The Ignition Problem for Chaplygin Gas System

Yujin Liu and Wenhua Sun

***Abstract*—The ignition problem for the Chaplygin system is considered. Under the entropy conditions, we obtain constructively the unique solution and discover that the combustion wave solutions may be extinguished for some cases. Especially, we obtain that the combustion wave occurs although there is no combustion before. The transition between deflagration and detonation is also shown.**

***Index Terms*—Ignition problem, Riemann problem, Entropy conditions, Detonation wave, Deflagration wave.**

I. INTRODUCTION

IN this paper we consider

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho E)_t + (\rho u E + p u)_x = 0, \\ q(x,t) = \begin{cases} 0 & \text{if } \sup\limits_{0\le y\le t} T(x,y) > T_i; \\ q(x,0) & \text{otherwise,} \end{cases} \end{cases} \tag{1}$$

where $\rho, u, p < 0$, T and T_i are the density, velocity, pressure, temperature and ignition temperature, respectively. The total energy $E = \frac{u^2}{2} + e + q$. The state equation is $p = -\frac{1}{\rho}$ and the internal energy $e = -\frac{p}{2\rho}$. The combustion process is exothermic [1]. For the results about Chaplygin gas, see [2], [3], [4], [5], [6], [7].

In [1], they began to study the CJ model. The authors [8] got constructively the unique Riemann solution. In [9], Liu and Sheng constructed the generalized Riemann solutions for

$$\begin{cases} u_t + p_x = 0, \\ \tau_t - u_x = 0, \\ E_t + (up)_x = 0, \\ q(x,t) = \begin{cases} 0, & \text{if } \sup\limits_{0\le y\le t} T(x,y) > T_i; \\ q(x,0), & \text{otherwise,} \end{cases} \end{cases} \tag{2}$$

where $\tau = \frac{1}{\rho}$, $p > 0$ and $E = \frac{u^2}{2} + \frac{p\tau}{\gamma - 1} + q$.

For the simplified scalar cases, the authors [10] studied

$$\begin{cases} (u + qz)_t + f(u)_x = 0, \\ z_t = -k\varphi(u)z. \end{cases} \tag{3}$$

Liu and Zhang [11] studied

$$\begin{cases} (u + qz)_t + f(u)_x = 0, \\ z(x,t) = \begin{cases} 0 & \text{if } \sup\limits_{0\le y\le t} u(x,y) > T_i; \\ z(x,0) & \text{otherwise.} \end{cases} \end{cases} \tag{4}$$

Many works [12], [13], [14], [15] have been done for the hyperbolic conservation law system.

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Based on the above results, they [16] studied the SZND model

$$\begin{cases} u_t + p_x = 0, \\ \tau_t - u_x = 0, \\ E_t + (up)_x = 0, \\ q(x,t) = -\frac{k}{t}\varphi(T)q, \end{cases} \tag{5}$$

and proved that the selfsimilar solutions of (2) are the limits of (5) as $k \to \infty$.

Hsu and Lin [17] studied (2) and (5) further and obtained when the selfsimilar solutions of (2) are the limits of (5) when $k \to \infty$.

In [18], the authors studied the ignition problem for the scalar nonconvex CJ model. Under the pointwise and global entropy conditions [19], they got the transitions between deflagration and detonation.

In [20], we obtain the unique Riemann solution of (1) with

$$(\tau, u, p, q) = \begin{cases} (\tau^-, u^-, p^-, q^-), & \text{when } x < 0, \\ (\tau^+, u^+, p^+, q^+), & \text{when } x > 0. \end{cases} \tag{6}$$

Now we investigate the ignition problem for (1) with

$$(u,p,\tau,q)(x,0) = \begin{cases} (u_-, p_-, \tau_-, q_-), & -\infty < x < -\varepsilon, \\ (\hat{u}, \hat{p}, \hat{\tau}, 0), & -\varepsilon < x < \varepsilon, \\ (u_+, p_+, \tau_+, q_+), & \varepsilon < x < +\infty, \end{cases} \tag{7}$$

where $(u_-, p_-, \tau_-) = (u_+, p_+, \tau_+) := (u_0, p_0, \tau_0)$, $q_- = q_+ := q_0 > 0$ and $\varepsilon > 0$ is small enough. We construct the unique solution of (1) and (7) in the (x,t) plane.

