Double Laplace Formable Transform Method for Solving PDEs

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Abstract—**This work introduces the double Laplace formable transform (DLFT), a novel double transform. A number of fundamental features and functions have been demonstrated using the newly defined double transform definition. Further, the existence, convolution, and derivative theorems of the DLFT are given and proven. Afterward, a few families of partial differential equations (PDEs) are solved using these results. There are new methods for solving PDE families. The latter solves the advection-diffusion equation, the Klein-Gordon equation, the telegraph equation, and other well-known PDEs precisely. According to the results, the DLFT is a useful and effective strategy for dealing with this kind of issue.**

Index Terms—**Formable transform, Laplace transform, partial differential equations, integral transform.**

I. Introduction

ONE of the most significant techniques used to solve PDEs is the use of integral transforms. Since PDEs may be used to describe a wide range of scientific processes and events, we can use integral transforms to alter these equations and get the exact solution of PDEs [1], [2], [3], [4], [5], [6], [7], [8].

Researchers and scientists have made great efforts to develop these techniques, and they use them to address contemporary scientific issues.

For instance, there are several transforms available: Fourier transform [9], natural transform [10], Sumudu transform [11], Elzaki transform [12], M-transform [13], Novel transform [14], including the Laplace integral transform (LIT) [15], polynomial transform [16], ARA transform [17], [18], [19], [20] and formable transform [21].

In recent years, double transforms have been successfully and widely utilized to solve different kinds of PDEs. In comparison to numerical approaches, they produced strong results [22], [23], [24], [25], [26]. Additional double transform extensions, including the double ARA transform [27], the double Sumudu transform [28], [29], double formable transform [30], [31], the double ARA transform [32], [33], the double Elzaki transform [34], and the Laplace-Sumudu transform [35], [36], [37] were presented by researchers and others.

The primary objective of this work is to provide a novel double transform DLFT, which is a combination of Laplace and Formable transforms. The new DLFT is a new transform that has the advantages of two transforms. We discuss some of essential properties of the DLFT and calculate them for certain basic functions. Several theorems related to DLFT are explored and utilized to establish new results. Moreover, the obtained results are tested by presenting formulas to solve families of PDEs with some applications.

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II. Laplace Integral Transform

The LIT is a mathematical concept that has applications in a wide range of fields, from physics and engineering to mathematics and statistics. It represents a powerful tool for understanding complex systems and analyzing the behavior of physical phenomena, and in this section, it is particularly useful in solving differential equations. In this section, we present the definition and some properties of the LIT; for more details, see [23].

Definition 2.1: Let $\emptyset\left(\xi\right)$ be a continuous function defined on the interval $[0,\infty)$, then LIT of $\emptyset\left(\xi\right)$ is given by

$$L_{\xi}\left[\emptyset\left(\xi\right)\right]=\Phi\left(v\right)=\int_{0}^{\infty}e^{-v\xi}\emptyset\left(\xi\right)d\xi,\Re v>0.$$

Definition 2.2: The single LIT of a function $\emptyset\left(\xi,\omega\right)$ with respect to ξ is given by

$$L_{\xi}\left[\emptyset\left(\xi,\omega\right)\right]=\Phi\left(v,\omega\right).$$

Now, we present some fundamental characteristics of LIT. Let $\emptyset\left(\xi\right)$ and $\psi\left(\xi\right)$ be a pair of continuous functions defined on $[0,\infty)$ where the LIT exists. Then,

i. $L_{\xi}\left[\alpha\emptyset\left(\xi\right)+\beta\psi\left(\xi\right)\right]=\alpha L_{\xi}\left[\emptyset\left(\xi\right)\right]+\beta L_{\xi}\left[\psi\left(\xi\right)\right]$, where α & $\beta\in\mathbb{R}$.
ii. $L_{\xi}\left[\emptyset'\left(\xi\right)\right]=vL_{\xi}\left[\emptyset\left(\xi\right)\right]-\emptyset\left(0\right)$.
iii. $L_{\xi}\left[\xi^{\alpha}\right]=\frac{\Gamma(\alpha+1)}{v^{\alpha+1}},\alpha+1>0$.

A. Laplace Transform of some basic functions

$$\begin{aligned}
&L_{\xi}[1]=1/v.\\
&L_{\xi}\left[\xi^{\alpha}\right]=\frac{\alpha!}{v^{\alpha+1}}.\\
&L_{\xi}\left[e^{a\xi}\right]=\frac{1}{v-a}.\\
&L_{\xi}\left[\sin\left(a\xi\right)\right]=\frac{a}{v^{2}+a^{2}}.\\
&L_{\xi}\left[\sinh\left(a\xi\right)\right]=\frac{a}{v^{2}-a^{2}}.\\
&L_{\xi}\left[\cos\left(a\xi\right)\right]=\frac{v}{v^{2}+a^{2}}.\\
&L_{\xi}\left[\cosh\left(a\xi\right)\right]=\frac{v}{v^{2}-a^{2}}.\\
&L_{\xi}\left[\xi\emptyset\left(\xi\right)\right]=-\frac{d(\Phi\left(v\right))}{dv}.\\
&L_{\xi}\left[\emptyset'\left(\xi\right)\right]=v\Phi\left(v\right)-\emptyset\left(0\right).
\end{aligned}\tag{1}$$

III. Formable Integral Transform

Formable transform is a mathematical concept that has applications in a wide range of fields. In this section, we present the definition and some characterestics of Formable transform, for more details see [11].

Definition 3.1: Let $\emptyset(\omega)$ be a continuous function defined on $[0,\infty)$, then Formable integral transform (FIT) of the function $\emptyset(\omega)$ of exponential order is defined as:

$$\mathcal{R}_\omega[\emptyset(\omega)] = \phi(s,u) = \frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}\emptyset(\omega)\,d\omega, \quad Res>0\ \& Reu>0. \tag{2}$$

Definition 3.2: The single FIT of a function $\emptyset(\xi,\omega)$ of two variables ξ and ω with respect to ω is given by

$$\mathcal{R}_\omega[\emptyset(\xi,\omega)] = \bar{\phi}(\xi,s,u) = \frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\omega, \quad s>0\ \&\ u>0. \tag{3}$$

Let $\emptyset(\xi)$ and $\psi(\xi)$ be a pair of continuous functions defined on $[0,\infty)$ where the FIT exists. Then

i. $\mathcal{R}_\omega[\alpha\emptyset(\omega)+\beta\psi(\omega)] = \alpha\mathcal{R}_\omega[\emptyset(\omega)] + \beta\mathcal{R}_\omega[\psi(\omega)]$, where α & $\beta\in\mathbb{R}$.
ii. $\mathcal{R}_\omega[\emptyset'(\omega)] = \frac{S}{u}\phi(s,u) - \frac{s}{u}\emptyset(0)$.
iii. $\mathcal{R}_\omega[\omega^n] = \frac{u^n n!}{s^n}$.

A. Some element functions

i. $\mathcal{R}_\omega[1]=1$.
ii. $\mathcal{R}_\omega[\omega^\alpha] = \frac{u^\alpha}{s^\alpha}\Gamma(\alpha+1),\ \alpha>0$.
iii. $\mathcal{R}_\omega[e^{a\omega}] = \frac{s}{s-au}.\mathcal{R}_\omega[\sin(a\omega)] = \frac{asu}{s^2+a^2u^2}$.
iv. $\mathcal{R}_\omega[\sinh(a\omega)] = \frac{asu}{s^2-a^2u^2}$.
v. $\mathcal{R}_\omega[\cos(a\omega)] = \frac{s^2}{s^2+a^2u^2}$.
vi. $\mathcal{R}_\omega[\cosh(a\omega)] = \frac{s^2}{s^2-a^2u^2}$.
vii. $\mathcal{R}_\omega[\omega\emptyset(\omega)] = \frac{u^2}{s}\frac{d(\phi(s,u))}{du} + \frac{u}{s}\phi(s,u)$.
viii. $\mathcal{R}_\omega[\emptyset'(\omega)] = \frac{s}{u}\phi(s,u) - \frac{s}{u}\emptyset(0)$.

