Analysis of Solutions for a Sixth-Order Boundary Value Problem and Numerical Search using Nonlinear Programming Techniques

André Luís Machado Martinez, Cristiane Aparecida Pendeza Martinez,
Thiago de Souza Pinto and Emerson Vitor Castelani

***Abstract*—We analyzed the existence of solutions for a sixth-order boundary value problem. Initially, we established the operator that redefines the problem as a fixed-point problem and elucidated its main properties. Subsequently, we investigated the existence of solutions within the function space $C^1[0,1]$, employing the robust framework of Krasnoselskii's fixed-point theorem. Additionally, we introduced an innovative numerical methodology to explore solutions to this problem. This approach involves discretizing the problem, thereby formulating a nonlinear system. We proposed a solution strategy through the application of mathematical programming techniques for the determination of numerical solutions.**

***Index Terms*—sixth-order, fixed point, Krasnoselskii, numerical solution.**

I. Introduction

In this study, we have examined the conditions requisite for the existence of solutions in the sixth-order boundary value problem. We consider a more generalized equation compared to the one investigated in [1] and [2]. The sixth-order boundary value problem considered here can be described as follows:

$$u^{(6)} + f(t, u, u') = 0, \quad 0 < t < 1, \tag{1}$$

with the boundary conditions:

$$u(0) = u'(0) = u''(0) = 0, \; u'(1) = u'''(1) = u^{(5)}(1) = 0. \tag{2}$$

Naturally, $f : \mathfrak{R}^3 \to \mathfrak{R}$ is a continuous function.

Within the existing literature, several studies have traditionally explored the sixth-order boundary value problem, focusing on both qualitative and quantitative aspects of the solutions. Noteworthy references in this area include [3], [4], [5], [6], [7], [8], [9], and the citations therein.

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André Luís Machado Martinez is an associate professor at Department of Mathematics of Federal Technological University of Parana, Alberto Carazzai Avenue, 1640, CEP 86300-000, Cornélio Procópio, Paraná, Brazil. (e-mail: martinez@utfpr.edu.br)

Cristiane Aparecida Pendeza Martinez is an associate professor at Department of mathematics of Federal Technological University of Parana, Alberto Carazzai Avenue, 1640, CEP 86300-000, Cornélio Procópio, Paraná, Brazil. (e-mail: crismartinez@utfpr.edu.br)

Thiago de Souza Pinto is an associate professor at Department of mathematics of Federal Technological University of Parana, Avenida Alberto Carazzai, 1640, CEP 86300-000, Cornélio Procópio, Paraná, Brazil. (email: thiagosp@utfpr.edu.br)

Emerson Vitor Castelani is an associate professor at Department of Mathematics of State University of Maringá, 5790 ,CEP 87020-900 Maringá, Paraná, Brazil. (e-mail: emersonvitor@gmail.com)

In [1] and [2], the authors analyzed the conditions necessary for solution existence through the application of fixed-point theorems. In [1], the authors considered a simplified version of the problem where f depends solely on u and t. They demonstrated the existence of a solution by employing Krasnoselskii's fixed point theorem.

In [2], the authors consider the same equation as given in (1)-(2), yet they demonstrate the existence of multiple solutions using the Avery-Peterson Theorem. It is important to draw a distinction here. Techniques grounded in fixed-point theorems within cones become more intricate when we encounter the presence of derivatives in the argument of f. This complexity essentially stems from the construction of fixed-point operators and, most crucially, the design of the cone in which this operator will operate. Moreover, even though the Avery-Peterson Theorem is a powerful tool for establishing the existence of multiple solutions, its assumptions end up demanding more conditions from the problem compared to the traditional Krasnoselskii Theorem (for examples of the potential use of both fixed point theorem mentionated here, we recommend [10] and [11]).

Hence, while in [1], the authors employ a simplified equation by omitting terms involving derivatives and utilize the Krasnoselskii Theorem, and in [2], the authors consider such terms but employ the Avery-Peterson theorem, a gap arises concerning the application of the Krasnoselskii Theorem in a more generalized equation like the one presented in (1)-(2).