This paper is organized as follows. II give some preliminaries. In Section III, we construct the all possible solutions of the ignition problem of (1) with (7). IV is the main conclusion.

II. PRELIMINARIES

In this section we give some preliminaries [20], [21], [22], [23]. Since

$$\lambda_{1,3} = u \pm \sqrt{\frac{-p}{\rho}}, \quad \lambda_2 = u, \tag{8}$$

$$\vec{\nu}_{1,3} = (1, \pm\frac{1}{\rho}\sqrt{\frac{-p}{\rho}}, \frac{-p}{\rho})^\top, \quad \vec{\nu}_2 = (1,0,0)^\top,$$

and $\nabla\lambda_i \cdot \vec{\nu}_i \equiv 0, i = 1,2,3$, (1) is linearly degenerate.

$\overrightarrow{R}(l)$(or $\overleftarrow{R}(l)$) (Fig. 1.(i) and Fig. 1.(ii)) are

$$\begin{cases} p\rho = p_l\rho_l, \\ u = u_l \pm \frac{p - p_l}{\sqrt{-p_l\rho_l}}, \quad (p > p_l, \text{or } p < p_l), \end{cases} \tag{9}$$

and $\overrightarrow{S}(l)$(or $\overleftarrow{S}(l)$) are

$$\begin{cases} p\rho = p_l\rho_l, \\ \frac{u - u_l}{p - p_l} = \pm\sqrt{-\frac{1}{p_l\rho_l}}, \quad (p_l > p, \text{or } p_l < p). \end{cases} \tag{10}$$

J is

$$\begin{cases} [u] = [p] = 0, \\ \rho_l \neq \rho_r. \end{cases} \tag{11}$$

Suppose

$$\rho = \rho_0 + \omega(t)\delta(x - x(t)), \quad \rho_0 = \begin{cases} \rho_l, & x < x(t), \\ \rho_r, & x > x(t), \end{cases} \tag{12}$$

$$u(x,t) = \begin{cases} u_l, & x < x(t), \\ u_\delta, & x = x(t), \\ u_r, & x > x(t), \end{cases} \tag{13}$$

$$p(x,t) = \begin{cases} p_l, & x < x(t), \\ 0, & x = x(t), \\ p_r, & x > x(t). \end{cases} \tag{14}$$

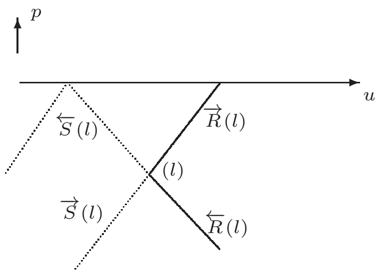

Fig. 1.(i) The backward wave curves in (u,p).

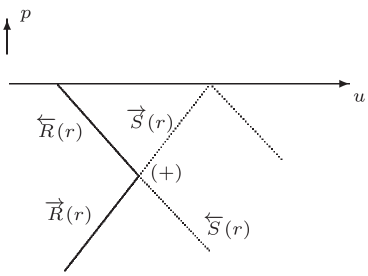

Fig. 1.(ii) The forward wave curves in (u,p).