IV. Double Laplace-Formable Transform

In this section, we present the new double transform DLFT and some basic properties.

Definition 4.1: The double Laplace-Formable transform of a continuous function $\emptyset(\xi,\omega)$ of two variables $\xi>0$ and $\omega>0$ is given by

$$\begin{aligned} L_\xi\mathcal{R}_\omega[\emptyset(\xi,\omega)] &= \Phi(v,s,u) \\ &= \frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega, \\ &\qquad Re\,[v]>0\ \&\ Re[u]>0, \end{aligned}$$

provided the existence of the integral.
The DLFT is a linear operator, because

$$\begin{aligned} L_\xi\mathcal{R}_\omega[\alpha\emptyset(\xi,\omega)\ &+\beta\psi(\xi,\omega)] \\ &= \alpha L_\xi\mathcal{R}_\omega[\emptyset(\xi,\omega)] + \beta L_\xi\mathcal{R}_\omega[\psi(\xi,\omega)], \end{aligned}$$

where α & $\beta\in\mathbb{R}$.
The inverse DLFT is given by

$$\begin{aligned} L_\xi^{-1}R_\omega^{-1}[\Phi(v,s,u)] &= \emptyset(\xi,\omega) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{v\xi}dv\ \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{e^{\frac{s\omega}{u}}}{s}\Phi(v,s,u)\,du. \end{aligned}$$

A. DLFT to some basic functions

i. Let $\emptyset(\xi,\omega)=\alpha$, $\xi>0$, $\omega>0$. Then

$$\begin{aligned} L_\xi\mathcal{R}_\omega[\alpha] &= \frac{s\alpha}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}d\xi d\omega \\ &= \alpha\int_0^\infty e^{-v\xi}d\xi\frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}d\omega = \frac{\alpha}{v}, \\ &\quad Re(v)>0. \end{aligned}$$

ii. Let $\emptyset(\xi,\omega)=\xi^\alpha\omega^\beta$, $\xi>0$, $\omega>0$. Then

$$\begin{aligned} L_\xi\mathcal{R}_\omega[\xi^\alpha\omega^\beta] &= \frac{s}{u}\int_0^\infty\int_0^\infty \xi^\alpha\omega^\beta e^{-v\xi-\frac{s\omega}{u}}d\xi d\omega \\ &= \int_0^\infty \xi^\alpha e^{-v\xi}d\xi\frac{s}{u}\int_0^\infty \omega^\beta e^{-\frac{s\omega}{u}}d\omega \\ &= L_\xi[\xi^\alpha]\,\mathcal{R}_\omega[\omega^\beta] \\ &= \frac{\Gamma(\alpha+1)}{v^{\alpha+1}}\frac{u^\beta\Gamma(\beta+1)}{s^\beta} \\ &= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,u^\beta}{s^\beta\ v^{\alpha+1}}, \end{aligned}$$

where $\alpha>-1$ and $\beta>-1$ are constants.

iii. Let $\emptyset(\xi,\omega)=e^{\alpha\xi+\beta\omega}$. Then

$$\begin{aligned} &L_\xi\mathcal{R}_\omega[e^{\alpha\xi+\beta\omega}] \\ &\quad= \frac{s}{\beta u}\int_0^\infty\int_0^\infty e^{\alpha\xi+\beta\omega}e^{-v\xi-\frac{s\omega}{u}}d\xi d\omega \\ &\quad= \int_0^\infty e^{\alpha\xi}e^{-v\xi}d\xi\frac{s}{u}\int_0^\infty e^{\beta\omega}e^{-\frac{s\omega}{u}}d\omega \\ &\quad= \frac{s}{(v-\alpha)(s-u\beta)}, \\ &\qquad\qquad v>0,\quad \frac{s}{u}>0. \end{aligned}$$

iv. Let $\emptyset(\xi,\omega)=sin(\alpha\xi+\beta\omega)$. Then

$$\begin{aligned} &L_\xi\mathcal{R}_\omega[\sin(\alpha\xi+\beta\omega)] \\ &\quad= L_\xi\mathcal{R}_\omega\left[\frac{e^{i(\alpha\xi+\beta\omega)}-e^{-i(\alpha\xi+\beta\omega)}}{2i}\right]. \end{aligned}$$

Now using linearity, we get

$$\begin{aligned} &L_\xi\mathcal{R}_\omega\left[\frac{e^{i(\alpha\xi+\beta\omega)}-e^{-i(\alpha\xi+\beta\omega)}}{2i}\right] \\ &\quad= L_\xi\mathcal{R}_\omega\left[\frac{e^{i(\alpha\xi+\beta\omega)}}{2i}\right] - L_\xi\mathcal{R}_\omega\left[\frac{e^{-i(\alpha\xi+\beta\omega)}}{2i}\right] \\ &\quad= L_\xi\left[\frac{e^{i\alpha\xi}}{2i}\right]\mathcal{R}_\omega\left[e^{i\beta\omega}\right] \\ &\qquad - L_\xi\left[\frac{e^{-i\alpha\xi}}{2i}\right]\mathcal{R}_\omega\left[e^{-i\beta\omega}\right]. \end{aligned}$$

So, we get

$$L_\xi\mathcal{R}_\omega[sin(\alpha\xi+\beta\omega)] = \frac{s^2\alpha+suv\beta}{(v^2+\alpha^2)(s^2+\beta^2)}.$$

Using the facts

$$\sin\xi = \frac{e^{i\xi}-e^{-i\xi}}{2i},\quad \cos\xi = \frac{e^{i\xi}+e^{-i\xi}}{2},$$
$$\sinh\xi = \frac{e^{\xi}-e^{-\xi}}{2},\quad \cosh\xi = \frac{e^{\xi}+e^{-\xi}}{2}.$$

We conclude that,

$$L_\xi\mathcal{R}_\omega[\cos(\alpha\xi+\beta\omega)] = \frac{s^2v-su\alpha\beta}{(v^2+\alpha^2)(s^2+u^2\beta^2)}.$$

$$L_\xi \mathcal{R}_\omega \left[\cosh\left(\alpha\xi+\beta\omega\right)\right] = \frac{s^2 v + su\alpha\beta}{\left(v^2-\alpha^2\right)\left(s^2-u^2\beta^2\right)}.$$

$$L_\xi \mathcal{R}_\omega \left[\sinh\left(\alpha\xi+\beta\omega\right)\right] = \frac{s^2 \alpha + suv\beta}{\left(v^2-\alpha^2\right)\left(s^2-u^2\beta^2\right)}.$$

v. Let $\emptyset(\xi,\omega) = \emptyset(\xi)\psi(\omega)$. Then

$$\begin{aligned} L_\xi \mathcal{R}_\omega\left[\emptyset(\xi,\omega)\right] &= L_\xi \mathcal{R}_\omega\left[\emptyset(\xi)\psi(\omega)\right] \\ &= \frac{s}{u}\int_0^\infty \int_0^\infty e^{-v\xi-\frac{s\omega}{u}} \emptyset(\xi)\psi(\omega)\, d\xi d\omega \\ &= \left(\int_0^\infty e^{-v\xi}\emptyset(\xi)\,d\xi\right) \\ &\quad \left(\frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}\psi(\omega)\,d\omega\right) \\ &= L_\xi\left[\emptyset(\xi)\right]\mathcal{R}_\omega\left[\psi(\omega)\right]. \end{aligned}$$

