Stress and Tension of Generalized Complements of Graphs

Madhumitha K. V, Swati Nayak, Sabitha D'Souza*

***Abstract*—Centrality measures are scalar values given to each node in the graph to quantify its importance based on an assumption. Stress and tension are the centrality index based on shortest path. In this paper, stress and tension of generalized complements of some standard graphs are calculated by counting the number of geodesics of different length.**

***Index Terms*—Geodesic, Stress, Tension, $k-$complement, $k(i)-$complement.**

I. Introduction

LET $G = (V, E)$ be a simple, finite, undirected and connected graph. The order and size of G is given by $|V| = n$ and $|E| = m$ respectively. The degree of a vertex v in a graph G, denoted by $\deg(v)$ is the number of edges incident on the vertex v. Let $P = \{v_o, v_1, \ldots, v_n\}$ be a v_ov_n path in G. The length of P, denoted by $l(P)$ is the number of edges in the v_ov_n path. Let $d(u, v)$ denote the distance between any two vertices u and v in G. The shortest path between any two vertices in G is called the geodesic. The diameter of G is the length of any longest geodesic, denoted by $diam(G)$. The maximum distance from vertex v to all other vertices in G is the eccentricity $e(v)$ of v. The complement $\overline{G}$ of G is the graph which has V as its vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. G is self complementary if $\overline{G}$ is isomorphic to G.

Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 1$. The $k-$complement G_k^P of G is obtained by removing the edges between V_i and V_j in G and adding edges between V_i and V_j which are not in G, for all V_i and V_j in P, where $i \neq j$.

For each set V_r in the partition P, remove the edges of G inside V_r and add the edges of $\overline{G}$ joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)-$complement of G with respect to P of V.

The $k - complement$ and $k(i) - complement$ of G are related as follows: [1]

1) $\overline{G_k^P} \cong G_{k(i)}^P$
2) $\overline{G_{k(i)}^P} \cong G_k^P$

In 1953, Alfonso Shimbel [2] defined the concept of stress of a vertex in a graph. Stress of a vertex v in a graph G is the number of shortest paths in G having v as an internal vertex and is denoted by $st(v_i)$. A graph G is $k-$stress regular, if all vertices of G have stress k. The total stress of a graph G is defined by $st(G) = \sum_{i=1}^{n} st(v_i)$. K. Bhargava et al. [3] determined the stress of vertices in some standard graphs. Raksha Poojary et al. [4] determined stress of paths, cycles, fans and wheels. They also determined the stress of a cut vertex of a graph G when G has at most 2 cut vertices. Shiny Joseph [5] determined total vertex stress in some merged graphs, cartesian product graphs and the join of two graphs. Raksha Poojary et al. [6] determined stress of wheel related graphs such as gear graph, helm graph, friendship graph, flower graph and sunflower graph.

K. Bhargava et al. [7] introduced the concept of tension on an edge in a graph. Let e be an edge in the graph G. The tension on e is defined as the number of geodesics in G passing through e. Total tension of G, denoted by $N_\tau(G)$, is defined as $N_\tau(G) = \sum_{e \in E} \tau(e)$.

In this paper, the stress and tension of generalized complement of a graph is computed in such a way that k-complement and $k(i)$-complement of the graph is connected. We compute stress and tension of generalized complements of path, cycles, complete bipartite, wheel graph and fan graph by calculating number of geodesics of different length.

Note that

- Geodesics of length n contribute $(n - 1)$ to the stress of a graph.
- Geodesics of length n contribute n to the tension of a graph.

II. Preliminary Definitions

1) A simple graph with $n \geq 3$ vertices forming a cycle of length n is called a cycle graph, denoted by C_n.
2) Wheel Graph W_{n+1} is formed by connecting a single universal vertex to all vertices of cycle.
3) Path graph is a graph whose vertices can be listed in the order $\{v_1, v_2, \ldots, v_n\}$ such that the edges are v_iv_{i+1}, where $1 \leq i \leq n - 1$.
4) The Fan graph $F_{1,n}$ is obtained by removing one peripheral edge from the wheel graph.
5) A complete bipartite graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset and every possible edge that could connect vertices in different subsets is part of the graph.

Theorem II.1. *[3], [7] For any graph with n vertices and diameter d, we have*

$$N_{str}(G) = \sum_{i=1}^{d} (i - 1) f_i \tag{1}$$

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$$N_\tau(G)=\sum_{i=1}^{d} i f_i \qquad (2)$$

where, f_i is the number of geodesics of length i in G.

III. Stress and Tension of $k-$complement of some standard graphs

Theorem III.1. *Let $P=\{V_1,V_2,\dots,V_k\}$ be a partition of path graph P_n of order n.*

1) *If any of the pendant vertex is in V_1 or $V_{\frac{n+1}{2}}$ and remaining $\frac{n-1}{2}$ partitions be K_2's. Then,*
   a) $St(P_n)^P_{\frac{n+1}{2}}=\frac{n^2-3n+2}{2}$.
   b) $N_\tau(P_n)^P_{\frac{n+1}{2}}=\frac{3n^2-8n+5}{2}$.
2) *If any of the non-pendant vertex is in V_i, $3\le i\le n-2$ and the remaining $\frac{n-1}{2}$ partitions be K_2's. Then,*
   a) $St(P_n)^P_{\frac{n+1}{2}}=\frac{n^2-3n-2}{2}$.
   b) $N_\tau(P_n)^P_{\frac{n+1}{2}}=\frac{3n^2-8n-3}{2}$.
3) *If $\langle V_i\rangle=K_2$; $i=1,2,\dots,\frac{n}{2}$. Then,*
   a) $St(P_n)^P_{\frac{n}{2}}=\frac{n^2-4n+4}{2}$.
   b) $N_\tau(P_n)^P_{\frac{n}{2}}=\frac{3n^2-10n+10}{2}$.