When $\rho_r \neq \rho_l$,

$$\omega(t) = \sqrt{\rho_l\rho_r(u_r - u_l)^2 - (\rho_r - \rho_l)(p_r - p_l)}\ t,$$

$$u_\delta = \frac{\rho_r u_r - \rho_l u_l + \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}}{\rho_r - \rho_l}.$$

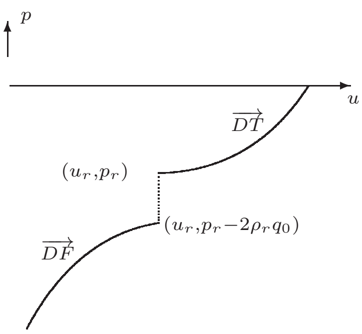

Fig. 2. The combustion wave curve in (u,p).

When $\rho_r = \rho_l$,

$$\omega(t) = (\rho_l u_l - \rho_r u_r)t,$$

$$u_\delta = \frac{1}{2}(u_r + u_l).$$

S_δ satisfies

$$u_r + \sqrt{-\frac{p_r}{\rho_r}} < \frac{\mathrm{d}x(t)}{\mathrm{d}t} < u_l - \sqrt{-\frac{p_l}{\rho_l}}. \tag{15}$$

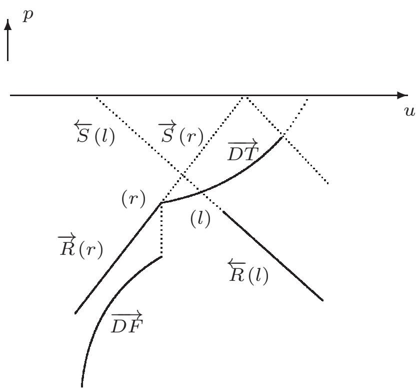

Fig. 3. The wave curves in Case 2.1.

From the R-H condition

$$\begin{cases} \zeta[u] = [p], \\ \zeta[\tau] = -[u], \\ \zeta[E] = [up], \end{cases}$$

we get

$$-\tau_r p + p_r \tau = 2q_0 > 0.$$

In (u,p) (Fig. 2.) we find

$$\overrightarrow{D}(r): \quad \frac{u - u_r}{p - p_r} = \sqrt{-\frac{2q_0 + \tau_r(p - p_r)}{p_r(p - p_r)}}, \tag{16}$$

$p_r < p < 0$ or $p < p_r - \frac{2q_0}{\tau_r}$.

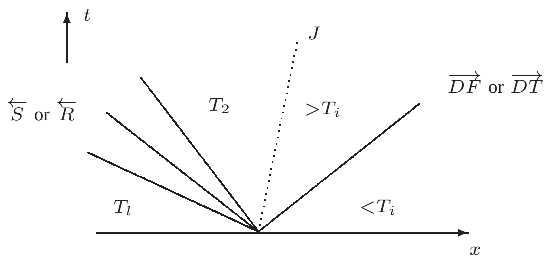

Fig. 4. combustion wave solution.

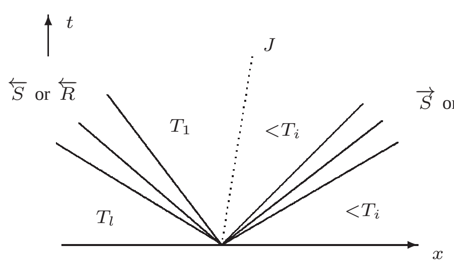

Fig. 5. non-combustion wave solution.

If $q_r = 0$, $q_l = 0$, it has been resolved in [23].

Case 2.1. When $q_l = 0,\ q_r = q_0 > 0$.

$\overleftarrow{W}(l) = \overleftarrow{W}_S(l) \cup S_\delta(l)$, $\overrightarrow{W}(r) = \overrightarrow{W}_S(r) \cup S_\delta(r) \cup \overrightarrow{DF}(r) \cup \overrightarrow{DT}(r)$ (Fig. 3.).

Subcase 2.1.1 $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$.

The *global entropy conditions (GEC) [20]:*

(i) η is as small as possible, here η is the oscillation frequency from $\{\eta \in R^1 : T(\eta) \le T_i\}$ to $\{\eta \in R^1 : T(\eta) > T_i\}$;

(ii) the combustion wave is as many as possible.