vi. Let $\emptyset(\xi,\omega) = J_0\left(\alpha\sqrt{\xi\omega}\right)$.then,

$$\begin{aligned} &L_\xi \mathcal{R}_\omega\left[J_0\left(\alpha\sqrt{\xi\omega}\right)\right] \\ &= \frac{s}{u}\int_0^\infty \int_0^\infty e^{-v\xi-\frac{s\omega}{u}} J_0\left(\alpha\sqrt{\xi\omega}\right) d\xi d\omega \\ &= \frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}d\omega \int_0^\infty e^{-v\xi} J_0\left(\alpha\sqrt{\xi\omega}\right) d\xi \\ &= \frac{s}{uv}\int_0^\infty e^{-\frac{s\omega}{u}} e^{-\frac{\alpha^2\omega}{4v}} d\omega \\ &= \mathcal{R}_\omega\left[L_\xi\left[J_0(\alpha\sqrt{\xi\omega})\right]\right] \\ &= \mathcal{R}_\omega\left[e^{-\frac{\alpha^2\omega}{4v}}\right] = \frac{4s}{4vs-\alpha^2 u}. \end{aligned}$$

Here J_0 denotes the modified order zero Bessel function.

B. Existence conditions of DLFT

Let $\emptyset(\xi,\omega)$ be a function that satisfies the following condition. If there exists a positive constant M such that $\forall\xi > X$ and $\forall\omega > Y$ we have

$$\left|\emptyset(\xi,\omega)\right| \le Me^{\alpha\xi+\beta\omega}, \tag{4}$$

$\forall\alpha > 0$ and $\beta > 0$ as $\xi \to \infty$ and $\omega \to \infty$.

Hence, we say $\emptyset(\xi,\omega) = O\left(e^{\alpha\xi+\beta\omega}\right)$ as $\xi \to \infty$ and $\omega \to \infty$, $v > \alpha$ and $\frac{s}{u} > \beta$.

Theorem 4.1: Let $\emptyset(\xi,\omega)$ be a continuous function in a region $(0,X)$ and $(0,Y)$ that satisfies the condition in equation (4). Then DLFT of $\emptyset(\xi,\omega)$ exists for all v and $\frac{s}{u}$ If $Re[v] > \alpha$ and $Re\left[\frac{s}{u}\right] > \beta$.

Proof: From the definition of DLFT, we have

$$\begin{aligned} \left|\Phi(v,s,u)\right| &= \left|\frac{s}{u}\int_0^\infty \int_0^\infty e^{-v\xi-\frac{s\omega}{u}} \emptyset(\xi,\omega)\, d\xi\ d\omega\right| \\ &\le \frac{s}{u}\int_0^\infty \int_0^\infty e^{-v\xi-\frac{s\omega}{u}} \left|\emptyset(\xi,\omega)\right| d\xi\ d\omega \\ &\le \frac{Ms}{u}\int_0^\infty e^{-(v-\alpha)\xi} d\xi \int_0^\infty e^{-\left(\frac{s}{u}-\beta\right)\omega} d\omega \\ &= \frac{Ms}{(v-\alpha)(s-u\beta)}. \end{aligned}$$

$Re[v] > \alpha$ and $Re\left[\frac{s}{u}\right] > \beta$. ■

V. Basic Properties of DLFT

In this part of the study, we introduce and prove some properties of the DLFT.

A. Shifting property

$$L_\xi \mathcal{R}_\omega\left[e^{\alpha\xi+\beta\omega}\emptyset(\xi,\omega)\right] = \frac{s}{(s-u\beta)}\Phi\left(v-\alpha, \frac{s-u\beta}{u}\right),$$

where,

$$L_\xi \mathcal{R}_\omega\left[\emptyset(\xi,\omega)\right] = \Phi(v,s,u).$$

Proof: From the definition of DLFT, we have

$$\begin{aligned} &L_\xi \mathcal{R}_\omega\left[e^{\alpha\xi+\beta\omega}\emptyset(\xi,\omega)\right] \\ &= \frac{s}{u}\int_0^\infty \int_0^\infty e^{-(v-\alpha)\xi-\left(\frac{s-u\beta}{u}\right)\omega}\ \emptyset(\xi,\omega)\, d\xi d\omega \\ &= \frac{s}{(s-u\beta)}\frac{(s-u\beta)}{u} \\ &\quad \int_0^\infty \int_0^\infty e^{-(v-\alpha)\xi-\left(\frac{s-u\beta}{u}\right)\omega}\emptyset(\xi,\omega)\, d\xi d\omega \\ &= \frac{s}{(s-u\beta)}\Phi\left(v-\alpha, \frac{s-u\beta}{u}\right). \end{aligned}$$

■

B. Derivatives properties

Let $\Phi(v,s,u) = L_\xi \mathcal{R}_\omega\left[\emptyset(\xi,\omega)\right]$. Then we have the following properties

i. $L_\xi \mathcal{R}_\omega\left[\frac{\partial\emptyset(\xi,\omega)}{\partial\xi}\right] = v\Phi(v,s,u) - \mathcal{R}_\omega\left[\emptyset(0,\omega)\right].$

ii. $L_\xi \mathcal{R}_\omega\left[\frac{\partial\emptyset(\xi,\omega)}{\partial\omega}\right] = \frac{s}{u}\Phi(v,s,u) - \frac{s}{u}L_\xi\left[\emptyset(\xi,0)\right].$

iii. $L_\xi \mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi^2}\right] = v^2\Phi(v,s,u) - v\mathcal{R}_\omega\left[\emptyset(0,\omega)\right]$ $-\mathcal{R}_\omega\left[\emptyset_\xi(0,\omega)\right].$

iv. $L_\xi \mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\omega^2}\right] = \frac{s^2}{u^2}\Phi(v,s,u) - \frac{s^2}{u^2}L_\xi\left[\emptyset(\xi,0)\right]$ $-\frac{s}{u}L_\xi\left[\emptyset_\omega(\xi,0)\right].$

v. $L_\xi \mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi\partial\omega}\right] = \frac{sv}{u}\Phi(v,s,u) - \frac{sv}{u}\Phi(\xi,0)$ $-\frac{s}{u}\mathcal{R}_\omega\left[\emptyset(0,\omega)\right] + \frac{s}{u}\emptyset(0,0).$

Now, we introduce the proof of that i, iii and v. We get the proof of ii and iv in a same way to i and iii, respectively.