Proof:

1) Let any of the pendant vertex be in V_1 or $V_{\frac{n+1}{2}}$ and remaining $\frac{n-1}{2}$ partitions be K_2's. Then, number of geodesics of length $1=\binom{n}{2}-(n-\frac{n+1}{2})$ and that of length $2=(n-\frac{n+1}{2})(n-2)$.
From II.1, we have

$$a)\ St(P_n)^P_{\frac{n+1}{2}}=0\left(\binom{n}{2}-\left(n-\frac{n+1}{2}\right)\right)+1\left(\left(n-\frac{n+1}{2}\right)(n-2)\right)=\left(n-\frac{n+1}{2}\right)(n-2)=\frac{n^2-3n+2}{2}.$$

$$b)\ N_\tau(P_n)^P_{\frac{n+1}{2}}=1\left(\binom{n}{2}-\left(n-\frac{n+1}{2}\right)\right)+2\left(\left(n-\frac{n+1}{2}\right)(n-2)\right)=1\left(\binom{n}{2}-\left(n-\frac{n+1}{2}\right)\right)+2\left(\frac{n^2-3n+2}{2}\right)=\frac{3n^2-8n+5}{2}.$$

2) Let any of the non-pendant vertex be in V_i, $3\le i\le n-2$ and the remaining $\frac{n-1}{2}$ partitions be K_2's. Then, number of geodesics of length $1=\binom{n}{2}-(n-\frac{n+1}{2})$. Number of geodesics of length $2=2(n-3)+(n-2)(n-(\frac{n+1}{2})-2)$.
From II.1, we have

$$a)\ St(P_n)^P_{\frac{n+1}{2}}=0\left(\binom{n}{2}-\left(n-\frac{n+1}{2}\right)\right)+1\left(2(n-3)+(n-2)\left(n-\left(\frac{n+1}{2}\right)-2\right)\right)=2(n-3)+(n-2)\left(n-\left(\frac{n+1}{2}\right)-2\right)=\frac{n^2-3n-2}{2}.$$

$$b)\ N_\tau(P_n)^P_{\frac{n+1}{2}}=1\left(\binom{n}{2}-\left(n-\frac{n+1}{2}\right)\right)+2\left(2(n-3)+(n-2)\left(n-\left(\frac{n+1}{2}\right)-2\right)\right)=1\left(\binom{n}{2}-\left(n-\frac{n+1}{2}\right)\right)+2\left(\frac{n^2-3n-2}{2}\right)=\frac{3n^2-8n-3}{2}.$$

3) Let $\langle V_i\rangle=K_2$; $i=1,2,\dots,\frac{n}{2}$. Then, number of geodesics of length $1=\binom{n}{2}-(n-(\frac{n}{2}+1))$.
Number of geodesics of length $2=(n-(\frac{n}{2}+1))(n-2)$.
From II.1, we have

$$a)\ St(P_n)^P_{\frac{n}{2}}=0\left(\binom{n}{2}-\left(n-\left(\frac{n}{2}+1\right)\right)\right)+1\left((n-2)\left(n-\left(\frac{n}{2}+1\right)\right)\right)=(n-2)\left(n-\left(\left(\frac{n}{2}\right)+1\right)\right)=\frac{n^2-4n+4}{2}.$$

$$b)\ N_\tau(P_n)^P_{\frac{n}{2}}=1\left(\binom{n}{2}-\left(n-\left(\frac{n}{2}+1\right)\right)\right)+2\left((n-2)\left(n-\left(\frac{n}{2}+1\right)\right)\right)=1\left(\binom{n}{2}-\left(n-\left(\frac{n}{2}+1\right)\right)\right)+2\left(\frac{n^2-4n+4}{2}\right)=\frac{3n^2-10n+10}{2}.$$

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Theorem III.2. *Let $P=\{V_1,V_2,\dots,V_k\}$ be a partition of C_n.*

1) *If any of the $\langle V_i\rangle=K_1$ and the remaining $\frac{n-1}{2}$ partitions be K_2's, then*
   a) $St(C_n)^P_{\frac{n+1}{2}}=\frac{n^2-n-6}{2}$.
   b) $N_\tau(C_n)^P_{\frac{n+1}{2}}=\frac{3n^2-4n-13}{2}$.
2) *If each $\langle V_i\rangle=K_2$, where $1\le i\le\frac{n}{2}$, then*

a) $St(C_n)^P_{\frac{n}{2}} = \frac{n^2}{2} - n$.
b) $N_\tau(C_n)^P_{\frac{n}{2}} = \frac{3n^2-6n}{2}$.

Proof:

1) Let any of the $\langle V_i \rangle = K_1$ and the remaining $\frac{n-1}{2}$ partitions be K_2's, $i \neq j$, $n \geq 3$. Then,
number of geodesics of length $1 = \binom{n}{2} - (n - (\frac{n+1}{2} - 1))$.
Number of geodesics of length $2 = 2(n-3) + (n-2)(n - (\frac{n+1}{2} - 1) - 2)$.
From II.1, we have

$$\begin{aligned} a)\ St(C_n)^P_{\frac{n+1}{2}} &= 0\left(\binom{n}{2} - \left(n - \left(\frac{n+1}{2} - 1\right)\right)\right) \\ &+ 1\left(2(n-3) + (n-2)\right. \\ &\left.\left(n - \left(\frac{n+1}{2} - 1\right) - 2\right)\right) \\ &= 2(n-3) \\ &+ (n-2)\left(n - \left(\frac{n+1}{2} - 1\right) - 2\right) \\ &= \frac{n^2-n-6}{2}. \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(C_n)^P_{\frac{n+1}{2}} &= 1\left(\binom{n}{2} - \left(n - \left(\frac{n+1}{2} - 1\right)\right)\right) \\ &+ 2\left(2(n-3)\right. \\ &\left. + (n-2)\left(n - \left(\frac{n+1}{2} - 1\right) - 2\right)\right) \\ &= 1\left(\binom{n}{2} - \left(n - \left(\frac{n+1}{2} - 1\right)\right)\right) \\ &+ 2\left(\frac{n^2-n-6}{2}\right) \\ &= \frac{3n^2-4n-13}{2}. \end{aligned}$$

