[2L(q^2)-L(q)]+C_4[3L(q^3)-2L(q^2)]+C_5[4L(q^4)-3L(q^3)]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}. \quad (12)$$

We apply (p,k) parametrization to the equation above using Lemma II.3. This yields a system of non-homogeneous linear equations where we equate coefficients involving k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 in the left-hand side to their counterparts in the right-hand side. These equations require solution to find the unknown values.

$$\begin{pmatrix} 0 & -1 & 1 & 1 & 1 \\ 24 & 10 & 14 & 12 & 2 \\ 36 & 12 & 24 & 12 & 0 \\ 18 & 4 & 14 & 12 & 8 \\ 3 & 2 & 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ -3 \\ -4 \\ 4 \end{pmatrix}.$$

On solving the above system, we get,

$$C_1=-\frac{7}{10}-u,\ C_2=-\frac{17}{20}+u,\ C_3=u,\ C_4=\frac{27}{10}\text{ and } C_5=-\frac{51}{20}.$$

By substituting the above statistics in (12) and further simplifying using Definition II.1, the equation (7) is obtained. Similarly, altering the right-hand side of (7) and subsequently using (12), results in equations (i) to (iv).

$$(i)\ \left(\frac{3}{40}-u\right)[L(-q)-L(q)]-\left(\frac{77}{80}-u\right)[2L(q^2)-L(-q)]+u[2L(q^2)-L(q)]+\frac{27}{40}[3L(q^3)-2L(q^2)]-\frac{51}{80}[4L(q^4)-3L(q^3)]=\frac{b(q)b(q^2)b(q^4)}{b(-q)}.$$

$$(ii)\ \left(\frac{2}{5}-u\right)[L(-q)-L(q)]-\left(\frac{29}{5}-u\right)[2L(q^2)-L(-q)]+u[2L(q^2)-L(q)]+\frac{18}{5}[3L(q^3)-2L(q^2)]-\frac{17}{5}[4L(q^4)-3L(q^3)]=\frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$(iii)\ \left(\frac{6}{5}-u\right)[L(-q)-L(q)]-\left(\frac{29}{10}-u\right)[2L(q^2)-L(-q)]+u[2L(q^2)-L(q)]+\frac{9}{15}[3L(q^3)-2L(q^2)]-\frac{17}{10}[4L(q^4)-3L(q^3)]=\frac{a(q)c^2(q)}{c(q^2)}.$$

$$(iv)\ \left(\frac{11}{30}-u\right)[L(-q)-L(q)]-\left(\frac{79}{60}-u\right)[2L(q^2)-L(-q)]+u[2L(q^2)-L(q)]+\frac{3}{10}[3L(q^3)-2L(q^2)]-\frac{37}{60}[4L(q^4)-3L(q^3)]=\frac{b^2(q)c^2(-q)}{2^{\frac{4}{3}}b(q^2)c(q^2)}.$$

On simplifying the above equations we get equations (8) to (11). ∎

Theorem III.3. *The connection between an infinite series and theta functions is as follows:*

$$[i]\ 1-3\sum_{1}^{\infty}\frac{r^3q^r}{1-q^r}+243\sum_{1}^{\infty}\frac{r^3q^{3r}}{1-q^{3r}}=a(q)b^3(q). \quad (13)$$

$$[ii]\ 1+24\sum_{1}^{\infty}\frac{r^3q^r}{1-q^r}+216\sum_{1}^{\infty}\frac{r^3q^{3r}}{1-q^{3r}}=[3a(q^3)-2b(q)]^2. \quad (14)$$

$$[iii]\ -3-72\sum_{1}^{\infty}\frac{r^3q^r}{1-q^r}-324\sum_{1}^{\infty}\frac{r^3q^{3r}}{1-q^{3r}}+\left[-1+24\sum_{1}^{\infty}\frac{2rq^{2r}}{1-q^{2r}}-\sum_{1}^{\infty}\frac{rq^r}{1-q^r}\right]^2+3\left[-1+24\sum_{1}^{\infty}\frac{3rq^{3r}}{1-q^{3r}}-\sum_{1}^{\infty}\frac{2rq^{2r}}{1-q^{2r}}\right]^2=\frac{b^8(q)}{b^4(q^2)}. \quad (15)$$