The temperature is respectively T_1, T_2 at the point $\star_1$, $\star_2$. (Fig. 3.).

(1) When $T_l > T_i$, $T_2 > T_i$, we get $\overrightarrow{S}\,or\,\overleftarrow{R}+J+\overrightarrow{DF}\,or\,\overrightarrow{DT}$ (Fig. 4.).

(2) When $T_l > T_i$, $T_2 \le T_i (\Rightarrow T_1 \le T_i)$, we get $\overrightarrow{S}\,or\,\overleftarrow{R} + J + \overrightarrow{S}\,or\,\overrightarrow{R}$ (Fig. 5.).

(3) When $T_l \le T_i$, $T_1 \le T_i$, we get $\overrightarrow{S}\,or\,\overleftarrow{R} + J + \overrightarrow{S}\,or\,\overrightarrow{R}$ (Fig. 5.).

(4) When $T_l \le T_i$, $T_1 > T_i (\Rightarrow T_2 > T_i)$, we get $\overrightarrow{S}\,or\,\overleftarrow{R} + J + \overrightarrow{DF}\,or\,\overrightarrow{DT}$ (Fig. 4.).

Subcase 2.1.2 $u_l - u_r \ge \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$. It is shown that we can get the S_δ solution [20], [21], [23].

Case 2.2. When $q_l > 0$, $q_r > 0$.

We know that $\overleftarrow{W}(l) = \overleftarrow{W}_S(l) \cup S_\delta(l) \cup \overleftarrow{DT}(l) \cup \overleftarrow{DF}(l)$, and $\overrightarrow{W}(r) = \overrightarrow{W}_S(r) \cup S_\delta(r) \cup \overrightarrow{DT}(r) \cup \overrightarrow{DF}(r)$ (Fig. 6.).

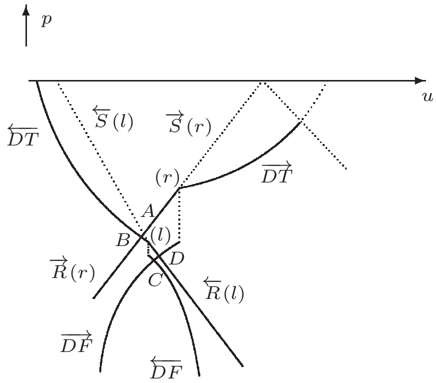

Fig. 6. The wave curves in Case 2.2.

Subcase 2.2.1 When $u_l - u_r < \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$.

Since $\eta = 0$ for the point A, $\eta = 2$ for the points B, C and D, we get $\overrightarrow{S}\,or\,\overleftarrow{R} + J + \overrightarrow{S}\,or\,\overrightarrow{R}$

Subcase 2.2.2 When $u_l - u_r \ge \sqrt{-\frac{p_l}{\rho_l}} + \sqrt{-\frac{p_r}{\rho_r}}$. It is the S_δ solution similarly.

Theorem 2.1 We get uniquely the Riemann solution of (1) and (6) under GEC.

III. Ignition problem for the system (1) with (7)

In what follows, we study the ignition problem (1) and (7) according to the different cases.

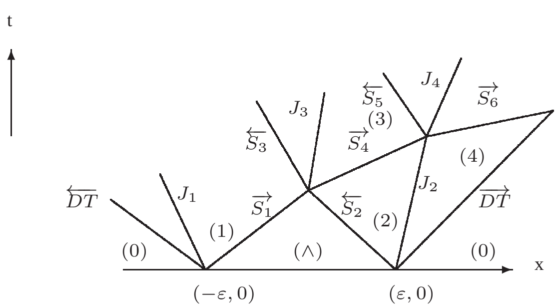

Fig. 7. The solution for Case 3.1.1.

Case 3.1. Solution at $(-\varepsilon, 0)$ is $\overleftarrow{DT} + J + \overrightarrow{S}$, and at $(\varepsilon, 0)$ is $\overleftarrow{S} + J + \overrightarrow{DT}$.