Concerning numerical aspects of the proposed problem, there are only a limited number of papers that delve into investigations of the sixth-order problem. Consequently, the exploration of numerical solutions for this type of problem remains relatively undeveloped. To address this scarcity, we propose an innovative approach that entails a numerical study employing nonlinear programming methods.

In a nutshell, we can succinctly summarize the contributions of this study as follows:

- A novel result establishing the existence of a solution for the equation defined by (1)-(2) through the application of the Krasnoselskii Theorem (Section 2).
- A fresh algorithm devised to tackle the problem (1)-(2), grounded in a nonlinear optimization method (Section 3).
- Instances that serve to illustrate both the existence and numerical facets (Section 2 and 3).

II. Existence of solution

As presented in [1], we can represent the problem (1)-(2) by representing it as a fixed point of the operator

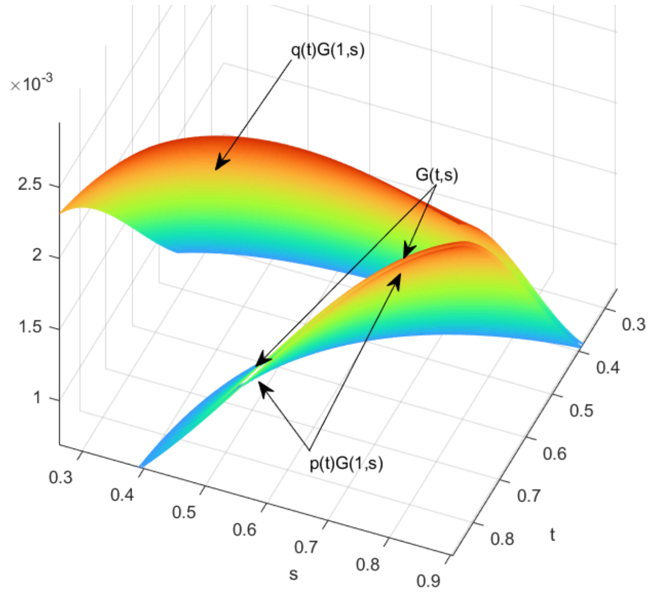

Fig. 1. Graph illustration of $G(t, s)$, $p(t)G(1, s)$ e. $p(t)G(1, s)$, in the figure we can see the relationship described in the equation (7).

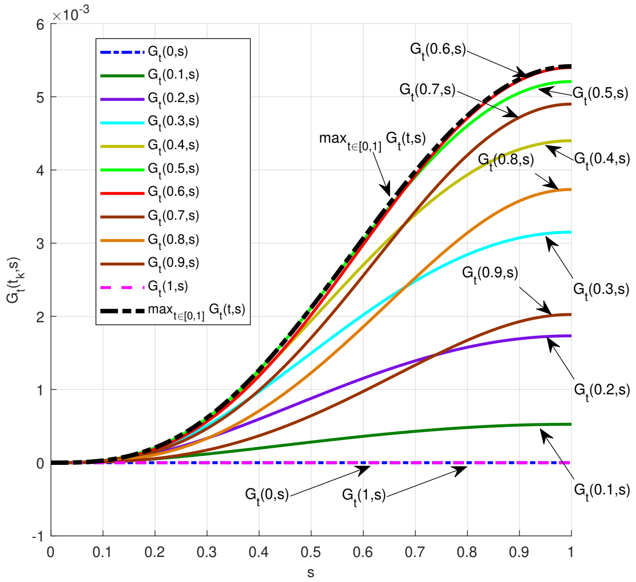

Fig. 2. Graph of $\frac{\partial}{\partial t}G(t, s) = G_t(t, s)$ for different values of t, highlighting the graph of $\max_{t\in[0,1]} \frac{\partial}{\partial t}G(t, s) = \max_{t\in[0,1]} G_t(t, s)$.