Proof:

$$C_1M(q)+C_2M(q^3)+C_3M(q^{12})+C_4\{L(q)-2L(q^2)\}^2+C_5\{L(q^2)-3L(q^3)\}^2+C_6\{3L(q^3)-4L(q^4)\}^2+C_7\{4L(q^4)-6L(q^6)\}^2+C_8\{L(q^4)-3L(q^{12})\}^2=a(q)b^3(q). \quad (16)$$

We apply (p,k) parametrization as per Lemma II.3 to modify the equation above. This leads to a set of non-homogeneous linear equations, achieved by equating coefficients of terms involving $k^4, pk^4, p^2k^4, p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on the left-hand side to their respective terms on the right-hand side, necessitating the solution for unknown values.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 4 & 4 \\ 124 & 4 & 4 & 28 & 4 & 4 & 16 & 16 \\ 964 & 4 & 4 & 244 & 28 & 4 & 16 & 16 \\ 2788 & 28 & -2 & 700 & 52 & 16 & -8 & -8 \\ 3910 & 70 & -5 & 970 & 154 & 28 & 4 & -14 \\ 2788 & 28 & -2 & 700 & 52 & -8 & 40 & 4 \\ 964 & 4 & \frac{1}{4} & 244 & 28 & 64 & 4 & 4 \\ 124 & 4 & \frac{1}{4} & 28 & 4 & -32 & -20 & -2 \\ 1 & 1 & \frac{1}{16} & 1 & 1 & 4 & 25 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ -8 \\ -\frac{13}{2} \\ 22 \\ -\frac{13}{2} \\ -8 \\ \frac{5}{2} \\ 1 \end{pmatrix}.$$

We note that, the system results in a unique solution,

$$C_1 = -\frac{1}{80},\ C_2 = \frac{81}{80},\ C_3 = 0,\ C_4 = 0,\ C_5 = 0,\ C_6 = 0,\ C_7 = 0,\ C_8 = 0.$$

Substituting these values in (16) yields, (13). Similarly, altering the right-hand side of (13) and subsequently using (16), results in equations (i) to (ii).

$$[i]\ \frac{1}{10}M(q) + \frac{9}{10}M(q^3) = \{3a(q^3) - 2b(q)\}^2.$$

$$[ii]\ -\frac{3}{10}M(q) - \frac{27}{10}M(q^3) + \{L(q) - 2L(q^2)\}^2 + 3\{L(q^2) - 3L(q^3)\}^2 = \frac{b^8(q)}{b^4(q^2)}.$$

On simplifying the above equations we get equations (14) to (15). ■

Theorem III.4. *The relation amongst an infinite series and theta functions holds:*

$$[i]\ -2 - 24\sum_1^\infty \frac{r^3q^r}{1-q^r} + 192\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} - 648\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} + 3\Bigg[-1 + 24\sum_1^\infty \frac{3rq^{3r}}{1-q^{3r}} - \sum_1^\infty \frac{2rq^{2r}}{1-q^{2r}}\Bigg]^2 = \frac{b^3(q)}{b^4(q^2)}. \tag{17}$$

$$[ii]\ 1 - 3\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} + 243\sum_1^\infty \frac{r^3q^{6r}}{1-q^{6r}} = a(q^2)b^3(q^2). \tag{18}$$

$$[iii] 1 - 3\sum_1^\infty \frac{r^3q^r}{1-q^r} + 243\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} = a(q)b^3(q). \tag{19}$$

$$[iv]\ 1 + 18\sum_1^\infty \frac{r^3q^r}{1-q^r} - 48\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} - 162\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} + 432\sum_1^\infty \frac{r^3q^{6r}}{1-q^{6r}} = a^3(q)a(q^2). \tag{20}$$

$$[v]\ 1 + 24\sum_1^\infty \frac{r^3q^r}{1-q^r} + 216\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} = [3a(q^3) - 2b(q)]^2. \tag{21}$$

$$[vi]\ 1 + 6\sum_1^\infty \frac{r^3q^r}{1-q^r} - 36\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} - 54\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} + 324\sum_1^\infty \frac{r^3q^{6r}}{1-q^{6r}} = a(q)a^3(q^2). \tag{22}$$

$$[vii]\ \frac{26}{27} + 8\sum_1^\infty \frac{r^3q^r}{1-q^r} - \frac{320}{9}\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} - \frac{248}{3}\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} + \frac{1024}{3}\sum_1^\infty \frac{r^3q^{6r}}{1-q^{6r}} + \frac{1}{27}\Bigg[-1 + 24\sum_1^\infty \frac{3rq^{3r}}{1-q^{3r}} - \sum_1^\infty \frac{2rq^{2r}}{1-q^{2r}}\Bigg]^2 = \frac{c^8(q)}{81c^4(q^2)}. \tag{23}$$