Proof: of i. The definition of DLFT implies

$$\begin{aligned} L_\xi \mathcal{R}_\omega\left[\frac{\partial\emptyset(\xi,\omega)}{\partial\xi}\right] &= \frac{s}{u}\int_0^\infty \int_0^\infty e^{-v\xi-\frac{s\omega}{u}} \frac{\partial\emptyset(\xi,\omega)}{\partial\xi} d\xi d\omega \\ &= \frac{s}{u}\int_0^\infty e^{-\frac{\omega}{u}} d\omega \int_0^\infty e^{-v\xi}\frac{\partial g(\xi,\omega)}{\partial\xi} d\xi, \end{aligned}$$

by integration by parts, we get

$$\begin{aligned} L_\xi \mathcal{R}_\omega\left[\frac{\partial\emptyset(\xi,\omega)}{\partial\xi}\right] &= \frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}} d\omega \\ &\quad \left(-\emptyset(0,\omega) + v\int_0^\infty e^{-v\xi}\emptyset(\xi,\omega) d\xi\right) \\ &= \frac{-s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}\emptyset(0,\omega)\, d\omega \\ &\quad + \frac{sv}{u}\int_0^\infty \int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega) d\xi d\omega \\ &= v\Phi(v,s,u) - \mathcal{R}_\omega\left[\emptyset(0,\omega)\right]. \end{aligned}$$

■

Proof: of iii.

$$\begin{aligned} L_\xi \mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi^2}\right] &= \frac{s}{u}\int_0^\infty \int_0^\infty e^{-v\xi-\frac{s\omega}{u}} \frac{\partial^2\emptyset(\xi,\omega)}{\partial^2\xi} d\xi d\omega \\ &= \frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}} d\omega \int_0^\infty e^{-v\xi}\frac{\partial^2\emptyset(\xi,\omega)}{\partial^2\xi} d\xi, \end{aligned}$$

by integration by parts, we get

$$L_\xi\mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi^2}\right]=\frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}d\omega$$
$$\left(-\emptyset_\xi(0,\omega)-v\emptyset(0,\omega)+v^2\int_0^\infty e^{-v\xi}\emptyset(\xi,\omega)d\xi\right)$$
$$=\frac{-s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}\emptyset_\xi(0,\omega)\,d\omega$$
$$-\frac{sv}{u}\int_0^\infty e^{-\frac{s\omega}{u}}\emptyset(0,\omega)\,d\omega$$
$$+\frac{sv^2}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)d\xi d\omega$$
$$=v^2\Phi(v,s,u)-v\mathcal{R}_\omega[\emptyset(0,\omega)]-\mathcal{R}_\omega[\emptyset_\xi(0,\omega)].$$

■

Proof: of v. The definition of DLFT implies

$$L_\xi\mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi\partial\omega}\right]$$
$$=\frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi\partial\omega}d\xi d\omega \tag{5}$$
$$=\frac{s}{u}\int_0^\infty e^{-v\xi}d\xi\int_0^\infty e^{-\frac{s\omega}{u}}\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi\partial\omega}d\omega,$$

by integration by parts, we get

$$L_\xi\mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi\partial\omega}\right]=\frac{s}{u}\int_0^\infty e^{-v\xi}d\xi$$
$$\left(-\emptyset_\xi(\xi,0)+\frac{s}{u}\int_0^\infty e^{-\frac{s\omega}{u}}\frac{\partial\emptyset(\xi,\omega)}{\partial\xi}d\omega\right)$$
$$=\frac{-s}{u}\int_0^\infty e^{-v\xi}\emptyset_\xi(\xi,0)\,d\xi \tag{6}$$
$$+\frac{s^2}{u^2}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\frac{\partial\emptyset(\xi,\omega)}{\partial\xi}d\xi d\omega$$
$$=-\frac{s}{u}L_\xi[\emptyset_\xi(\xi,0)]+\frac{s}{u}L_\xi\mathcal{R}_\omega\left[\frac{\partial\emptyset(\xi,\omega)}{\partial\xi}\right].$$

Using the fact in Equation (5) and Equation (6), we get

$$L_\xi\mathcal{R}_\omega\left[\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi\partial\omega}\right]=\frac{sv}{u}\Phi(v,s,u)-\frac{sv}{u}\Phi(\xi,0)$$
$$-\frac{s}{u}\mathcal{R}_\omega[\emptyset(0,\omega)]+\frac{s}{u}\emptyset(0,0).$$

■

Remark 5.1: Let $\Phi(v,s,u)$ be the DLFT of the function $\emptyset(\xi,\omega)$ then the DLFT of the n-th partial derivative with respect to ξ and ω are

i. $L_\xi\mathcal{R}_\omega\left[\frac{\partial^n\emptyset(\xi,\omega)}{\partial\xi^n}\right]=v^n\Phi(v,s,u)-\sum_{k=0}^{n-1}v^{n-k-1}\mathcal{R}_\omega\left[\frac{\partial^k}{\partial\xi^k}\emptyset(0,\omega)\right].$

ii. $L_\xi\mathcal{R}_\omega\left[\frac{\partial^n\emptyset(\xi,\omega)}{\partial\omega^n}\right]=\left(\frac{s}{u}\right)^n\Phi(v,s,u)-\frac{s}{u}\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1}L_\xi\left[\frac{\partial^k}{\partial\omega^k}\emptyset(\xi,0)\right].$

C. DLFT of a periodic function

Theorem 5.1: If $\emptyset(\xi,\omega)$ is a periodic function of periods α and β such that $\emptyset(\xi+\alpha,\omega+\beta)=\emptyset(\xi,\omega)$ for all ξ,ω, and if DLFT of $\emptyset(\xi,\omega)$. Then,

$$L_\xi\mathcal{R}_\omega[\emptyset(\xi,\omega)]=\left(1-e^{-v\alpha-\frac{s\beta}{u}}\right)^{-1}$$
$$\left(\frac{s}{u}\int_0^\alpha\int_0^\beta e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega\right).$$

Proof: From the definition DLFT, we have

$$L_\xi\mathcal{R}_\omega[\emptyset(\xi,\omega)]=\frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega$$
$$=\frac{s}{u}\int_0^\alpha\int_0^\beta e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega$$
$$+\frac{s}{u}\int_\alpha^\infty\int_\beta^\infty e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega.$$

Let $\xi=\alpha+p$ and $\omega=\beta+q$ on the second double integral, we get

$$\Phi(v,s,u)=L_\xi\mathcal{R}_\omega[\emptyset(\xi,\omega)]$$
$$=\frac{s}{u}\int_0^\alpha\int_0^\beta e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega$$
$$+\frac{s}{u}\int_0^\infty\int_0^\infty e^{-v(\alpha+p)-\frac{s(\beta+q)}{u}}\emptyset(p+\alpha,q+\beta)\,dpdq. \tag{7}$$

Using the function's periodicity $\emptyset(\xi,\omega)$, Equation (7) becomes

$$\Phi(v,s,u)=\frac{s}{u}\int_0^\alpha\int_0^\beta e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega$$
$$+e^{-v\alpha-\frac{s\beta}{u}}\frac{s}{u}\int_0^\infty\int_0^\infty e^{-vp-\frac{sq}{u}}\emptyset(p,q)\,dpdq.$$

From the definition of DLFT, we get

$$\Phi(v,s,u)=\frac{s}{u}\int_0^\alpha\int_0^\beta e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi,\omega)\,d\xi d\omega$$
$$+e^{-v\alpha-\frac{s\beta}{u}}\Phi(v,s,u). \tag{8}$$

Equation (8) can be simplified into

$$\Phi(v,s,u)=\left(1-e^{-v\alpha-\frac{s\beta}{u}}\right)^{-1}$$
$$\left(\frac{s}{u}\int_0^\alpha\int_0^\beta e^{-v\xi-\frac{sy}{u}}\emptyset(\xi,\omega)\,d\xi d\omega\right).$$

■

D. Convolution theorem of DLFT

Theorem 5.2: Let $\Phi(v,s,u)=L_\xi\mathcal{R}_\omega[\emptyset(\xi,\omega)]$. Then,

$$L_\xi\mathcal{R}_\omega[\emptyset(\xi-\delta,\omega-\epsilon)\psi(\xi-\delta,\omega-\epsilon)]$$
$$=e^{-s\delta-\frac{s\epsilon}{\omega}}\Phi(v,s,u), \tag{9}$$

where $\psi(\xi,\omega)$ is the Heaviside unit step function provided by

$$\psi(\xi-\delta,\omega-\epsilon)=\begin{cases}1 & \xi>\delta,\ \omega>\epsilon,\\ 0 & \text{Otherwise.}\end{cases}$$