$T : C^1[0, 1] \to C^1[0, 1]$. This operator forms the bedrock of our exploration into the existence of a solution within the function spaces $C^1[0, 1]$, and its formulation is presented as follows:

$$Tu(t) = \int_0^1 G(t, s)f(s, u, u')ds \tag{3}$$

where G is the Green's function:

$$G(t, s) = \left(\frac{t^3}{2} - \frac{t^4}{8}\right)\frac{(1-s)^4}{24} - \left(\frac{t^3}{12} - \frac{t^4}{16}\right)\frac{(1-s)^2}{2} + \frac{t^3}{48} - \frac{5t^4}{192} + \frac{t^5}{120} - \frac{(t-s)^5}{120}H(t-s), \tag{4}$$

$$\frac{\partial}{\partial t}G(t, s) = \left(\frac{3t^2}{2} - \frac{t^3}{2}\right)\frac{(1-s)^4}{24} + \left(-\frac{t^2}{4} + \frac{t^3}{4}\right)\frac{(1-s)^2}{2} + \frac{t^2}{16} - \frac{5t^3}{48} + \frac{t^4}{24} - \frac{(t-s)^4}{120}H(t-s), \tag{5}$$

where

$$H(\zeta) = \begin{cases} 1, & \zeta \geq 0 \\ 0, & \zeta < 0 \end{cases}. \tag{6}$$

In sequence, we have listed some properties of the Green function G and its derivative $\frac{\partial}{\partial t}G$ that will be useful. As presented in [2], we enunciate the Property 1.

Propriety 1. *How* $G(1, s) = \frac{s^3}{960}(20 - 25s + 8s^2) \geq 0$ *following as presented in [1] there are polynomials* $p(t)$ *and* $q(t)$ *such that:*

$$p(t)G(1, s) \leq G(t, s) \leq q(t)G(1, s), \tag{7}$$

where $p(t) = 4t - 4t^2 + t^4$, $q(t) = \frac{t^3}{3}(20 - 25t + 8t^2)$.

Figure 1 illustrates a comparison among the graphs of $G(t, s)$, $p(t)G(1, s)$, *and* $p(t)G(1, s)$. *This analysis allows us to validate Property 1.*

As $\frac{\partial}{\partial t}G$ *is restricted to the interval* $[0, 1] \times [0, 1]$, *we can conclude that:*

$$\frac{\partial}{\partial t}G(0, t) \leq \frac{\partial}{\partial t}G(t, s) \leq \max_{t\in[0,1]}\left[\frac{\partial}{\partial t}G(t, s)\right], \tag{8}$$

Figure 2 illustrates the graph of $\frac{\partial}{\partial t}G(t, s)$ for various values of t: $t = 0, 0.1, 0.2, \dots, 1$. Additionally, the figure highlights the plot of $\max_{t\in[0,1]} \frac{\partial}{\partial t}G(t, s)$. It can also be observed that the function $\max_{t\in[0,1]} \frac{\partial}{\partial t}G(;s)$ presents changes in its graph as t varies, starting as a null function at $t = 0$, approaching the maximum around $t \in [0.5, 0.6]$, and then returning to a null function at $t = 1$.

To determine the existence of multiple solutions, let's consider the cone

$$E = \{u \in C^1[0, 1] : u(0) = u'(0) = 0,\ u(t) \geq 0, \forall t \in [0, 1]\},$$

where $C^1[0, 1]$ is the Banach space of continuously differentiable functions in $[0, 1]$ equipped with

$$\|u\|_E = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

Remark 1. *If* $u \in E$ *then* Tu *satisfies the condition* $Tu(0) = 0$. *Moreover* $\|(Tu)'\|_\infty \geq \|Tu\|_E$ *and* $\|u\|_\infty \leq \|u\|_E = \|u'\|_\infty$. *In fact, since* $u(0) = 0$ *one has from the mean value theorem that*

$$u(t) = u(t) - u(0) \leq t\|u'\|_\infty,\ 0 \leq t \leq 1.$$

Therefore,

$$\|u\|_\infty \leq \|u'\|_\infty$$

.

To establish the continuity and complete continuity of the integral operator T, we will present the Proposition 1, as in [2].