2) Let each $\langle V_i \rangle = K_2$ where $1 \leq i \leq \frac{n}{2}$. Then, we have number of geodesics of length $1 = \binom{n}{2} - (n - (\frac{n}{2}))$.
Number of geodesics of length $2 = (n - \frac{n}{2})(n-2)$.
From II.1, we have

$$\begin{aligned} a)\ St(C_n)^P_{\frac{n}{2}} &= 0\left(\binom{n}{2} - \left(n - \frac{n}{2}\right)\right) \\ &+ 1\left((n-2)\left(n - \frac{n}{2}\right)\right) \\ &= (n-2)\left(n - \frac{n}{2}\right) \\ &= \frac{n^2}{2} - n. \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(C_n)^P_{\frac{n}{2}} &= 1\left(\binom{n}{2} - \left(n - \frac{n}{2}\right)\right) \\ &+ 2\left((n-2)\left(n - \frac{n}{2}\right)\right) \\ &= 1\left(\binom{n}{2} - \left(n - \frac{n}{2}\right)\right) + 2\left(\frac{n^2}{2} - n\right) \\ &= \frac{3n^2-6n}{2}. \end{aligned}$$

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Theorem III.3. *Let $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of friendship graph F_n on $2n+1$ vertices.*

If any of the $\langle V_i \rangle$ is C_3 and the remaining $n-1$'s are all K_2 's, then
1) $St(F_n)^P_n = 4(n-1)$.
2) $N_\tau(F_n)^P_n = 2n^2 + 7n - 6$.

Proof: Let any of the $\langle V_i \rangle$ be C_3 and the remaining $n-1$'s are all K_2 's. Then, we have
number of geodesics of length $1 = \binom{n}{2} - 2(n-1)$.
Number of geodesics of length $2 = 2(2(n-1))$.
From II.1, we have

$$\begin{aligned} 1)\ \ St(F_n)^P_n &= 0(n(2n+1) - 2(n-1)) + 1(2(2(n-1))) \\ &= 2(2(n-1)) \\ &= 4(n-1). \end{aligned}$$

$$\begin{aligned} 2)\ \ N_\tau(F_n)^P_n &= 1(n(2n+1) - 2(n-1)) + 2(2(2(n-1))) \\ &= 1(n(2n+1) - 2(n-1)) + 2(4(n-1)) \\ &= 2n^2 + 7n - 6. \end{aligned}$$

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Theorem III.4. *Let $P = \{V_1, V_2, \ldots, V_{n-1}\}$ be a partition of star graph $K_{1,n-1}$ with n vertices.*
If any of the $\langle V_i \rangle$ is K_2, where $1 \leq i \leq n-1$ including the universal vertex and the remaining $\langle V_{n-2} \rangle$ partition consists of all other pendant vertices, then
1) $St(K_{1,n-1})^P_{n-1} = n-2$.
2) $N_\tau(K_{1,n-1})^P_{n-1} = \frac{n^2+n-4}{2}$.

Proof: Let any of the $\langle V_i \rangle$ be K_2, where $1 \leq i \leq n-1$ including the universal vertex and the remaining $\langle V_{n-2} \rangle$ partition consists of all other pendant vertices. Then, we have
number of geodesics of length $1 = \frac{(n-1)(n-2)}{2} + 1$.
Number of geodesics of length $2 = n-2$.
From II.1, we have

$$\begin{aligned} 1)\ \ St(K_{1,n-1})^P_{n-1} &= 0\left(\frac{(n-1)(n-2)}{2} + 1\right) + 1(n-2) \\ &= n-2. \end{aligned}$$

$$\begin{aligned} 2)\ \ N_\tau(K_{1,n-1})^P_{n-1} &= 1\left(\frac{(n-1)(n-2)}{2} + 1\right) + 2(n-2) \\ &= \frac{n^2+n-4}{2}. \end{aligned}$$

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Theorem III.5. *Let $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of crown graph S^0_n.*
If $\langle V_i \rangle = K_2$, $1 \leq i \leq n$, then
1) $St(S^0_n)^P_n = 4n(n-2)$.
2) $N_\tau(S^0_n)^P_n = 9n^2 - 15n$.

Proof: Let $\langle V_i \rangle = K_2$, $1 \leq i \leq n$. Then,
number of geodesics of length $1 = \frac{2n(2n-1)}{2} - n(n-2)$.
Number of geodesics of length $2 = 4n(n-2)$.
From II.1, we have

$$\begin{aligned} 1)\ \ Str(S^0_n)^P_n &= 0(n(2n-1) - n(n-2)) + 1(4n(n-2)) \\ &= 4n(n-2). \end{aligned}$$

$$\begin{aligned} 2)\ \ N_\tau(S^0_n)^P_n &= 1(n(2n-1) - n(n-2)) + 2(4n(n-2)) \\ &= 9n^2 - 15n. \end{aligned}$$

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Theorem III.6. *Let $P = \{V_1, V_2, \ldots, V_{m+n-1}\}$ be a partition of bistar $B(m,n)$ graph.*

If $\langle V_1 \rangle$ is K_2 which is the edge between two central vertices and all other $\langle V_{m+n-2} \rangle$ partitions are the pendant vertices, then

1) $St(B(m,n))^P_{m+n-1} = 2mn - m - n$.
2) $N_\tau(B(m,n))^P_{m+n-1} = \frac{m^2+n^2+10mn-7m-7n+4}{2}$.