Proof: The definition of DLFT implies

$$L_\xi\mathcal{R}_\omega[\emptyset(\xi-\delta,\omega-\epsilon)\psi(\xi-\delta,\omega-\epsilon)]$$
$$=\frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}$$
$$(\emptyset(\xi-\delta,\omega-\epsilon)\psi(\xi-\delta,\omega-\epsilon))\,d\xi d\omega \tag{10}$$
$$=\frac{s}{u}\int_\delta^\infty\int_\epsilon^\infty e^{-v\xi-\frac{s\omega}{u}}\emptyset(\xi-\delta,\omega-\epsilon)\,d\xi d\omega.$$

Letting $\xi-\delta=p$ and $\omega-\epsilon=q$ in Equation (10), we get

$$L_\xi\mathcal{R}_\omega[\emptyset(\xi-\delta,\omega-\epsilon)\psi(\xi-\delta,\omega-\epsilon)]$$
$$=\frac{s}{u}\int_0^\infty\int_0^\infty e^{-v(\delta+p)-\frac{s(\epsilon+q)}{u}}\emptyset(p,q)\,dpdq,$$

above equation can be written as

$$\begin{aligned} & L_\xi \mathcal{R}_\omega \left[\emptyset\left(\xi-\delta,\omega-\epsilon\right)\psi\left(\xi-\delta,\omega-\epsilon\right)\right] \\ & \quad = e^{-v\delta-\frac{s\epsilon}{u}}\left(\frac{s}{u}\int_0^\infty\int_0^\infty e^{-vp-\frac{sq}{u}}\emptyset\left(p,q\right)dpdq\right) \\ & \quad = e^{-v\delta-\frac{s\epsilon}{u}}\Phi\left(v,s,u\right). \end{aligned}$$

■

Theorem 5.3: (Convolution theorem)
If $L_\xi \mathcal{R}_\omega\left[\emptyset\left(\xi,\omega\right)\right] = \Phi\left(v,s,u\right)$ and $L_\xi \mathcal{R}_\omega\left[\psi\left(\xi,\omega\right)\right] = \Psi\left(v,s,u\right)$.Then,

$$L_\xi \mathcal{R}_\omega\left[\left(\emptyset**\psi\right)\left(\xi,\omega\right)\right] = \left(\frac{u}{s}\right)\Phi\left(v,s,u\right)\Psi\left(v,s,u\right),$$

where,

$$\left(\emptyset**\psi\right)\left(\xi,\omega\right) = \int_0^\xi\int_0^\omega \emptyset\left(\xi-\delta,\omega-\epsilon\right)\psi\left(\delta,\epsilon\right)d\delta d\epsilon.$$

Proof:

$$\begin{aligned} & L_\xi \mathcal{R}_\omega\left[\left(\emptyset**\psi\right)\left(\xi,\omega\right)\right] \\ & \quad = \frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\left(\emptyset**\psi\right)\left(\xi,\omega\right)d\xi d\omega \\ & \quad = \frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}} \\ & \left(\int_0^\xi\int_0^\omega \emptyset\left(\xi-\delta,\omega-\epsilon\right)\psi\left(\delta,\epsilon\right)d\delta d\epsilon\right)d\xi d\omega. \end{aligned} \tag{11}$$

By using the Heaviside unit step function, Equation(9) may be expressed in writing as

$$\begin{aligned} L_\xi \mathcal{R}_\omega\left[\left(\emptyset**\psi\right)\left(\xi,\omega\right)\right] = & \frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}} \\ & \left(\int_0^\infty\int_0^\infty \emptyset\left(\xi-\delta,\omega-\epsilon\right)\right. \\ & \left.\Psi\left(\xi-\delta,\omega-\epsilon\right)\psi\left(\delta,\epsilon\right)d\delta d\epsilon\right)d\xi d\omega, \end{aligned}$$

thus,

$$\begin{aligned} & L_\xi \mathcal{R}_\omega\left[\left(\emptyset**\psi\right)\left(\xi,\omega\right)\right] = \int_0^\infty\int_0^\infty \psi\left(\delta,\epsilon\right)d\delta d\epsilon \\ & \left(\frac{s}{u}\int_0^\infty\int_0^\infty e^{-v\xi-\frac{s\omega}{u}}\right. \\ & \left.\emptyset\left(\xi-\delta,\omega-\epsilon\right)\Psi\left(\xi-\delta,\omega-\epsilon\right)d\xi d\omega\right) \\ & \quad = \int_0^\infty\int_0^\infty \psi\left(\delta,\epsilon\right)d\delta d\epsilon\left(e^{-v\delta-\frac{s\epsilon}{u}}\Phi\left(v,s,u\right)\right) \\ & \quad = \Phi\left(v,s,u\right)\int_0^\infty\int_0^\infty \psi\left(\delta,\epsilon\right)\ e^{-v\delta-\frac{s\epsilon}{u}}d\delta\ d\epsilon \\ & \quad = \frac{u}{s}\Phi\left(v,s,u\right)\Psi\left(v,s,u\right). \end{aligned}$$

■

Now, some results are presented in Table I below.

TABLE I
DLFT OF SOME FUNCTION.

$\emptyset\left(\xi,\omega\right)$	$\phi\left(v,s,u\right)$
$\alpha,\ \alpha\in R$	$\frac{\alpha}{v}$
$\xi^\alpha\omega^\beta$	$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)u^\beta}{s^\beta\ v^{\alpha+1}}$
$e^{\alpha\xi+\beta\omega}$	$\frac{s}{(v-\alpha)(s-u\beta)}$
$\sin\left(\alpha\xi+\beta\omega\right)$	$\frac{s}{2i(v-\alpha i)(s-iu\beta)}-\frac{s}{2i(v+\alpha i)(s+i\beta u)}$
$\cos\left(\alpha\xi+\beta\omega\right)$	$\frac{s}{2(v-\alpha i)(s-iu\beta)}+\frac{s}{2(v+\alpha i)(s+i\beta u)}$
$\sinh\left(\alpha\xi+\beta\omega\right)$	$\frac{s}{2(v-\alpha)(s-u\beta)}-\frac{s}{2(v+\alpha)(s+\beta u)}$
$\cosh\left(\alpha\xi+\beta\omega\right)$	$\frac{s}{2(v-\alpha)(s-u\beta)}+\frac{s}{2(v+\alpha)(s+\beta u)}$
$J_0\left(\alpha\sqrt{\xi\omega}\right)$	$\frac{4s}{4vs-\alpha^2u}$
$\left(\emptyset\ \psi\right)\left(\xi-\delta,\omega-\epsilon\right)$	$e^{-s\delta-\frac{s\epsilon}{\omega}}\Phi\left(v,s,u\right)$
$\left(\emptyset**\psi\right)\left(\xi,\omega\right)$	$\frac{u}{s}\Phi\left(v,s,u\right)\Psi\left(v,s,u\right)$

VI. APPLICATIONS ON DLFT

This section contains two parts, in the first one, we present a formula that solves a family of PDEs using DLFT. In the second part, we employ the DLFT to resolve some examples.