$$[viii]\ \frac{1}{216} + \frac{40}{9}\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} + \frac{22}{3}\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} - \frac{32}{3}\sum_1^\infty \frac{r^3q^{6r}}{1-q^{6r}} - \frac{1}{216}\Bigg[-1 + 24\sum_1^\infty \frac{3rq^{3r}}{1-q^{3r}} - \sum_1^\infty \frac{2rq^{2r}}{1-q^{2r}}\Bigg]^2 = \frac{c^2(q)c^2(q^2)}{81}. \tag{24}$$

$$[ix]\ \frac{1}{72} + \frac{16}{3}\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} + 22\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} - 24\sum_1^\infty \frac{r^3q^{6r}}{1-q^{6r}} - \frac{1}{72}\Bigg\{-1 + 24\sum_1^\infty \frac{3rq^{3r}}{1-q^{3r}} - \sum_1^\infty \frac{2rq^{2r}}{1-q^{2r}}\Bigg\}^2 = \frac{4}{27}a(q)c^3(q^2). \tag{25}$$

$$[x]\ -\frac{1}{1728} - \frac{1}{18}\sum_1^\infty \frac{r^3q^{2r}}{1-q^{2r}} + \frac{1}{12}\sum_1^\infty \frac{r^3q^{3r}}{1-q^{3r}} - \frac{1}{6}\sum_1^\infty \frac{r^3q^{6r}}{1-q^{6r}} + \frac{1}{1728}\Bigg[-1 + 24\sum_1^\infty \frac{3rq^{3r}}{1-q^{3r}} - \sum_1^\infty \frac{2rq^{2r}}{1-q^{2r}}\Bigg]^2 = \frac{c^8(q)}{81c^4(q)}. \tag{26}$$

Proof: Let us persume that,

$$
\begin{aligned}
&C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)+C_5M(q^{12})\\
&\quad+C_6\{L(-q)-2L(q^2)\}^2+C_7\{2L(q^2)-3L(q^3)\}^2\\
&\quad+C_8\{3L(q^3)-4L(q^4)\}^2=\frac{b^3(q)}{b^4(q^2)}. \qquad (27)
\end{aligned}
$$

Following the (p,k) parametrization of the expression above using Lemma II.3, and by equating the coefficients of$k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on both sides, we acquire

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
124 & 4 & 4 & 4 & 4 & -20 & 4 & 4\\
964 & 64 & 4 & 4 & 4 & 76 & 28 & 4\\
2788 & 178 & 28 & -2 & -2 & 232 & 52 & 16\\
3910 & 235 & 70 & -5 & -5 & 220 & 154 & 28\\
2788 & 178 & 28 & -2 & -2 & 136 & 52 & -8\\
964 & 64 & 4 & 4 & \frac{1}{4} & 64 & 28 & 64\\
124 & 4 & 4 & 4 & \frac{1}{4} & 16 & 4 & -32\\
1 & 1 & 1 & 1 & \frac{1}{16} & 4 & 1 & 4
\end{pmatrix}
\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8 \end{pmatrix}
=
\begin{pmatrix} 1\\ -8\\ 28\\ -56\\ 70\\ -56\\ 28\\ -8\\ 1 \end{pmatrix}.
$$

We note that, the system results in a unique solution,

$$
C_1=-\frac{1}{10},\ C_2=\frac{4}{5},\ C_3=-\frac{27}{20},\ C_4=0,\ C_5=0,\ C_6=0,\ C_7=3,\ C_8=0.
$$

Substituting these values in (27) yields, (17). Similarly, altering the right-hand side of (17) and subsequently using (27), results in equations (i) to (ix).