Proposition 1. *The operator* T *is continuous and completely continuous.*

Proof: Continuity can be inferred by employing the Lebesgue dominated convergence theorem, combined with the fact that

$$|T(u)(t) - T(u_n)(t)| \leq \int_0^1 G(t, s)|f(s, u(s), u'(s))$$

$$-f(s, u_n(s), u'_n(s))|ds,$$

$$\leq \int_0^1 G(t,s)\left|f(s,u(s),u'(s)) - f(s,u_n(s),u'_n(s))\right| ds,$$

$$\leq \int_0^1 q(t)G(1,s)\left|f(s,u,u'(s)) - f(s,u_n(s),u'_n(s))\right| ds,$$

$$\leq \int_0^1 G(1,s)\left|f(s,u(s),u'(s)) - f(s,u_n(s),u'_n(s))\right| ds.$$

with $u_n, u \in E$. To show complete continuity we will use the Arzela-Ascoli's theorem. Let $\Omega \subseteq E$ be bounded, in other words, there exists $\Lambda_0 > 0$ with $\|u\| \leq \Lambda_0$ for each $u \in \Omega$. Now if $u \in \Omega$, we have

$$|(Tu)(t)| \leq \int_0^1 |G(t,s)| H_{\Lambda_0}(s) ds$$

where H_{Λ_0} is determined by the bounded set and the function u. It is immediate that $H_{\Lambda_0}(s) \in L^1[0,1]$. Consequently, it follows that $T(\Omega)$ forms a bounded and equicontinuous family over the interval $[0,1]$. As a result, the Arzelà-Ascoli theorem can be applied to establish the complete continuity of the operator $T: E \to E$. ∎

Our existence result will be demonstrated using the Krasnoselskii fixed point theorem for the compression of a cone, as detailed in [1].Consequently,

Theorem 1. *(Krasnoselskii) Let E be a cone in a Banach space C and let Ω_1 and Ω_2 be open subsets of C, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $T: E \cap (\Omega_2\ \overline{\Omega}_1) \to E$ is a completely continuous operator such that*

- $\|Tu\| \geq \|u\|$ *if* $u \in E \cap \partial\Omega_1$ *and*
- $\|Tu\| \leq \|u\|$ *if* $u \in E \cap \partial\Omega_2$.

Then T has a fixed point in $E \cap (\Omega_2 - \overline{\Omega}_1)$.

To establish the existence of solutions, it is necessary to consider some fundamental assumptions.
(H1) Assuming the existence of a positive constant d_1 for the problem defined in (1)-(2), such that

- For all $(s,u,v) \in [0,1] \times [0,d_1] \times [0,d_1]$ then $0 \leq f(s,u,v) \leq \frac{d}{r_1}$;
- $r_1 = \int_0^1 \max_{t\in[0,1]} \left[\frac{\partial}{\partial t} G(t,s)\right] ds$;
- Exist $s \in (0,1)$, such that $f(s,0,0) > 0$.

The lemma presented below will be crucial for the demonstration of our main result.

Lemma 1. *Suppose that (H1) holds. Then there exists positive constant $d_2 < d_1$ such that*

$$\|Tu\|_E \geq \|u\|_E$$

for all $u \in E$ satisfying $\|u\|_E = r$.

Proof: Suppose the inequality $\|Tu\|_E \geq \|u\|_E$ is false for all $u \in E \cap \Omega_1$ with $\|u\|_E = d_2$, $0 < d_2 < d_1$. Then for each positive integer n with $\frac{1}{n} > d_1$, there exists $u_n \in E$ such that

$$\|u_n\|_E = \frac{1}{n} \text{ and } \|Tu_n\|_E < \frac{1}{n}.$$

Consequently, we have constructed a sequence u_n such that $u_n \to 0$ and $Tu_n \to 0$. Since T is continuous, one has $T0 = 0$, which is a contradiction in view of (H1). In fact,

$$T0 = \int_0^1 G(t,s) f(s,0,0) ds \geq \int_0^1 p(t) G(1,s) f(s,0,0) ds,$$

since $p(t)G(1,s)f(s,0,0) \geq 0, \forall s \in [0,1]$. Now, the functions p and $G(1,s)$ do not cancel each other in $(0,1)$. Therefore, since $f(s,0,0)$ is not identically zero, we obtain that $T0$ is not identically zero. Consequently, there must exist a value d_2 such that $0 < d_2 < d_1$, satisfying the conditions $\|Tu\|_E \geq \|u\|_E$, $\|u\|_E = d_2$. ∎