Proof: Let $\langle V_1 \rangle$ be K_2 which is the edge between two central vertices and all other $\langle V_{m+n-2} \rangle$ partitions are the pendant vertices. Then, we have

number of geodesics of length $1 = \frac{(m+n)(m+n-1)}{2} - (m+n-2)$.

Number of geodesics of length $2 = (m-1)n + (n-1)m$.

From II.1, we have

1)
$$\begin{aligned} St(B(m,n))^P_{m+n-1} &= 0\Bigg(\frac{(m+n)(m+n-1)}{2} \\ &-(m+n-2)) \\ &+ 1((m-1)n + (n-1)m) \\ &= (m-1)n + (n-1)m \\ &= 2mn - m - n. \end{aligned}$$

2)
$$\begin{aligned} N_\tau(B(m,n))^P_{m+n-1} &= 1\Bigg(\frac{(m+n)(m+n-1)}{2} \\ &-(m+n-2)) \\ &+ 2((m-1)n + (n-1)m) \\ &= 1\Bigg(\frac{(m+n)(m+n-1)}{2} \\ &-(m+n-2)) + 2(2mn - m - n) \\ &= \frac{m^2+n^2+10mn-7m-7n+4}{2}. \end{aligned}$$

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Theorem III.7. *Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of the complete bipartite graph $K_{m,n}$.*

1) *If $m = n$ and $\langle V_i \rangle = K_2$, $1 \le i \le m$, then*
   a) $St(K_{m,n})^P_m = 2m(m-1)$.
   b) $N_\tau(K_{m,n})^P_m = 4m^2 + mn - 4m$.
2) *If $m = n-1$ and $\langle V_1 \rangle = K_{1,2}$ and $\langle V_i \rangle = K_2$ for $i = 2, \ldots, m$, then*
   a) $St(K_{m,n})^P_m = 2m^2 + 2m - 3$.
   b) $N_\tau(K_{m,n})^P_m = 4m^2 + 4m + mn - 6$.
3) *If $m < n$ and $\langle V_1 \rangle = K_{1,2}$, $\langle V_i \rangle = K_2$, where $2 \le i \le m$ and the remaining $n-m-1$ partitions are all singleton partites, then*
   a) $St(K_{m,n})^P_{n-1} = m^2 - m + 2n + mn - 5$.
   b) $N_\tau(K_{m,n})^P_{n-1} = \frac{5m^2+n^2+4mn-3m+7n-20}{2}$.

Proof:

1) Let $m = n$, $\langle V_i \rangle = K_2$, where $1 \le i \le m$, then number of geodesics of length $1 = m^2$.
   Number of geodesics of length $2 = 2m(m-1)$.
   From II.1, we have
   a)
   $$\begin{aligned} St(K_{m,n})^P_m &= 0(m^2) + 1(2m(m-1)) \\ &= 2m(m-1). \end{aligned}$$
   b)
   $$\begin{aligned} N_\tau(K_{m,n})^P_m &= 1(m^2) + 2(2m(m-1)) \\ &= 4m^2 + mn - 4m. \end{aligned}$$
2) Let $m = n-1$, $\langle V_1 \rangle = K_{1,2}$ and $\langle V_i \rangle = K_2$ for $i = 2, \ldots, m$. Then, number of geodesics of length $1 = mn$. Number of geodesics of length $2 = 1(m) + (m-1)3 + m(m-1)2$.
   From II.1, we have
   a)
   $$\begin{aligned} St(K_{m,n})^P_m &= 0(mn) + 1(1(m) + (m-1)3 \\ &+ m(m-1)2) \\ &= 1(m) + (m-1)3 + m(m-1)2 \\ &= 2m^2 + 2m - 3. \end{aligned}$$
   b)
   $$\begin{aligned} N_\tau(K_{m,n})^P_m &= 1(mn) + 2(1(m) + (m-1)3 \\ &+ m(m-1)2) \\ &= 1(mn) + 2(2m^2 + 2m - 3) \\ &= 4m^2 + 4m + mn - 6. \end{aligned}$$
3) If $m < n$, let $\langle V_1 \rangle = K_{1,2}$, $\langle V_i \rangle = K_2$, where $2 \le i \le m$ and the remaining $n-m-1$ partitions are all singleton partites. Then, number of geodesics of length $1 = \frac{(m+n)(m+n-1)}{2} - m(n-1)$. Number of geodesics of length $2 = 1(n-1) + (m-1)3 + (m-1)(n-m-1) + 2((n-m-1) + m(m-1))$.
   From II.1, we have
   a)
   $$\begin{aligned} St(K_{m,n})^P_{n-1} &= 0\Bigg(\frac{(m+n)(m+n-1)}{2} \\ &-m(n-1)) \\ &+ 1(1(n-1) + (m-1)3 \\ &+ (m-1)(n-m-1) \\ &+ 2((n-m-1) + m(m-1))) \\ &= (n-1) + (m-1)3 \\ &+ (m-1)(n-m-1) \\ &+ 2((n-m-1) + m(m-1))) \\ &= m^2 - m + 2n + mn - 5. \end{aligned}$$
   b)
   $$\begin{aligned} N_\tau(K_{m,n})^P_{n-1} &= 1\Bigg(\frac{(m+n)(m+n-1)}{2} \\ &-m(n-1)) \\ &+ 2(1(n-1) + (m-1)3 \\ &+ (m-1)(n-m-1) \\ &+ 2((n-m-1) + m(m-1))) \\ &= \frac{5m^2+n^2+4mn-3m+7n-20}{2}. \end{aligned}$$

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Theorem III.8. *Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of the wheel graph W_n.*

1) *For odd n, let $\langle V_1 \rangle = C_3$ including central vertex and remaining $\frac{n-3}{2}$ partites induce K_2. Then,*
   a) $St(W_n)^P_{\frac{n-1}{2}} = \frac{n^2-13}{2}$.
   b) $N_\tau(W_n)^P_{\frac{n-1}{2}} = \frac{3n^2-4n-19}{2}$.