Case 3.1.1 When $p_1 + p_2 - \hat{p} < 0$, we know that $\overrightarrow{S_1}$ interacts with $\overleftarrow{S_2}$ at the point (x_1, t_1) and $\overrightarrow{S_1}\overleftarrow{S_2} \to \overleftarrow{S_3} J_3 \overrightarrow{S_4}$ (Fig. 7.).

Since $\overleftarrow{S_3} : \frac{dx}{dt} = u_1 - \sqrt{-\frac{p_1}{\rho_1}}$ and $\sigma_{J_1} : \frac{dx}{dt} = u_1$, it follows that $\overleftarrow{S_3}$ can not overtake J_1 which tells the combustion wave $\overleftarrow{DT}$ can persist after the perturbation.

Since $\overrightarrow{S_4}$ can overtake J_2 at the point (x_2, t_2), if $p_* < 0$, i.e., $2\sqrt{\rho_4} p_3 + (\sqrt{\rho_2} - \sqrt{\rho_4}) p_2 < 0$, we know that $\overrightarrow{S_4} J_2 \to \overleftarrow{S_5} J_4 \overrightarrow{S_6}$. Notice that the new shock wave $\overleftarrow{S_5}$ can not overtake J_3 in the finite time. Since the new shock wave $\overrightarrow{S_6}$ can overtake the detonation wave $\overrightarrow{DT}$ at the point (x_3, t_3) (Fig. 8.).

In the (u, p) (Fig. 9.), there are at most two intersection points, and we continue to discuss as follows according to GEC.

(1) As $T_- > T_i$, $T_2 > T_i$, we get $\overleftarrow{S}\,J\,\overrightarrow{DT}$.

(2) As $T_- > T_i$, $T_2 \le T_i (\Rightarrow T_1 \le T_i)$, we get $\overleftarrow{S}\,J\,\overrightarrow{S}$.

(3) As $T_- \le T_i$, $T_1 \le T_i$, we get $\overleftarrow{S}\,J\,\overrightarrow{S}$.

(4) As $T_- \le T_i$, $T_1 > T_i (\Rightarrow T_2 > T_i)$, we get $\overleftarrow{S}\,J\,\overrightarrow{DT}$.

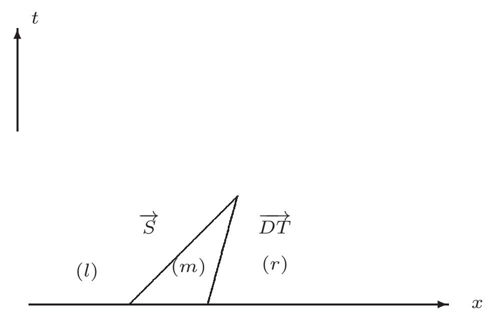

Fig. 8. The interaction of $\overrightarrow{S}$ and $\overrightarrow{DT}$.

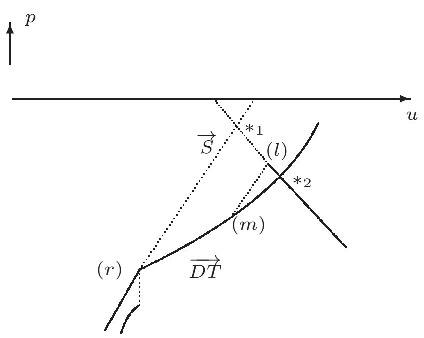

Fig. 9. The wave curves in (u, p).

After the above interaction process, we get that $\overrightarrow{S_6}\overrightarrow{DT} \to \overleftarrow{S}\,J\,\overrightarrow{DT}$ or $\overleftarrow{S}\,J\,\overrightarrow{S}$ (Fig. 7.). Notice that the new shock wave can not overtake J_4 in the finite time. The above discussions imply that $\overrightarrow{DT}$ can persist or be extinguished by such perturbation.