$$[i]\ -\frac{1}{80}M(q^2)+\frac{81}{80}M(q^6)=a(q^2)b^3(q^2).$$

$$[ii]\ -\frac{1}{80}M(q)+\frac{81}{80}M(q^3)=a(q)b^3(q).$$

$$[iii]\ \frac{3}{40}M(q)-\frac{1}{5}M(q^2)-\frac{27}{40}M(q^3)+\frac{9}{5}M(q^6)=a^3(q)a(q^2).$$

$$[iv]\ \frac{1}{10}M(q)+\frac{9}{10}M(q^3)=[3a(q^3)-2b(q)]^2.$$

$$[v]\ \frac{1}{40}M(q)-\frac{3}{20}M(q^2)-\frac{9}{40}M(q^3)+\frac{27}{20}M(q^6)=a(q)a^3(q^2).$$

$$[vi]\ \frac{1}{30}M(q)-\frac{4}{27}M(q^2)-\frac{31}{90}M(q^3)+\frac{64}{45}M(q^6)+\frac{1}{27}\{2L(q^2)-3L(q^3)\}^2=\frac{c^8(q)}{81c^4(q^2)}.$$

$$[vii]\ \frac{1}{54}M(q^2)+\frac{11}{360}M(q^3)-\frac{2}{45}M(q^6)-\frac{1}{216}\{2L(q^2)-3L(q^3)\}^2=\frac{c^2(q)c^2(q^2)}{81}.$$

$$[viii]\ \frac{1}{45}M(q^2)+\frac{11}{120}M(q^3)-\frac{1}{10}M(q^6)-\frac{1}{72}\{2L(q^2)-3L(q^3)\}^2=\frac{4}{27}a(q)c^3(q^2).$$

$$[ix]\ -\frac{1}{4320}M(q^2)+\frac{1}{2880}M(q^3)-\frac{1}{1440}M(q^6)+\frac{1}{1728}\{2L(q^2)-3L(q^3)\}^2=\frac{c^8(q)}{81c^4(q)}.$$

On simplifying the above equations we get equations (18) to (26). ■

Theorem III.5. *The connection between an infinite series and theta functions is as follows:*

$$
\begin{aligned}
&[i]\ (1-4u)+240\sum_1^\infty\Bigg[\Big(-\frac{1}{10}-\frac{2u}{5}\Big)\frac{r^3q^r}{1-q^r}\\
&+\Big(\frac{9}{10}-\frac{18u}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}\Bigg]+u\Bigg[-2+24\sum_1^\infty\Bigg[\frac{3rq^{3r}}{1-q^{3r}}\\
&-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2=[3a(q^3)-2b(q)]^2. \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
&[ii]\ (1-4u)+240\sum_1^\infty\Bigg[-\frac{2u}{5}\frac{r^3q^r}{1-q^r}-\frac{1}{80}\frac{r^3q^{2r}}{1-q^{2r}}\\
&-\frac{18u}{5}\frac{r^3q^{3r}}{1-q^{3r}}+\frac{81}{80}\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]+u\Bigg[-2\\
&+24\sum_1^\infty\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2=a(q^2)b^3(q^2). \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
&[iii]\ (1-4u)+240\sum_1^\infty\Bigg[\Big(-\frac{1}{80}-\frac{2u}{5}\Big)\frac{r^3q^r}{1-q^r}\\
&+\Big(\frac{81}{80}-\frac{18u}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}\Bigg]+u\Bigg[-2+24\sum_1^\infty\Bigg[\frac{3rq^{3r}}{1-q^{3r}}\\
&-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2=a(q)b^3(q). \qquad (30)
\end{aligned}
$$

$$[iv]\ (1-4u)+240\sum_1^\infty\Bigg[\Big(\frac{3}{40}-\frac{2u}{5}\Big)\frac{r^3q^r}{1-q^r}$$

$$-\frac{1}{5}\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{27}{40}+\frac{18u}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}+\frac{9}{5}\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]$$
$$+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2=a^3(q)a(q^2). \tag{31}$$

$$[v]\ (1-4u)+240\sum_{1}^{\infty}\Bigg[\left(-\frac{1}{40}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r}$$
$$-\frac{3}{20}\frac{r^3q^{2r}}{1-q^{2r}}+\left(\frac{9}{40}-\frac{18u}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}+\frac{27}{20}\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]$$
$$+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2=a(q)a^3(q^2). \tag{32}$$

Proof:

$$C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)+C_5M(q^{12})$$
$$+C_6\{L(q)-3L(q^3)\}^2+C_7\{3L(q^3)-4L(q^4)\}^2$$
$$+C_8\{4L(q^4)-6L(q^6)\}^2=[3a(q^3)-2b(q)]^2. \tag{33}$$