2) *If n is even, $\langle V_1\rangle = C_3$ including the central vertex and let $\langle V_{\frac{n}{2}}\rangle = K_1$ the remaining $\frac{n-4}{2}$ partites induce K_2 's, then*
   a) $St(W_n)^P_{\frac{n}{2}} = \frac{n^2+n-20}{2}$.
   b) $N_\tau(W_n)^P_{\frac{n}{2}} = \frac{3n^2-2n-34}{2}$.

Proof:

1) If n is odd, let $\langle V_1\rangle = C_3$ including the central vertex. Remaining $\frac{n-3}{2}$ partites induce K_2's. Then, number of geodesics of length $1 = \frac{n(n-1)}{2} - \left(\frac{n-1}{2} + n - 3\right)$. Number of geodesics of length $2 = 2(1) + (n-5)2 + (\frac{n-1}{2})(n-3)$.
   From II.1, we have

$$\begin{aligned} a)\ St(W_n)^P_{\frac{n-1}{2}} &= 0\left(\frac{n(n-1)}{2} - \left(\frac{n-1}{2} + n - 3\right)\right) \\ &+ 1\left(2(1) + (n-5)2 + \left(\frac{n-1}{2}\right)(n-3)\right) \\ &= 2(1) + (n-5)2 + \left(\frac{n-1}{2}\right)(n-3) \\ &= \frac{n^2-13}{2}. \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(W_n)^P_{\frac{n-1}{2}} &= 1\left(\frac{n(n-1)}{2} - \left(\frac{n-1}{2} + n - 3\right)\right) \\ &+ 2\left(2(1) + (n-5)2 + \left(\frac{n-1}{2}\right)(n-3)\right) \\ &= \frac{3n^2-4n-19}{2}. \end{aligned}$$

2) If n is even, let $\langle V_1\rangle = C_3$ including the central vertex and let $\langle V_{\frac{n}{2}}\rangle = K_1$ then the remaining $\frac{n-4}{2}$ partites induce K_2 's. Then, number of geodesics of length $1 = \frac{n(n-1)}{2} - (\frac{n}{2} + n - 3)$. Number of geodesics of length $2 = 2(1) + (n-5)2 + 2(n-4) + (\frac{n}{2} - 2)(n-3)$.
   From II.1, we have

$$\begin{aligned} a)\ St(W_n)^P_{\frac{n}{2}} &= 0\left(\frac{n(n-1)}{2} - \left(\frac{n}{2} + n - 3\right)\right) \\ &+ 1\left(2(1) + (n-5)2 + 2(n-4) + \left(\frac{n}{2} - 2\right)(n-3)\right) \\ &= 2(1) + (n-5)2 + 2(n-4) + \left(\frac{n}{2} - 2\right)(n-3) \\ &= \frac{n^2+n-20}{2}. \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(W_n)^P_{\frac{n}{2}} &= 1\left(\frac{n(n-1)}{2} - \left(\frac{n}{2} + n - 3\right)\right) \\ &+ 2\left(2(1) + (n-5)2 + 2(n-4) + \left(\frac{n}{2} - 2\right)(n-3)\right) \\ &= \frac{3n^2-2n-34}{2}. \end{aligned}$$

■

Theorem III.9. *Let $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of fan graph $F_{1,n}$ with $n+1$ vertices.*

If $\langle V_1\rangle = K_2$ which is an edge connecting the universal vertex with any other vertex and $\langle V_i\rangle = K_1$, where $1 \le i \le n-1$, then

1) $St(F_{1,n})^P_n = n^2 - 2n - 2$.
2) $N_\tau(F_{1,n})^P_n = \left(\sum_{i=3}^{n}(n+1-i)\right) + 2n^2 - 5n$.

Proof: Let $\langle V_1\rangle = K_2$ which is an edge connecting the universal vertex with any other vertex and $\langle V_i\rangle = K_1$, where $1 \le i \le n-1$. Then, number of geodesics of length $1 = \left(\sum_{i=3}^{n}(n+1-i)\right) + 1$. Number of geodesics of length $2 = 2(n-3) + (n-3)(n-4) + (n-2)(1)$. Number of geodesics of length $3 = 1(n-3)$.

From II.1, we have

$$\begin{aligned} 1)\ St(F_{1,n})^P_n &= 0\left(\left(\sum_{i=3}^{n}(n+1-i)\right) + 1\right) \\ &+ 1(2(n-3) + (n-3)(n-4) \\ &+ (n-2)(1)) + 2(1(n-3)) \\ &= 1(2(n-3) + (n-3)(n-4) \\ &+ (n-2)(1)) \\ &+ 2(1(n-3)) \\ &= n^2 - 2n - 2. \end{aligned}$$

$$\begin{aligned} 2)\ N_\tau(F_{1,n})^P_n &= 1\left(\left(\sum_{i=3}^{n}(n+1-i)\right) + 1\right) \\ &+ 2(2(n-3) + (n-3)(n-4) \\ &+ (n-2)(1)) + 3(1(n-3)) \\ &= \left(\left(\sum_{i=3}^{n}(n+1-i)\right) + 1\right) \\ &+ 2n^2 - 5n - 1 \\ &= \left(\sum_{i=3}^{n}(n+1-i)\right) + 2n^2 - 5n. \end{aligned}$$