Theorem 2. *Suppose that* **(H1)** *holds. Then 1 and 2 has a positive solution $u \in E$ and $d_2 \leq \|u\|_E \leq d_1$*

Proof: Consider $u \in E$ with $|u|_E \leq d_1$. By applying **(H1)**, we can obtain:

$$\begin{aligned}
\|Tu\|_E &= \max_{t\in[0,1]} \left|\frac{\partial}{\partial t}(Tu)(t)\right|, \\
&\leq \max_{t\in[0,1]} \int_0^1 \left|\frac{\partial}{\partial t} G(t,s)\right| |f(s,u(s),u'(s))| ds \\
&\leq \int_0^1 \max_{t\in[0,1]} \left[\frac{\partial}{\partial t} G(t,s)\right] |f(s,u(s),u'(s))| ds \\
&\leq \frac{d}{r_1} \int_0^1 \max_{t\in[0,1]} \left[\frac{\partial}{\partial t} G(t,s)\right] ds \\
&\leq d.
\end{aligned}$$

By combining Lemma 1 and Krasnoselskii Theorem, we can conclude that the problem (1)-(2) possesses a positive solution, which corresponds to a fixed point of the operator T. ∎

Example 1. *Let us consider (1)-(2) with*

$$f(t,u,v) = t + 20u^4 + 30v^4.$$

Choosing the constant $d = 2$. Calculating

$$r_1 = \int_0^1 \max_{t\in[0,1]} \left[\frac{\partial}{\partial t} G(t,s)\right] ds = 0.00239740,$$

*it is straightforward to verify that under these conditions, the hypothesis **(H1)** is satisfied.*

III. Numerical solution

In this section, we present a method based on optimization techniques that uses the finite difference approximation for the derivatives of the problem (1)-(2). That is, the approach involves discretize the domain of the desired solution using a mesh and replacing the derivatives with approximations obtained via finite differences. By considering the problem (1)-(2) defined on a mesh, we can define a problem equivalent to a system of nonlinear equations, which can be solved using optimization techniques. To discretize the Sixth-Order Boundary Value Problem, we need to approximate the derivatives that compose the problem defined in (1) and (2).

We will consider f a real function differentiable at a point x and in a neighborhood of this point. Also, we will use approximations based on classical finite difference schemes. For the first derivative of u we can use these formulas:
Central Finite difference:

$$u'(t) \approx \overline{u}^{(1)}(t,h) = \frac{u(t+h) - u(t-h)}{2h},$$

Finite difference forward :

$$u'(t) \approx \overline{u}_{+}^{(1)}(t,h) = \frac{u(t+h)-u(t)}{h},$$

Finite difference backward:

$$u'(t) \approx \overline{u}_{-}^{(1)}(t,h) = \frac{u(t)-u(t-h)}{h}.$$

For the second order derivative we will also consider approximations obtained by finite differences.
Finite Difference forward:

$$u''(x) \approx \overline{u}_{+}^{(2)}(t,h) = \frac{u(t+2h)-2u(t+h)+u(t)}{h^2}.$$

For the third order derivative we will also consider approximations obtained by finite differences.
Finite difference backward:

$$u'''(x) \approx \overline{u}_{-}^{(3)}(t,h) = \frac{1}{h^3}[u(t)-3u(t-h) +3u(t-2h)-u(t-3h)].$$

For the fifth order derivative we will also consider approximations obtained by finite differences.
Finite difference backward:

$$u^{(5)}(x) \approx \overline{u}_{-}^{(5)}(t,h) = \frac{1}{h^5}[(-u(t)+5u(t-h)-10u(t-2h) +10u(t-3h)-5u(t-4h)+u(t-5h))].$$