Following the (p,k) parametrization of the expression above, as defined in Lemma II.3, and by setting the coefficients of $k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 equal on both sides, we arrive at the following result:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 4 & 1 & 4\\ 124 & 4 & 4 & 4 & 4 & 64 & 4 & 16\\ 964 & 64 & 4 & 4 & 4 & 400 & 4 & 16\\ 2788 & 178 & 28 & -2 & -2 & 1216 & 16 & -8\\ 3910 & 235 & 70 & -5 & -5 & 1816 & 28 & 4\\ 2788 & 178 & 28 & -2 & -2 & 1216 & -8 & 40\\ 964 & 64 & 4 & 4 & \frac{1}{4} & 400 & 64 & 4\\ 124 & 4 & 4 & 4 & \frac{1}{4} & 64 & -32 & -20\\ 1 & 1 & 1 & 1 & \frac{1}{16} & 4 & 4 & 25 \end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8 \end{pmatrix}=\begin{pmatrix} 1\\ 16\\ 100\\ 304\\ 454\\ 304\\ 100\\ 16\\ 1 \end{pmatrix}.$$

We note that, the system results in an infinitely many solutions,

$$C_1=\left(\frac{1}{10}-\frac{2u}{5}\right),\ C_2=0,\ C_3=\left(\frac{9}{10}-\frac{18u}{5}\right),\ C_4=0,$$
$$C_5=0,\ C_6=u,\ C_7=0,\ C_8=0.$$

Substituting these values in (33) yields, (28). Altering the right-hand side of (28) and subsequently using (33) results in equations (i) to (iv).

$$[i]\ -\frac{2u}{5}M(q)-\frac{1}{80}M(q^2)-\frac{18u}{5}M(q^3)+\frac{81}{80}M(q^6)$$
$$+u\{L(q)-3L(q^3)\}^2=a(q^2)b^3(q^2).$$

$$[ii]\ \left(-\frac{1}{80}-\frac{2u}{5}\right)M(q)+\left(\frac{81}{80}-\frac{18u}{5}\right)M(q^3)+u\{L(q)$$
$$-3L(q^3)\}^2=a(q)b^3(q).$$

$$[iii]\ \left(\frac{3}{40}-\frac{2u}{5}\right)M(q)-\frac{1}{5}M(q^2)-\left(\frac{27}{40}+\frac{18u}{5}\right)M(q^3)$$
$$+\frac{9}{5}M(q^6)+u\{L(q)-3L(q^3)\}^2=a^3(q)a(q^2).$$

$$[iv]\ \left(\frac{1}{40}-\frac{2u}{5}\right)M(q)-\frac{3}{20}M(q^2)-\left(\frac{9}{40}-\frac{18u}{5}\right)M(q^3)$$
$$+\frac{27}{20}M(q^6)+u\{L(q)-3L(q^3)\}^2=a(q)a^3(q^2).$$

On simplifying the above equations we get equations (29) to (32). ■

Theorem III.6. *The relation amongst an infinite series and theta functions holds:*

$$[i]\ (1-4u-v)+240\sum_{1}^{\infty}\Bigg[-\frac{2u}{5}\frac{r^3q^r}{1-q^r}$$
$$-\left(\frac{1}{80}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{18u}{5}+\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}$$
$$+\left(\frac{81}{80}-\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}$$
$$-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2+v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}$$
$$-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2=a(q^2)b^3(q^2). \tag{34}$$

$$[ii]\ (1-4u-v)+240\sum_{1}^{\infty}\Bigg[\left(-\frac{1}{80}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r}$$
$$+\left(\frac{81}{80}-\frac{18u}{5}-\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}+\left(\frac{81}{80}-\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]$$
$$+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2$$
$$+v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2=a(q)b^3(q). \tag{35}$$

$$[iii]\ (1-4u-v)+240\sum_{1}^{\infty}\Bigg[\left(\frac{3}{40}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r}$$
$$-\left(\frac{1}{5}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{27}{40}+\frac{18u}{5}+\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}$$

$$+\left(\frac{9}{5}-\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]$$
$$+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2$$
$$+v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2$$
$$=a^3(q)a(q^2). \tag{36}$$

$$[iv]\ (1-4u-v)+240\sum_{1}^{\infty}\Bigg[\left(-\frac{1}{10}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r}$$
$$+\left(\frac{9}{10}-\frac{18u}{5}-\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}-\left(\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]$$
$$+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2$$
$$+v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2$$
$$=[3a(q^3)-2b(q)]^2. \tag{37}$$

$$[v]\ (1-4u-v)+240\sum_{1}^{\infty}\Bigg[\left(\frac{1}{40}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r}$$
$$-\left(\frac{3}{20}\right)\frac{r^3q^{2r}}{1-q^{2r}}+\left(\frac{9}{40}-\frac{18u}{5}-\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}$$
$$+\left(\frac{27}{20}-\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]$$
$$+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2$$
$$+v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2$$
$$=a(q)a^3(q^2). \tag{38}$$