A. DLFT for Solving PDEs

Let us take the following PDE,

$$\begin{aligned} A\frac{\partial^2\emptyset\left(\xi,\omega\right)}{\partial\xi^2}+B\frac{\partial^2\emptyset\left(\xi,\omega\right)}{\partial\omega^2}+C\frac{\partial\emptyset\left(\xi,\omega\right)}{\partial\xi} \\ +D\frac{\partial\emptyset\left(\xi,\omega\right)}{\partial\omega}+E\emptyset\left(\xi,\omega\right)=u\left(\xi,\omega\right), \end{aligned} \tag{12}$$

with the initial conditions (ICs)

$$\emptyset\left(\xi,0\right)=f_1\left(\xi\right),\frac{\partial\emptyset\left(\xi,0\right)}{\partial\omega}=f_2\left(\xi\right), \tag{13}$$

and the boundary conditions (BCs)

$$\emptyset\left(0,\omega\right)=h_1\left(\omega\right),\frac{\partial\emptyset\left(0,\omega\right)}{\partial\xi}=h_2\left(\omega\right), \tag{14}$$

where the source term is $u\left(\xi,\omega\right)$, the constants are A,B,C,D, and E, and the unknown function is $\emptyset\left(\xi,\omega\right)$.

Using DLFT on Equation (12) and simplifying the resultant one is the fundamental principle behind this technique. The single LIT is applied on the conditions in Equation (13)

$$L_\xi\left[f_1\left(\xi\right)\right]=F_1\left(v\right), \qquad L_\xi\left[f_2\left(\xi\right)\right]=F_2\left(v\right), \tag{15}$$

and the single FIT is applied on the conditions in Equation (14)

$$\mathcal{R}_\omega\left[h_1\left(\omega\right)\right]=H_1\left(s,u\right), \qquad \mathcal{R}_\omega\left[h_2\left(\omega\right)\right]=H_2\left(s,u\right). \tag{16}$$

Applying DLFT on Equation (12), we have

$$\begin{aligned} & AL_\xi\mathcal{R}_\omega\left[\frac{\partial^2\emptyset\left(\xi,\omega\right)}{\partial\xi^2}\right]+B\ L_\xi\mathcal{R}_\omega\left[\frac{\partial^2\emptyset\left(\xi,\omega\right)}{\partial\omega^2}\right] \\ & \quad +C\ L_\xi\mathcal{R}_\omega\left[\frac{\partial\emptyset\left(\xi,\omega\right)}{\partial\xi}\right]+D\ L_\xi\mathcal{R}_\omega\left[\frac{\partial\emptyset\left(\xi,\omega\right)}{\partial\omega}\right] \\ & \quad +E\ L_\xi\mathcal{R}_\omega\left[\emptyset\left(\xi,\omega\right)\right]=L_\xi\mathcal{R}_\omega\left[u\left(\xi,\omega\right)\right], \end{aligned} \tag{17}$$

the ICs (13) and the BCs (15) yield that

$$\begin{aligned} & A\left(v^2\Phi(v,s,u) - v\mathcal{R}_\omega[\emptyset(0,\omega)] - \mathcal{R}_\omega\left[\frac{\partial\emptyset(0,\omega)}{\partial\xi}\right]\right) \\ & + B\left(\frac{s^2}{u^2}\Phi(v,s,u) - \frac{s^2}{u^2}L_\xi[\emptyset(\xi,0)] \right.\\ & \left. - \frac{s}{u}L_\xi\left[\frac{\partial\emptyset(\xi,0)}{\partial\omega}\right]\right) \\ & + C\ (v\Phi(v,s,u) - \mathcal{R}_\omega[\emptyset(0,\omega)]) \\ & + D\left(\frac{s}{u}\Phi(v,s,u) - \frac{s}{u}L_\xi[\emptyset(\xi,0)]\right) \\ & + E\ \Phi(v,s,u) = U(v,s,u). \end{aligned} \tag{18}$$

Substituting Equations (14) and (16) in (18) we get

$$\begin{aligned} & Av^2\Phi(v,s,u) - AvH_1(u) - AH_2(u) + B\frac{s^2}{u^2}\Phi(v,s,u) \\ & - B\frac{s^2}{u^2}F_1(v) - B\frac{s}{u}F_2(v) + Cv\Phi(v,s,u) \\ & - CH_1(u) + D\frac{s}{u}\Phi(v,s,u) - D\frac{s}{u}F_1(v) \\ & + E\Phi(v,s,u) = U(v,s,u). \end{aligned} \tag{19}$$

By simplifying Equation (19) we get

$$\begin{aligned} & \Phi(v,s,u)\left[A\ v^2 + B\ \frac{s^2}{u^2} + C\ v + D\frac{s}{u} + E\right] \\ & = A\ v\ H_1(s,u) + A\ H_2(s,u) + B\frac{s^2}{u^2}F_1(v) \\ & + B\frac{s}{u}F_2(v) + C\ H_1(s,u) + D\frac{s}{u}F_1(v) + U(v,s,u), \end{aligned} \tag{20}$$

after a few calculations, Equation (20) become

$$\begin{aligned} G(v,s,u) = & \frac{H_{\mathbf{1}}(s,u)(Av+c) + H_{\mathbf{2}}(s,u)(A)}{Av^{\mathbf{2}} + B\frac{S^{\mathbf{2}}}{u^{\mathbf{2}}} + Cv + D\frac{s}{u} + E} \\ & + \frac{F_{\mathbf{1}}(v)\left(B\frac{s^{\mathbf{2}}}{u^{\mathbf{2}}} + \frac{DS}{u}\right) + F_{\mathbf{2}}(v)\left(\frac{BS}{u}\right)}{Av^{\mathbf{2}} + B\frac{S^{\mathbf{2}}}{u^{\mathbf{2}}} + Cv + D\frac{s}{u} + E} \\ & + \frac{U(v,s,u)}{Av^{\mathbf{2}} + B\frac{S^{\mathbf{2}}}{u^{\mathbf{2}}} + Cv + D\frac{s}{u} + E}. \end{aligned} \tag{21}$$

In order to find the original equation's solution for Equation(20), apply the inverse DLFT to both sides of Equation (21), yielding:

$$\begin{aligned} \emptyset(\xi,\omega) = & L_\xi^{-1}\mathcal{R}_\omega^{-1}\left[\frac{H_{\mathbf{1}}\ (s,u)(Av+c) + H_{\mathbf{2}}(s,u)(A)}{Av^{\mathbf{2}} + B\frac{S^{\mathbf{2}}}{u^{\mathbf{2}}} + Cv + D\frac{s}{u} + E}\right] \\ & + \left[\frac{F_{\mathbf{1}}(v)\left(B\frac{s^{\mathbf{2}}}{u^{\mathbf{2}}} + \frac{DS}{u}\right) + F_{\mathbf{2}}(v)\left(\frac{BS}{u}\right) + U(v,s,u)}{Av^{\mathbf{2}} + B\frac{S^{\mathbf{2}}}{u^{\mathbf{2}}} + Cv + D\frac{s}{u} + E}\right]. \end{aligned} \tag{22}$$

B. *Illustrative problems*

DLFT is used to illustrate and resolve a few problems in this section. By solving these instances and obtaining the precise answers, the effectiveness and simplicity of the suggested approach are shown.