■

IV. Stress and Tension of $k(i)$−complement of some standard graphs

Theorem IV.1. *Let $P = \{V_1, V_2\}$ be a partition of P_n.*

1) *Let n be odd and $P = \{V_1, V_2\}$ be a partition of $V(P_n)$ such that $\langle V_1\rangle$ consists of all the $(\frac{n+1}{2})$ alternating vertices of P_n and all the remaining $(\frac{n-1}{2})$ vertices of P_n be $\langle V_2\rangle$. Then,*
   a) $St(P_n)^P_{2(i)} = n^2 - 5n + 6$.
   b) $N_\tau(P_n)^P_{2(i)} = \frac{9n^2-38n+45}{4}$.
2) *Let n be even and $P = \{V_1, V_2\}$ be a partition of $V(P_n)$ such that $\langle V_1\rangle$ consists of all the $\frac{n}{2}$ alternating vertices of P_n and all the remaining $\frac{n}{2}$ vertices of P_n be $\langle V_2\rangle$. Then,*
   a) $St(P_n)^P_{2(i)} = n^2 - 5n + 6$.
   b) $N_\tau(P_n)^P_{2(i)} = \frac{9n^2-38n+44}{4}$.

Proof:

1) For odd n, consider the partition of $V(P_n)$ as follows:

$\langle V_1\rangle$ consists of all the $\frac{n+1}{2}$ alternating vertices of P_n and the remaining $\frac{n-1}{2}$ vertices of P_n are in $\langle V_2\rangle$. Then, number of geodesics of length $1 = \frac{n^2+2n-3}{4}$. Number of geodesics of length $2 = (n-3)3 + (\frac{n^2-8n+15}{4})4$.
From II.1, we have

a) $$\begin{aligned} St(P_n)^P_{2(i)} &= 0\left(\frac{n^2+2n-3}{4}\right) \\ &+ 1\left((n-3)3 + \left(\frac{n^2-8n+15}{4}\right)4\right) \\ &= (n-3)3 + \left(\frac{n^2-8n+15}{4}\right)4 \\ &= n^2-5n+6. \end{aligned}$$

b) $$\begin{aligned} N_\tau(P_n)^P_{2(i)} &= 1\left(\frac{n^2+2n-3}{4}\right) \\ &+ 2\left((n-3)3 + \left(\frac{n^2-8n+15}{4}\right)4\right) \\ &= 1\left(\frac{n^2+2n-3}{4}\right) + 2(n^2-5n+6) \\ &= \frac{9n^2-38n+45}{4}. \end{aligned}$$

2) For even n, consider the partition of $V(P_n)$ as follows: $\langle V_1\rangle$ consists of all the $\frac{n}{2}$ alternating vertices of P_n then all the remaining $\frac{n}{2}$ vertices of P_n are in $\langle V_2\rangle$. Then, number of geodesics of length $1 = \frac{1}{2}(n+(n-2)(\frac{n}{2}+1))$. Number of geodesics of length $2 = 1(2) + (n-4)3 + (\frac{n^2}{4} - 2n + 4)4$.
From II.1, we have

a) $$\begin{aligned} St(P_n)^P_{2(i)} &= 0\left(\frac{1}{2}\left(n+(n-2)\left(\frac{n}{2}+1\right)\right)\right) \\ &+ 1\,(1(2) + (n-4)3 \\ &+ \left(\frac{n^2}{4} - 2n + 4\right)4\Big) \\ &= 1(2) + (n-4)3 + \left(\frac{n^2}{4} - 2n + 4\right)4 \\ &= n^2-5n+6. \end{aligned}$$

b) $$\begin{aligned} N_\tau(P_n)^P_{2(i)} &= 1\left(\frac{1}{2}\left(n+(n-2)\left(\frac{n}{2}+1\right)\right)\right) \\ &+ 2\,(1(2) + (n-4)3 \\ &+ \left(\frac{n^2}{4} - 2n + 4\right)4\Big) \\ &= \frac{1}{2}\left(n+(n-2)\left(\frac{n}{2}+1\right)\right) \\ &+ 2(n^2-5n+6) \\ &= \frac{9n^2-38n+44}{4}. \end{aligned}$$

■

Theorem IV.2. *Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of C_n.*

1) *If n is even, let $P = \{V_1, V_2\}$ be a partition of $V(C_n)$ such that $\langle V_1\rangle$ consists of $\frac{n}{2}$ alternating vertices and the remaining $\frac{n}{2}$ vertices be in $\langle V_2\rangle$. Then,*

   a) $St(C_n)^P_{2(i)} = \frac{n^2-4n}{4}$.

   b) $N_\tau(C_n)^P_{2(i)} = \frac{3n^2-6n}{4}$.

2) *If n is odd, let $P = \{V_1, V_2, V_3\}$ be a partition of $V(C_n)$ such that $\langle V_1\rangle$ consists of K_1, $\langle V_2\rangle$ consists of all the $\frac{n-1}{2}$ alternating vertices and the remaining $\frac{n-1}{2}$ vertices be in $\langle V_3\rangle$. Then,*

   a) $St(C_n)^P_{3(i)} = n^2-4n+3$.

   b) $N_\tau(C_n)^P_{3(i)} = \frac{9n^2-32n+27}{4}$.