Proof:

$$C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)+C_5M(q^{12})$$
$$+C_6\{L(q)-3L(q^3)\}^2+C_7\{3L(q^3)-4L(q^4)\}^2$$
$$+C_8\{L(q^3)-2L(q^6)\}^2=a(q^2)b^3(q^2). \tag{39}$$

Subsequently upon (p,k) parametrization of the above expression using Lemma II.3 and by equating the coefficients of $k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on either sides, we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 4 & 1 & 1 \\ 124 & 4 & 4 & 4 & 4 & 64 & 4 & 4 \\ 964 & 64 & 4 & 4 & 4 & 400 & 4 & 4 \\ 2788 & 178 & 28 & -2 & -2 & 1216 & 16 & 4 \\ 3910 & 235 & 70 & -5 & -5 & 1816 & 28 & 10 \\ 2788 & 178 & 28 & -2 & -2 & 1216 & -8 & 4 \\ 964 & 64 & 4 & 4 & \frac{1}{4} & 400 & 64 & 4 \\ 124 & 4 & 4 & 4 & \frac{1}{4} & 64 & -32 & 4 \\ 1 & 1 & 1 & 1 & \frac{1}{16} & 4 & 4 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{pmatrix}=\begin{pmatrix} 1 \\ 4 \\ \frac{13}{4} \\ -\frac{17}{4} \\ -8 \\ -\frac{17}{4} \\ \frac{13}{4} \\ 4 \\ 1 \end{pmatrix}.$$

We note that, the system results in an infinitely many solutions,

$$C_1=\left(-\frac{2u}{5}\right),\ C_2=\left(-\frac{1}{80}\right),$$
$$C_3=\left(-\frac{18u}{5}-\frac{v}{5}\right),\ C_4=\left(\frac{81}{80}-\frac{4v}{5}\right),$$
$$C_5=0,\ C_6=u,\ C_7=0,\ C_8=v.$$

Substituting these values in (39) yields, (34). On changing the right hand side in (34), and applying (39), we obtain equations (i) to (iv) below.

$$[i]\ -\left(\frac{1}{80}+\frac{2u}{5}\right)M(q)+\left(\frac{81}{80}-\frac{18u}{5}-\frac{v}{5}\right)M(q^3)$$
$$-\left(\frac{4v}{5}\right)M(q^6)+u\{L(q)-3L(q^3)\}^2$$
$$+v\{L(q^3)-2L(q^6)\}^2=a(q)b^3(q).$$

$$[ii]\ \left(\frac{3}{40}-\frac{2u}{5}\right)M(q)-\left(\frac{1}{5}\right)M(q^2)-\left(\frac{27}{40}+\frac{18u}{5}\right.$$
$$\left.+\frac{v}{5}\right)M(q^3)+\left(\frac{9}{5}-\frac{4v}{5}\right)M(q^6)+u\{L(q)-3L(q^3)\}^2$$
$$+v\{L(q^3)-2L(q^6)\}^2=a^3(q)a(q^2).$$

$$[iii]\ \left(\frac{1}{10}-\frac{2u}{5}\right)M(q)+\left(\frac{9}{10}-\frac{18u}{5}-\frac{v}{5}\right)M(q^3)$$
$$-\left(\frac{4v}{5}\right)M(q^6)+u\{L(q)-3L(q^3)\}^2$$
$$+v\{L(q^3)-2L(q^6)\}^2=\{3a(q^3)-2b(q)\}^2.$$

$$[iv]\ \left(\frac{1}{40}-\frac{2u}{5}\right)M(q)-\left(\frac{3}{20}\right)M(q^2)-\left(\frac{9}{40}\right.$$
$$\left.+\frac{18u}{5}+\frac{v}{5}\right)M(q^3)+\left(\frac{27}{20}-\frac{4v}{5}\right)M(q^6)$$
$$+u\{L(q)-3L(q^3)\}^2+v\{L(q^3)-2L(q^6)\}^2$$
$$=a(q)a^3(q^2).$$