If $p_* \ge 0$, i.e., $2\sqrt{\rho_4} p_3 + (\sqrt{\rho_2} - \sqrt{\rho_4}) p_2 \ge 0$, we know that $\overrightarrow{S_4} J_2 \to \overrightarrow{S}_{\delta_1}$. For this case, we get that $\overrightarrow{S}_{\delta_1} \overrightarrow{DT} \to \overrightarrow{S}_{\delta_2}$ or $\overleftarrow{DT}\,J\,\overrightarrow{DT}$. Notice that the new detonation wave can not overtake J_3 in the finite time. The above analysis shows that $\overrightarrow{DT}$ may be extinguished or persist after the perturbation.

Case 3.1.2 When $p_1 + p_2 - \hat{p} \ge 0$, we obtain that $\overrightarrow{S_1}\overleftarrow{S_2} \to \overrightarrow{S}_{\delta_1}$ (Fig. 10.).

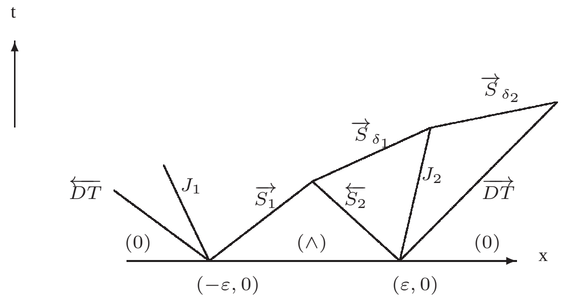

Fig. 10. The solution for Case 3.1.2.

For this case, we know that when $\rho_1 \neq \rho_2$,

$$\omega(t) = \sqrt{\rho_1\rho_2(u_2-u_1)^2 - (\rho_2-\rho_1)(p_2-p_1)}\ (t-t_1)$$

$$u_\delta = \frac{\rho_2 u_2 - \rho_1 u_1 + \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}}{\rho_2 - \rho_1},$$

when $\rho_1 = \rho_2$,

$$\omega(t) = (\rho_1 u_1 - \rho_2 u_2)(t-t_1)$$

$$u_\delta = \frac{1}{2}(u_1+u_2).$$

Consider that for $\overrightarrow{S}_{\delta_1}$, it holds

$$u_r + \sqrt{-\frac{p_r}{\rho_r}} < \frac{\mathrm{d}x(t)}{\mathrm{d}t} < u_l - \sqrt{-\frac{p_l}{\rho_l}}, \qquad (17)$$

and for J_2 it holds $\frac{dx}{dt} = u_l$, $\overrightarrow{S}_{\delta_1}$ will overtake J_2 in the finite time. It follows that there are two possibilities. If $p_* < 0$, we know $\overrightarrow{S}_{\delta_1} J_2 \rightarrow \overleftarrow{S}\, or\, \overleftarrow{R} + \overrightarrow{S}\, or\, \overrightarrow{R}$, and notice that the result may not contain the contact discontinuity. The discussions that follows are similar with Case 3.1.1 which yield that $\overrightarrow{DT}$ may be extinguished or persist after the perturbation.

If $p_* \geq 0$, we have $\overrightarrow{S}_{\delta_1} J_2 \rightarrow \overrightarrow{S}_{\delta_2}$ and $\overrightarrow{S}_{\delta_2}\overrightarrow{DT} \rightarrow \overrightarrow{S}_{\delta_2}$ or $\overleftarrow{DT} + J + \overrightarrow{DT}$, and observe that $\overrightarrow{DT}$ may persist or be extinguished by such small perturbation.

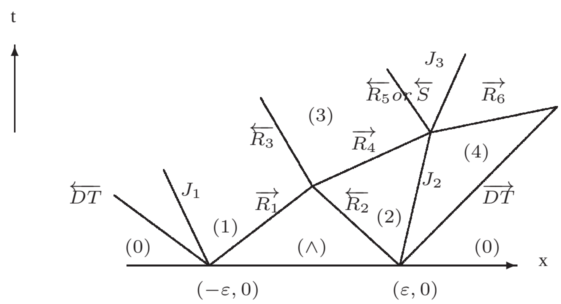

Fig. 11. The solution for Case 3.2.