Problem 6.1. Take the wave equation

$$\frac{\partial^2\emptyset(\xi,\omega)}{\partial\omega^2} = \frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi^2}, \quad \xi \geq 0, \qquad \omega \geq 0, \tag{23}$$

with the ICs

$$\emptyset(\xi,0) = \sin\xi, \quad \frac{\partial\emptyset(\xi,0)}{\partial\omega} = 2, \tag{24}$$

and the BCs

$$\emptyset(0,\omega) = 2\omega, \quad \frac{\partial\emptyset(0,\omega)}{\partial\xi} = \cos\omega. \tag{25}$$

Solution 6.1. Firstly, we consider

$$f_1(\xi) = \sin\xi, \quad f_2(\xi) = 2, \tag{26}$$

$$h_1(\omega) = 2\omega, \quad h_2(\omega) = \cos\omega. \tag{27}$$

Applying the single LIT on $f_1(\xi)$ and $f_2(\xi)$ in Equation (26), to get

$$F_1(v) = \frac{1}{v^2+1}, \quad F_2(v) = \frac{2}{v}.$$

Applying the single FIT on $h_1(\omega)$ and $h_2(\omega)$ in Equation (27), to get

$$H_1(s,u) = \frac{2u}{s}, \quad H_2(s,u) = \frac{s^2}{u^2+s^2}.$$

Rewrite Equation (23) in the form

$$\frac{\partial^2\emptyset(\xi,\omega)}{\partial\xi^2} - \frac{\partial^2\emptyset(\xi,\omega)}{\partial\omega^2} = 0.$$

By comparing the general formula in Equation (21), we get $A = 1, B = -1, C = D = E = 0$ and $U(v,s,u) = 0$. Substituting the values of the functions F_1, F_2, H_1 and H_2 and above values in the general formula in Equation (21), we obtain

$$\Phi(v,s,u) = \frac{\frac{2uv}{s} + \frac{s^2}{s^2+u^2} - \frac{s^2}{u^2(v^2+1)} - \frac{2s}{uv}}{v^2 - \frac{s^2}{u^2}}.$$

Now, simplifying and taking the DLFT inverse to both sides of the above equation, we get

$$\begin{aligned} \emptyset(\xi,\omega) & = L_\xi^{-1}\mathcal{R}_\omega^{-1}\left[\frac{2u}{sv} + \frac{s^2}{(v^2+1)(u^2+s^2)}\right] \\ & = 2\omega\sin\xi\ \ \cos\omega. \end{aligned}$$

Below, we sketch the 3D plot of the solution of wave Equation (23) in Fig 1, with the ICs (24) and the BCs (25).

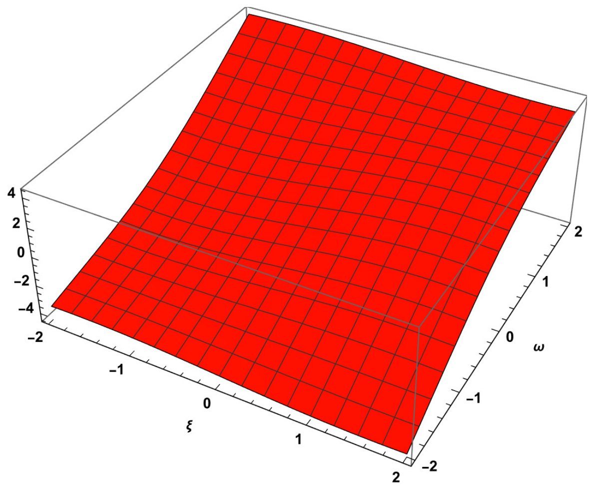

Fig. 1. The solution of problem 2

Problem 6.2. Take the heat equation

$$\frac{\partial \emptyset(\xi,\omega)}{\partial \omega}=\frac{\partial^2 \emptyset(\xi,\omega)}{\partial \xi^2}-3\emptyset(\xi,\omega)+3, \xi\geq 0, \omega\geq 0, \quad (28)$$

with the IC

$$\emptyset(\xi,0)=1+\sin\xi, \quad (29)$$

and the BCs

$$\emptyset(0,\omega)=1, \quad \frac{\partial \emptyset(0,\omega)}{\partial \xi}=e^{-4\omega}. \quad (30)$$

Solution 6.2. Now, we have

$$f_1(\xi)=1+\sin\xi, \quad (31)$$

$$h_1(\omega)=2\omega, \quad h_2(\omega)=\cos\omega. \quad (32)$$

Applying the single LIT on $f_1(\xi)$ in Equation (31), we get

$$F_1(v)=\frac{1}{v}+\frac{1}{v^2+1}.$$

Applying the single FIT on $h_1(\omega)$ and $h_2(\omega)$ in Equation (32), we get

$$H_1(s,u)=1, \quad H_2(s,u)=\frac{s}{s+4u}.$$

Substituting the values of the functions F_1, H_1 and H_2 in the general formula in Equation (22), we get

$$\begin{aligned}\emptyset(\xi,\omega)&=L_\xi^{-1}\mathcal{R}_\omega^{-1}\left[\frac{1}{v}+\frac{s}{(v^2+1)(s+4u\ )}\right]\\&=1+e^{-4\omega}\sin\xi.\end{aligned}$$

Below, we sketch the 3D plot of the solution of heat Equation (28) in Fig 2, with the IC (29) and the BCs (30).

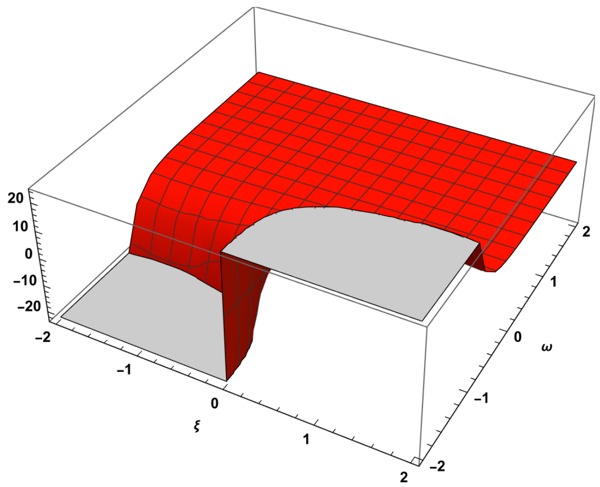

Fig. 2. The solution of problem 2

Problem 6.3. Take the telegraph equation

$$\frac{\partial^2 \emptyset(\xi,\omega)}{\partial \xi^2}=\frac{\partial^2 \emptyset(\xi,\omega)}{\partial \omega^2}+\frac{\partial \emptyset(\xi,\omega)}{\partial \omega}-\emptyset(\xi,\omega), \quad (33)$$
$$\xi\geq 0,\ \omega\geq 0,$$

with the ICs

$$\emptyset(\xi,0)=e^{\xi}, \quad \frac{\partial \emptyset(\xi,0)}{\partial \omega}=-2e^{\xi}, \quad (34)$$

and the BCs

$$\emptyset(0,\omega)=e^{-2\omega}, \quad \frac{\partial \emptyset(0,\omega)}{\partial \xi}=e^{-2\omega}, \quad (35)$$

Solution 6.3. Here, we have

$$f_1(\xi)=e^{\xi}, \quad f_2(\xi)=-2e^{\xi}, \quad (36)$$

$$h_1(\omega)=e^{-2\omega}, \quad h_2(\omega)=e^{-2\omega}. \quad (37)$$

Applying the single LIT on $f_1(\xi)$ and $f_2(\xi)$ in Equation (36), we get $F_1(v)=\frac{1}{v-1}$, $F_2(s)=\frac{-2}{v-1}$.
Applying the single FIT on $h_1(y)$ and $h_2(y)$ in Equation (37), we get

$$H_1(s,u)=\frac{s}{s+2u}, \quad H_2(s,u)=\frac{s}{s+2u}.$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 in the general formula in Equation (22), to get

$$\emptyset(\xi,\omega)=L_\xi^{-1}\mathcal{R}_\omega^{-1}\left[\frac{s}{(v-1)(s+2u\ )}\right]=e^{\xi-2\omega}.$$

Below, we sketch the 3D plot of the solution of telegraph Equation (33) in Fig 3, with the ICs (34) and the BCs (35).