Simplifying the above equations, we get equations (35) to (38). ■

IV. Evaluation of Convolution sum $\sum_{r=i+3j}\delta(i)\delta(j)$ using the existing convolution for $r=3i+6j$

Let $\mathbb{N}$ be the set of natural numbers. For $k,r\in\mathbb{N}$, we define

$$\delta_k(r)=\sum_{d/r}d^k.$$

where d runs through the non-negative integeral divisors of r.
For $i,\ j,\ r\in\mathbb{N}$ with $i\leq j$, the convolution sum is defined as,

$$W_{i,j}(r):=\sum_{il+jk=r}\delta(l)\delta(k).$$

For all r, the convolution $\sum_{li+kj=r}\delta(i)\delta(j)$ has been explicitly evaluated for various i and j values, by Alaca et al. [1], [2], [3], [4], [5], [6], H. C. Vidya and B. R. Srivtasa Kumar [17], Williams et al. [20], [21] and E. X. W. Xia and O. X. M. Yao [22]. The claims of J. W. L. Glaisher [10] substantiates our proof,

$$L^2(q)=1+\sum_{r=1}^{\infty}(240\delta_3(r)-288r\delta_1(r))q^r. \tag{40}$$

Theorem IV.1. *For any $r\in\mathbb{N}$ and $u\in\mathbb{R}-\{0\}$, we have*

$$\begin{aligned}\sum_{i+3j=r}\delta(i)\delta(j)&=\left(\frac{1}{24}-\frac{r}{12}\right)\delta_1(r)+\left(\frac{1}{24}-\frac{3r}{4}\right)\delta_1\left(\frac{r}{3}\right)\\&+\left(\frac{1}{1152u}+\frac{1}{24}\right)\delta_3(r)-\frac{1}{1152u}\delta_3\left(\frac{r}{2}\right)\\&+\left(\frac{3}{8}-\frac{9}{128u}\right)\delta_3\left(\frac{r}{3}\right)+\left(\frac{9}{128u}\right)\delta_3\left(\frac{r}{6}\right)\\&+\frac{1}{3456u}\left(A(r)-B(r)\right),\end{aligned} \tag{41}$$

where $1+\sum_{r=1}^{\infty}A(r)q^r$
$=\frac{(-q;q^2)^2_\infty(-q^3;q^6)^2_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty(q;q)^6_\infty}{(q^3;q^3)^3_\infty}$
$+4q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty(q;q)^6_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty(q^3;q^3)^3_\infty}$ *and*
$1+\sum_{r=1}^{\infty}B(r)q^r$
$=\frac{(-q^2;q^4)^2_\infty(-q^6;q^{12})^2_\infty(q^4;q^4)_\infty(q^{12};q^{12})_\infty(q^2;q^2)^6_\infty}{(q^6;q^6)^3_\infty}$
$+4q^2\frac{(q^8;q^8)_\infty(q^{24};q^{24})_\infty(q^2;q^2)^6_\infty}{(q^4;q^8)_\infty(q^{12};q^{24})_\infty(q^6;q^6)^3_\infty}.$

Proof: Applying Definition II.1 to equation (39) and reorganizing we obtain,

$$\begin{aligned}&\left[\frac{1}{40}-\frac{2u}{5}\right]\left[1+240\sum_{r=1}^{\infty}\delta_3(r)q^r\right]\\&-\left[\frac{3}{20}\right]\left[1+240\sum_{r=1}^{\infty}\delta_3\left(\frac{r}{2}\right)q^r\right]\\&-\left[\frac{9}{40}+\frac{18u}{5}+\frac{v}{5}\right]\left[1+240\sum_{r=1}^{\infty}\delta_3\left(\frac{r}{3}\right)q^r\right]\\&+\left[\frac{27}{20}-\frac{4v}{5}\right]\left[1+240\sum_{r=1}^{\infty}\delta_3\left(\frac{r}{6}\right)q^r\right]\\&+u\Bigg[1+\sum_{r=1}^{\infty}\left[240\delta_3(r)-288r\delta_1(r)\right]q^r-6\\&+144\sum_{r=1}^{\infty}\left[\delta_1(r)+\delta_1\left(\frac{r}{3}\right)\right]q^r\\&-3456u\sum_{r=i+3j}\delta(i)\delta(j)q^r\\&+9+\sum_{r=1}^{\infty}\left[2160\delta_3\left(\frac{r}{3}\right)-2592r\delta_1\left(\frac{r}{3}\right)\right]q^r\Bigg]\\&v\Bigg[1+\sum_{r=1}^{\infty}\left[240\delta_3\left(\frac{r}{3}\right)-288r\delta_1\left(\frac{r}{3}\right)\right]q^r-4\\&+96\sum_{r=1}^{\infty}\left[\delta_1\left(\frac{r}{3}\right)+\delta_1\left(\frac{r}{6}\right)\right]q^r\\&-2304v\sum_{r=i+3j}\delta(i)\delta(j)q^r\\&+4+\sum_{r=1}^{\infty}\left[960\delta_3\left(\frac{r}{6}\right)-1152r\delta_1\left(\frac{r}{6}\right)\right]q^r\Bigg]\\&=1+\sum_{r=1}^{\infty}B(r)q^r,\end{aligned}$$