For the sixth order derivative of a real function using finite differences is given by:

$$u^{(6)}(x) \approx \overline{u}^{(6)}(t,h) = \frac{1}{h^6}[-u(t+3h)+6u(t+2h) -15u(t+h)+20u(t)-15u(t-h)+6u(t-2h)-u(t-3h)].$$

Considering points just to the right, we can use the formula:

$$u^{(6)}(x) \approx \overline{u}_{+}^{(6)}(t,h) = \frac{1}{h^6}[(-u(t+6h)+6u(t+5h) -15u(t+4h)+20u(t+3h)-15u(t+2h)+6u(t+h)-u(t))].$$

For points on the left, we use the formula:

$$u^{(6)}(x) \approx \overline{u}_{-}^{(6)}(t,h) = \frac{1}{h^6}[(-u(t)+6u(t-h)-15u(t-2h) +20u(t-3h)-15u(t-4h)+6u(t-5h)-u(t-6h))].$$

In all the formulas above, h represents the distance between consecutive points.

To understand how our numerical approach works, it is necessary to understand the discretized problem model in terms of optimization. In this sense, let's consider $\{t_j, j = 0,1,...,n\}$ a discretization of $[0,1]$ by an equal spaced mesh where $h = t_{j+1} - t_j$, $j = 0,1,...,n-1$ and $u_j \approx u(t_j)$, $j = 0,1,...,n$, and define the vector $\mathbf{u} = (u_0, u_1, \ldots, u_n)$. Replacing the classical finite difference schemes in (1) and (2) we obtain the nonlinear system $R(\mathbf{u}) = 0$, where $R : \Re^{n+1} \to \Re^{n+7}$ is defined as:

$$R_i(\mathbf{u}) = \overline{u}_{+}^{(6)}(t_i,h) + f(t_i,\mathbf{u},\overline{u}_{+}^{(1)}(t_i,h)) = 0,\ i = 0,1,2$$

$$R_i(\mathbf{u}) = \overline{u}^{(6)}(t_i,h) + f(t_i,\mathbf{u},\overline{u}^{(1)}(t_i,h)) = 0,\ 3 \le i \le n-3$$

$$R_i(\mathbf{u}) = \overline{u}_{-}^{(6)}(t_i,h) + f(t_i,\mathbf{u},\overline{u}_{-}^{(1)}(t_i,h)) = 0,\ n-2 \le i \le n$$

$$R_{n+1}(\mathbf{u}) = u_0 = 0$$

$$R_{n+2}(\mathbf{u}) = \overline{u}_{+}^{(1)}(t_0,h) = 0$$

$$R_{n+3}(\mathbf{u}) = \overline{u}_{+}^{(2)}(t_0,h) = 0$$

$$R_{n+4}(\mathbf{u}) = \overline{u}_{-}^{(1)}(t_n,h) = 0$$

$$R_{n+5}(\mathbf{u}) = \overline{u}_{-}^{(3)}(t_n,h) = 0$$

$$R_{n+6}(\mathbf{u}) = \overline{u}_{-}^{(5)}(t_n,h) = 0$$

The nonlinear system $R(\mathbf{u}) = 0$ gives rise to a set of $n+7$ equations. Assuming $u_0 = 0$, our goal is to determine $u_1, u_2, ..., u_n$, thus seeking to find $n+1$ variables. Traditionally, numerical solutions rely on fixed-point methods. In this case, the method is defined by an iterative sequence based on operator (3). We recommend referring to [2], [12], and [13] for further insight.

In this article, we propose a method that hinges on the Gauss-Newton approach [14]. It's worth noting that the expansive potential of the Gauss-Newton method extends to wider applications as well, as exemplified in works such as [15] and [16].

An algorithm for solving the nonlinear system is described below.