Proof:

1) For even n, consider the partition of $V(C_n)$ as follows: $\langle V_1\rangle$ consists of $\frac{n}{2}$ alternating vertices and the remaining $\frac{n}{2}$ vertices be in $\langle V_2\rangle$. Then,
number of geodesics of length $1 = \frac{n}{2}(\frac{n}{2}+1)$.
Number of geodesics of length $2 = \frac{n(n-1)}{2} - \frac{n}{2}(\frac{n}{2}+1)$.
From II.1, we have

a) $$\begin{aligned} St(C_n)^P_{2(i)} &= 0\left(\frac{n}{2}\left(\frac{n}{2}+1\right)\right) \\ &+ 1\left(\frac{n(n-1)}{2} - \frac{n}{2}\left(\frac{n}{2}+1\right)\right) \\ &= \frac{n(n-1)}{2} - \frac{n}{2}\left(\frac{n}{2}+1\right) \\ &= \frac{n^2-4n}{4}. \end{aligned}$$

b) $$\begin{aligned} N_\tau(C_n)^P_{2(i)} &= 1\left(\frac{n}{2}\left(\frac{n}{2}+1\right)\right) \\ &+ 2\left(\frac{n(n-1)}{2} - \frac{n}{2}\left(\frac{n}{2}+1\right)\right) \\ &= 1\left(\frac{n}{2}\left(\frac{n}{2}+1\right)\right) + 2\left(\frac{n^2-4n}{4}\right) \\ &= \frac{3n^2-6n}{4}. \end{aligned}$$

2) For odd n, consider the partition of $V(C_n)$ as follows: $\langle V_1\rangle$ consists of K_1, $\langle V_2\rangle$ consists of all the $\frac{n-1}{2}$ alternating vertices and the remaining $\frac{n-1}{2}$ vertices be in $\langle V_3\rangle$. Then, number of geodesics of length $1 = 1+\frac{n^2-1}{4}$. Number of geodesics of length $2 = (n-4)3 + 2(2) + (n-5)1 + (\frac{n^2}{4} - 2n + 4)4$.
From II.1, we have

a) $$\begin{aligned} St(C_n)^P_{3(i)} &= 0\left(1 + \frac{n^2-1}{4}\right) \\ &+ 1\,((n-4)3 + 2(2) + (n-5)1 \\ &+ \left(\frac{n^2}{4} - 2n + 4\right)4\Big) \\ &= (n-4)3 + 2(2) + (n-5)1 \\ &+ \left(\frac{n^2}{4} - 2n + 4\right)4 \\ &= n^2-4n+3. \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(C_n)^P_{3(i)} &= 1\left(1+\frac{n^2-1}{4}\right) \\ &+2\left((n-4)3+2(2)+(n-5)1\right. \\ &\left.+\left(\frac{n^2}{4}-2n+4\right)4\right) \\ &= 1\left(1+\frac{n^2-1}{4}\right)+2(n^2-4n+3) \\ &= \frac{9n^2-32n+27}{4}. \end{aligned}$$

■

Theorem IV.3. *Let $P=\{V_1,V_2,\dots,V_k\}$ be a partition of $V(K_{m,n})$ such that*

1) *If $m=n$, m and n are even then, $V_i=\{v_i,v_j\}$ be such that both $i,j\in\{1,2,\dots,m\}$ or $i,j\in\{m+1,m+2,\dots,m+n\}$, then*
   a) *$St(K_{m,n})^P_{m(i)}=m^2(m-2)$.*
   b) *$N_\tau(K_{m,n})^P_{m(i)}=m^3+2m^2-3m$.*
2) *If $m=n$, m and n are odd then, $\langle V_1\rangle=v_i$; $1\le i\le m$, $\langle V_2\rangle=v_{m+i}$; $1\le i\le n$ and $\langle V_i\rangle=\{v_i,v_j\}$ be such that both $i,j\in\{1,2,\dots,m\}$ or $i,j\in\{m+1,m+2,\dots,m+n\}$, then*
   a) *$St(K_{m,n})^P_{m(i)}=(m-1)^2m$.*
   b) *$N_\tau(K_{m,n})^P_{m(i)}=\frac{4m^3-7m^2+5m+mn+n-2}{2}$.*
3) *If $m\neq n$ then partition of $V(K_{m,n})$ be as follows: $\langle V_1\rangle=K_{1,1}$ and the remaining $\langle V_i\rangle$ be single vertex where $2\le i\le(m+n-1)$, then*
   a) *$St(K_{m,n})^P_{m+n-1(i)}=3(m-1)(n-1)+\frac{(m-1)(m-2)}{2}(n)+\frac{(n-1)(n-2)}{2}(m)$.*
   b) *$N_\tau(K_{m,n})^P_{m+n-1(i)}=(mn-1)+7(m-1)(n-1)+2\left(\frac{(m-1)(m-2)}{2}(n)+\frac{(n-1)(n-2)}{2}(m)\right)$.*

Proof:

1) If $m=n$, m and n are even then, let $P=\{V_1,V_2,\dots,V_m\}$ a partition of $V(K_{m,n})$ such that $\langle V_i\rangle=\{v_i,v_j\}$, where both $i,j\in\{1,2,\dots,m\}$ or $i,j\in\{m+1,m+2,\dots,m+n\}$. Then, we have number of geodesics of length $1=(m+1)m$. Number of geodesics of length $2=m^2(m-2)$.
   From II.1, we have

$$\begin{aligned} a)\ St(K_{m,n})^P_{m(i)} &= 0((m+1)m)+1(m^2(m-2)) \\ &= m^2(m-2). \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(K_{m,n})^P_{m(i)} &= 1(m^2(m+1))+2(m(m-2)) \\ &= m^3+2m^2-3m. \end{aligned}$$