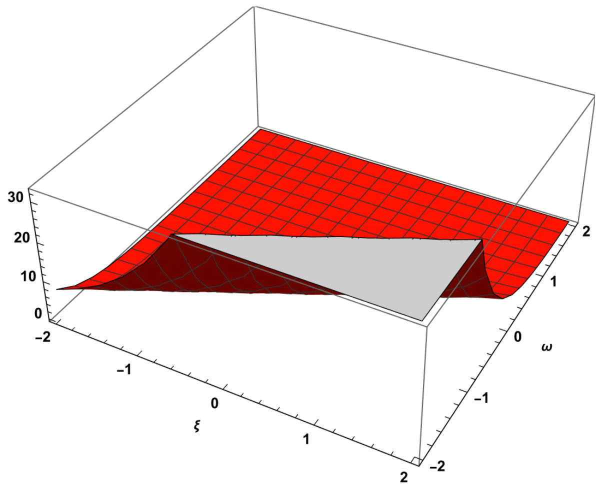

Fig. 3. The solution of problem 3

Problem 6.4. Take the Klein-Gordon equation

$$\frac{\partial^2 \emptyset(\xi,\omega)}{\partial \xi^2}-\emptyset(\xi,\omega)=\frac{\partial^2 \emptyset(\xi,\omega)}{\partial \omega^2}-\cos\xi\ \cos\omega, \quad (38)$$

with the ICs

$$\emptyset(\xi,0)=cos\xi, \quad \frac{\partial \emptyset(\xi,0)}{\partial \omega}=0, \quad (39)$$

and the BCs

$$\emptyset(0,\omega)=cos\omega, \quad \frac{\partial \emptyset(0,\omega)}{\partial \xi}=0. \quad (40)$$

Solution 6.4. Applying the single LIT on $f_1(\xi)$ and $f_2(\xi)$ in Equation (39), we get

$$F_1(v)=\frac{v}{v^2+1}, \qquad F_2(v)=0.$$

Applying the single FIT on $h_1(\omega)$ and $h_2(\omega)$ in Equation (40), we get

$$H_1(s,u)=\frac{s^2}{s^2+u^2}, \qquad H_2(s,u)=0.$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 in the general formula in Equation (22), to get

$$\emptyset(\xi,\omega)=L_\xi^{-1}\mathcal{R}_\omega^{-1}\left[\frac{vs^2}{(v^2+1)(s^2+1)}\right]=\cos\xi\cos\omega.$$

Below, we sketch the 3D plot of the solution of Klein-Gordon Equation (38) in Fig. 4, with the ICs (39) and the BCs (40).

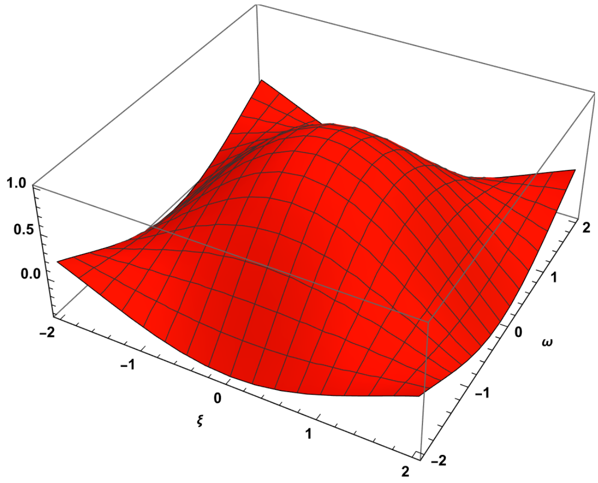

Fig. 4. The solution of problem 4

Problem 6.5. Consider the Homogenous-Laplace equation

$$\frac{\partial^2 \emptyset(\xi,\omega)}{\partial \xi^2} + \frac{\partial^2 \emptyset(\xi,\omega)}{\partial \omega^2} = 0, \tag{41}$$

with the ICs

$$\emptyset(\xi,0) = 0, \qquad \frac{\partial \emptyset(\xi,0)}{\partial \omega} = \cos(\xi), \tag{42}$$

and the BCs

$$\emptyset(0,\omega) = \sinh(\omega), \qquad \frac{\partial \emptyset(0,\omega)}{\partial \xi} = 0. \tag{43}$$

Solution 6.5. Applying the single LIT on $f_1(\xi)$ and $f_2(\xi)$ in Equation (42), we get

$$F_1(v) = 0, \qquad F_2(v) = \frac{v}{v^2+1}.$$

Applying the single FIT on $h_1(\omega)$ and $h_2(\omega)$ in Equation (43), we get

$$H_1(s,u) = \frac{su}{s^2-u^2}, \qquad H_2(s,u) = 0.$$

Substituting the values of the functions F_1, F_2, H_1 and H_2 in the general formula in Equation (22), to get

$$\emptyset(\xi,\omega) = L_\xi^{-1}\mathcal{R}_\omega^{-1}\left[\frac{vsu}{(v^2+1)(s^2-u^2)}\right] = \cos\xi\sinh\omega.$$

Below, we sketch the 3D plot of the Homogenous-Laplace Equation (41) in Fig. 5, with the ICs (42) and the BCs (43).

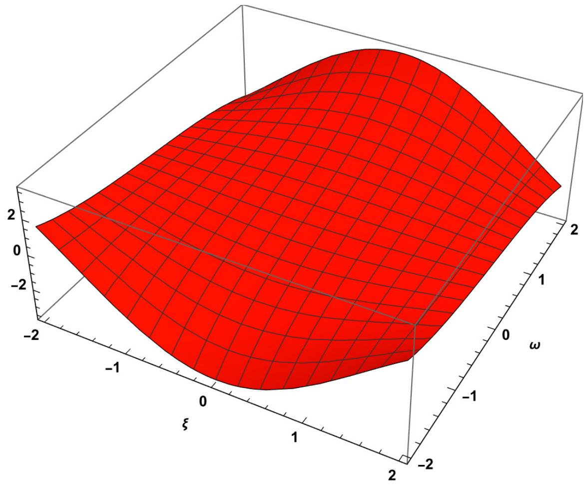

Fig. 5. The solution of problem 5

VII. A FEW OBSERVATIONS REGARDING THE NEW APPROACH

In this section of the essay, we provide some remarks about the new transform DLFT, some advantages and disadvantages.

- The primary aim of this study is to propose an innovative method for solving PDEs by converting them into algebraic equations and subsequently use the newly devised double transform technique to determine the solution in a different domain. Subsequently, the inverse transform is employed to derive the solutions of the target equations in the original space. Furthermore, solutions are provided for a specific category of IEs.
- It is difficult to solve PDEs when using the single Laplace and Formable transformations. Hence, the new DLFT aids researchers in resolving such problems.
- In contrast to other approaches, DLFT provides a rapid convergence of the precise answer without requiring any complicated calculation.
- The source terms for solving nonhomogeneous equations must meet the DLFIT existence requirements.
- This transform's inability to be used directly to resolve nonlinear PDEs is a drawback. It must be paired with other popular iterative techniques, like the homotropy approach and the differential transform method.

VIII. CONCLUSIONS

This study introduced the DLFT, which is a novel transform technique that combines the LIT and the FIT. The definition and the basic properties were presented and proved in this research. To get solutions for some families of PDEs, the basic features of the suggested double transform were presented and discussed. Further discussions and applications of the new DLFT results for solving fractional PDEs and nonlinear fractional PDEs [39], [40], [41], [42], [43] are planned to be obtained in future work.

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