Algorithm 1 Gauss-Newton

1: Define an uniformly distributed mesh $\{t_j\}$ in $[0,1]$;
2: Define an initial approximation $\mathbf{u}^0$ (so that $u_j^0 \approx u(t_j)$) and tolerance $\varepsilon > 0$;
3: k=0;
4: **while** $\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_\infty > \varepsilon$ or $k = 0$ **do**
5: Compute vector $\mathbf{R}_k = R(\mathbf{u}^k)$ and matrix

$$\mathbf{A}_k = \begin{bmatrix} \nabla R_0(\mathbf{u}^k) \\ \nabla R_1(\mathbf{u}^k) \\ \vdots \\ \nabla R_{n+6}(\mathbf{u}^k) \end{bmatrix}$$

6: Find Δ_k such that:

$$(\mathbf{A}_k^T \mathbf{A}_k)\Delta_k = -\mathbf{A}_k^T \mathbf{R}_k$$

7: To satisfy the Armijo's condition, we need to determine the appropriate value of α_k.
8: Compute $\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha_k \Delta_k$.
9: **end while**
10: output: $\mathbf{u}^k$.

Subsequently, examples are provided to demonstrate the potential of Algorithm 1.

Example 2. *Consider in problem (1) - (2):*

$$f(t,u,v) = u^2 + v^2$$

The analytical solution of (1) - (2) is $u^*(t) = 0$. The results of applying Algorithm 1 in this example are presented in Table I, and illustrated in Figure 2.

Example 3. *Now let us consider the same function given in Example 1.*

The simulation results were obtained using MatLab 8.0 software running on Windows 10 Home Single Language operating system.

We apply the Algorithm 1 considering the stopping criterion the condition $\|u^k - u^{k-1}\| < 10^{-4}$ and consider $h = 0.05$, $t_{j+1} - t_j = h, \forall j$.

In Table I, we have the results obtained through the application of Algorithm 1 to Examples 2 and 3. It is evident

TABLE I
PERFORMANCE OF ALGORITHM 1

Problem	Iterations	Precision ($\|u^k - u^*\|$)	$\|u^k - u^{k-1}\|$
Example 2	5	3.5696e-14	4.1475e-09
Example 3	7	-	3.4531e-06

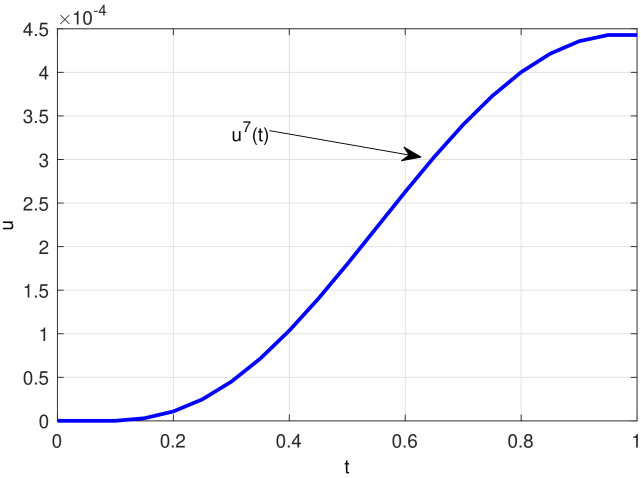

Fig. 3. Graph illustration of the solution provided by Algorithm 1 to Example 3.

that the method exhibits good performance, achieving a high degree of accuracy even with a minimal number of iterations. This outcome substantiates the effectiveness of the strategy involving problem (1)-(2) discretization and the subsequent application of non-linear programming methods.

The graph depicted in Figure 3 illustrates the approximate solution, revealing that it adheres closely to the anticipated conditions of the problem (1)-(2). This alignment with expectations bolsters the credibility of the obtained solution.

IV. CONCLUSION

This work presents a study where we investigate whether the problem (1), (2) admits a solution under specific conditions for the function f, as stipulated by Krasnoselskii theorem. Our approach involves thorough analysis, the implementation of a method rooted in non-linear programming techniques, rigorous testing, and the presentation of nontrivial examples that were meticulously examined.