2) If $m=n$, m and n are odd then let $P=\{V_1,V_2,\dots,V_{m+1}\}$ a partition of $V(K_{m,n})$ such that $\langle V_1\rangle=v_i$; $1\le i\le m$, $V_2=v_{m+i}$; $1\le i\le n$ and $V_i=\{v_i,v_j\}$ be such that both $i,j\in\{1,2,\dots,m\}$ or $i,j\in\{m+1,m+2,\dots,m+n\}$ then, number of geodesics of length $1=\frac{2m+(m+n-2)(m+1)}{2}$. Number of geodesics of length $2=(m-1)^2m$.
   From II.1, we have

$$\begin{aligned} a)\ St(K_{m,n})^P_{m+1(i)} &= 0\left(\frac{2m+(m+n-2)(m+1)}{2}\right) \\ &+1\left((m-1)^2m\right) \\ &= (m-1)^2m. \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(K_{m,n})^P_{m+1(i)} &= 1\left(\frac{2m+(m+n-2)(m+1)}{2}\right) \\ &+2\left((m-1)^2m\right) \\ &= \frac{4m^3-7m^2+5m+mn+n-2}{2}. \end{aligned}$$

3) If $\langle V_1\rangle=K_{1,1}$ and the remaining $\langle V_i\rangle$ be single vertex where $2\le i\le m+n-1$, then number of geodesics of length $1=mn-1$. Number of geodesics of length $2=(m-1)(n-1)+\frac{(m-1)(m-2)}{2}(n)+(n-1)(m-1)+\frac{(n-1)(n-2)}{2}(m)$. Number of geodesics of length $3=(m-1)(n-1)$.
   From II.1, we have

$$\begin{aligned} a)\ St(K_{m,n})^P_{m+n-1(i)} &= 0(mn-1)+1\left((m-1)(n-1)\right. \\ &+\frac{(m-1)(m-2)}{2}(n) \\ &+(n-1)(m-1) \\ &\left.+\frac{(n-1)(n-2)}{2}(m)\right) \\ &+2(m-1)(n-1) \\ &= (m-1)(n-1) \\ &+\frac{(m-1)(m-2)}{2}(n) \\ &+(n-1)(m-1) \\ &+\frac{(n-1)(n-2)}{2}(m) \\ &+2(m-1)(n-1) \\ &= 3(m-1)(n-1) \\ &+\frac{(m-1)(m-2)}{2}(n) \\ &+\frac{(n-1)(n-2)}{2}(m). \end{aligned}$$

$$\begin{aligned} b)\ N_\tau(K_{m,n})^P_{m+n-1(i)} &= 1(mn-1) \\ &+2\left((m-1)(n-1)\right. \\ &+\frac{(m-1)(m-2)}{2}(n) \\ &+(n-1)(m-1) \\ &\left.+\frac{(n-1)(n-2)}{2}(m)\right) \\ &+3(m-1)(n-1) \\ &= (mn-1) \\ &+7(m-1)(n-1) \\ &+2\left(\frac{(m-1)(m-2)}{2}(n)\right. \\ &\left.+\frac{(n-1)(n-2)}{2}(m)\right). \end{aligned}$$

■

Theorem IV.4. *For wheel graph W_n partition the vertex set into 2 partitions i.e., $\langle V_1\rangle$ consists of K_1 including universal vertex and $\langle V_2\rangle$ consists of all other vertices. Then,*

1) *$St(W_n)^P_{2(i)}=(n-1)(n-4)$.*
2) *$N_\tau(W_n)^P_{2(i)}=\frac{5n^2-23n+18}{2}$.*

Proof: Let the vertex set be partitioned into 2 partitions i.e., $\langle V_1\rangle$ consists of K_1 including universal vertex and $\langle V_2\rangle$

consists of all other vertices. Then, number of geodesics of length $1 = \frac{n(n-1)}{2} - (n-1)$. Number of geodesics of length $2 = (n-1)(n-4)$.

From II.1, we have

$$1)\ St(W_n)^P_{2(i)} = 0\left(\frac{n(n-1)}{2} - (n-1)\right) + 1((n-1)(n-4)) = (n-1)(n-4).$$

$$2)\ N_\tau(W_n)^P_{2(i)} = 1\left(\frac{n(n-1)}{2} - (n-1)\right) + 2((n-1)) = \frac{5n^2-23n+18}{2}.$$

■

Theorem IV.5. *Let $P = \{V_1, V_2\}$ be a partition of $V(F_{1,n})$ such that let $\langle V_1\rangle$ consists of K_1 i.e., the vertex which is adjacent to all other vertices and $\langle V_2\rangle$ consists of all the remaining n vertices. Then,*

1) $St(F_{1,n})^P_{2(i)} = n^2 - 4n + 5$.
2) $N_\tau(F_{1,n})^P_{2(i)} = \frac{5n^2-17n+22}{2}$.

Proof: Let $P = \{V_1, V_2\}$ be a partition of $V(F_{1,n})$ such that $\langle V_1\rangle$ consists of K_1 i.e., the vertex which is adjacent to all other vertices and $\langle V_2\rangle$ consists of all the remaining n vertices. Then, number of geodesics of length $1 = \frac{(n-1)(n-2)}{2} + n$. Number of geodesics of length $2 = 2(n-2) + (n-3)(n-3)$.

From II.1, we have

$$1)\ St(F_{1,n})^P_{2(i)} = 0\left(\frac{(n-1)(n-2)}{2} + n\right) + 1(2(n-2) + (n-3)(n-3)) = (2(n-2) + (n-3)(n-3)) = n^2 - 4n + 5.$$

$$2)\ N_\tau(F_{1,n})^P_{2(i)} = 1\left(\frac{(n-1)(n-2)}{2} + n\right) + 2(2(n-2) + (n-3)(n-3)) = 1\left(\frac{(n-1)(n-2)}{2} + n\right) + 2(n^2 - 4n + 5) = \frac{5n^2-17n+22}{2}.$$

■

V. Conclusion

Generalised complements of a graph depends on partition of a vertex set. Also, stress and tension of the generalised complements of the graph varies with respect to the partitions of V of G. In this paper, we have obtained stress and tension of k and $k(i)$ complement of few standard graphs such as path, cycle, complete bipartite graph, wheel graph and fan graph.

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