Rearranging,

$$\begin{aligned}&1+\sum_{r=1}^{\infty}\left[\Big(144u-288ru\Big)\delta_1(r)\right]q^r\\&+\sum_{r=1}^{\infty}\left[\Big(144u+96v-288rv-2592ru\Big)\delta_1\left(\frac{r}{3}\right)\right]q^r\\&+\sum_{r=1}^{\infty}\left[(96v-1152rv)\delta_1\left(\frac{r}{6}\right)\right]q^r\\&+\sum_{r=1}^{\infty}\left[(144u+6)\delta_3(r)-36\delta_3\left(\frac{r}{2}\right)\right]q^r\\&+\sum_{r=1}^{\infty}\left[(1296u+192v-54)\delta_3\left(\frac{r}{3}\right)\right]q^r\\&+\sum_{r=1}^{\infty}\left[(768v+324)\delta_3\left(\frac{r}{6}\right)\right]q^r\\&-3456u\sum_{r=i+3j}\delta(i)\delta(j)q^r-2304v\sum_{r=i+3j}\delta(i)\delta(j)q^r\\&=1+\sum_{r=1}^{\infty}B(r)q^r.\end{aligned}$$

From [19], the convolution sum for $r = 3i + 6j$ is given by,

$$\sum_{3i+6j=r} \delta(i)\delta(j) = \frac{1}{24}\delta_1\left(\frac{r}{3}\right) + \frac{1}{24}\delta_1\left(\frac{r}{6}\right) + \frac{1}{12}\delta_3\left(\frac{r}{3}\right) + \frac{1}{3}\delta_3\left(\frac{r}{6}\right) - \frac{1}{8}r\left(\delta_1\left(\frac{r}{3}\right) + 4\delta_1\left(\frac{r}{6}\right)\right) - \frac{1}{2304u}\left(3\delta_3(r) + 243\delta_3\left(\frac{r}{3}\right) + A(r)\right). \tag{42}$$

Therefore,

$$1 + \sum_{r=1}^{\infty}\left[(144 - 288ru)\delta_1(r) + (144u - 2592ru)\delta_1\left(\frac{r}{3}\right)\right]q^r + \sum_{r=1}^{\infty}\left[(144u + 9)\delta_3(r) - 36\delta_3\left(\frac{r}{2}\right)\right]q^r + \sum_{r=1}^{\infty}\left[(1296u - 297)\delta_3\left(\frac{r}{3}\right) + (324)\delta_3\left(\frac{r}{6}\right) + A(r)\right]q^r - 3456u\sum_{r=i+3j}\delta(i)\delta(j)q^r = 1 + \sum_{r=1}^{\infty}B(r)q^r.$$

Hence,

$$\sum_{i+3j=r}\delta(i)\delta(j) = \left(\frac{1}{24} - \frac{r}{12}\right)\delta_1(r) + \left(\frac{1}{24} - \frac{3r}{4}\right)\delta_1\left(\frac{r}{3}\right) + \left(\frac{1}{1152u} + \frac{1}{24}\right)\delta_3(r) - \frac{1}{1152u}\delta_3\left(\frac{r}{2}\right) + \left(\frac{3}{8} - \frac{9}{128u}\right)\delta_3\left(\frac{r}{3}\right) + \left(\frac{9}{128u}\right)\delta_3\left(\frac{r}{6}\right) + \frac{1}{3456u}\left(A(r) - B(r)\right),$$

where $1 + \sum_{r=1}^{\infty}A(r)q^r = a(q)b^3(q)$ and $1 + \sum_{r=1}^{\infty}B(r)q^r = a(q^2)b^3(q^2)$.

■

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