It's noteworthy that the scope of the problem under consideration extends beyond that of the investigations conducted in the previous works [1] and [2]. In this study, we introduce an additional layer of complexity by considering the impact of the increment of u' in the function f within (1). This augmentation necessitates enhanced control over the constituent functions of the problem to demonstrate the existence of a solution. Remarkably, the results we have obtained demonstrate the existence of solutions akin to those highlighted in the simpler problem presented in [1].

REFERENCES

[1] B. Yang, "Positive solutions to a nonlinear sixth order boundary value problem," *Differential Equations & Applications*, vol. 11, no. 2, pp. 307–317, 2019.

[2] A. L. M. Martinez, C. A. P. Martinez, G. M. Bressan, R. M. Souza, and E. W. Stiegelmeier, "Multiple solutions for a sixth order boundary value problem," *Trends in Computational and Applied Mathematics*, vol. 22, pp. 2–12, 2021.

[3] M. Adjustovs and A. Lepins, "Extremal solutions of a boundary value problem for a sixth-order equation.," *Differential Equations*, vol. 50, no. 2, pp. 141–146, 2014.

[4] R. P. Agarwal, B. Kovacs, and D. O'Regan, "Positive solutions for a sixth-order boundary value problem with four parameters," *Boundary Value Problems*, vol. 2013, no. 1, pp. 1–22, 2013.

[5] R. P. Agarwal, B. Kovacs, and D. O' Regan, "Existence of positive solution for a sixth–order differential system with variable parameters," *Journal of Applied Mathematics and Computing*, vol. 44, pp. 437–454, 2014.

[6] T. Garbuza, "On solutions of one 6-th order nonlinear boundary value problem," *Mathematical Modelling and Analysis*, vol. 13, no. 3, pp. 349–355, 2008.

[7] K. Ghanbari and H. Mirzaei, "On the isospectral sixth order sturm-liouville equation," *J. Lie Theory*, vol. 23, no. 4, pp. 921–935, 2013.

[8] T. Gyulov, "Trivial and nontrivial solutions of a boundary value problem for a sixth-order ordinary differential equation," *Comptes Rendus-Academie Bulgarie Des Science*, vol. 58, no. 9, pp. 1013–1018, 2005.

[9] M. Möller and B. Zinsou, "Sixth order differential operators with eigenvalue dependent boundary conditions," *Applicable Analysis and Discrete Mathematics*, pp. 378–389, 2013.

[10] A. L. M. Martinez and E. V. Castelani, "Existence and multiplicity of positive solutions for a m-point boundary value problem," *IAENG International Journal of Applied Mathematics*, vol. 38, no. 4, pp. 180–183, 2008.

[11] M. Hu and L. Wang, "Positive periodic solutions in shifts delta (+/-) for a neutral dynamic equation on time scales," *IAENG International Journal of Applied Mathematics*, vol. 48, no. 2, pp. 1–6, 2018.

[12] C. A. P. Martinez, A. L. M. Martinez, G. M. Bressan, E. V. Castelani, and R. M. Souza, "Multiple solutions for a fourth order equation with nonlinear boundary conditions: theoretical and numerical aspects," *Differential Equations & Applications*, vol. 11, no. 3, pp. 335–348, 2019.

[13] A. L. M. Martinez, E. V. Castelani, C. A. P. Martinez, G. M. Bressan, and R. M. Souza, "Multiple solutions to a third-order three-point nonhomogeneous boundary value problem aided by nonlinear programming methods.," *Differential Equations & Applications*, vol. 13, no. 1, pp. 35–49, 2021.

[14] C. T. Kelley, *Iterative methods for optimization*. SIAM, 1999.

[15] M. Zeng and H. Fu, "A broyden trust region quasi-newton method for nonlinear equations," *IAENG International Journal of Computer Science*, vol. 46, no. 3, pp. 454–458, 2019.

[16] S. Surulere, M. Y. Shatalov, A. Mkolesia, T. Malange, and A. Adeniji, "The integral-differential and integral approach for the exact solution of the hybrid functional forms for morse potential.," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp. 242–250, 2020.