Theorem 3.1 For this case, we find that after the small perturbation, the backward detonation wave $\overleftarrow{DT}$ must be able to keep after the perturbation, while the forward detonation wave $\overrightarrow{DT}$ may be extinguished.

Case 3.2. Solution at $(-\varepsilon, 0)$ is $\overleftarrow{DT} + J + \overrightarrow{R}$, and at $(\varepsilon, 0)$ is $\overleftarrow{R} + J + \overrightarrow{DT}$.

$\overrightarrow{R_1}$ will intersect with $\overleftarrow{R_2}$ at the point (x_1, t_1) and a new Riemann problem is formed. Due to $p_1 < \hat{p}$ and $p_2 < \hat{p}$, we know that $\overrightarrow{R_1}\overleftarrow{R_2} \rightarrow \overleftarrow{R_3}\overrightarrow{R_4}$ (Fig. 11.).

Since for $\overleftarrow{R_3}$ we have $\frac{dx}{dt} = \hat{u} - \sqrt{-\frac{\hat{p}}{\hat{\rho}}}$, and for J_1 we know $\frac{dx}{dt} = \hat{u}$, the rarefaction wave $\overleftarrow{R_3}$ can not overtake J_1 in the finite time. Due to $\frac{dx}{dt} = u_2 + \sqrt{-\frac{p_2}{\rho_2}}$ for $\overrightarrow{R_4}$, and $\frac{dx}{dt} = u_2$ for J_2, it follows that $\overrightarrow{R_4}$ can overtake J_2 at the finite time (x_2, t_2), and

$$\overrightarrow{R_4}J_2 \rightarrow \overleftarrow{R_5} or \overleftarrow{S} + J_3 + \overrightarrow{R_6}.$$

After the interaction of $\overleftarrow{S}\, or \overleftarrow{R_5}$ and $\overleftarrow{R_3}$, we get a new backward rarefaction wave, but this new backward rarefaction wave can not overtake J_1 in the finite time, thus $\overleftarrow{DT}$ can persist after the perturbation. On the other hand, since $\overrightarrow{R_6}$ can overtake $\overrightarrow{DT}$ at the finite time (x_3, t_3) and after the interaction we get that the result is $\overleftarrow{R} + J + \overrightarrow{S}$, i.e., $\overrightarrow{DT}$ can be extinguished after the perturbation.

Theorem 3.2 In this case, we conclude that after the small perturbation, the backward detonation wave $\overleftarrow{DT}$ can persist, while the forward detonation wave $\overrightarrow{DT}$ may be extinguished.

Case 3.3. Solution at $(-\varepsilon, 0)$ is $\overleftarrow{DF} + J + \overrightarrow{S}$, and at $(\varepsilon, 0)$ is $\overleftarrow{S} + J + \overrightarrow{DF}$.

Case 3.3.1 If $p_1 + p_2 - \hat{p} < 0$, it follows that $\overrightarrow{S_1}\overleftarrow{S_2} \rightarrow \overleftarrow{S_3}J_3\overrightarrow{S_4}$ (Fig. 12.).

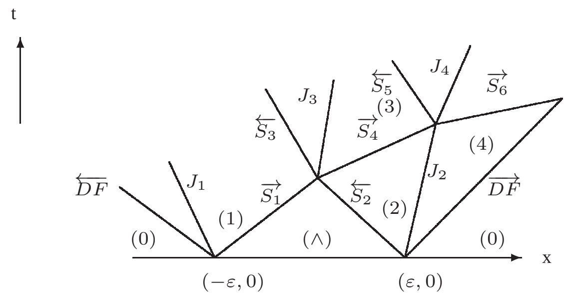

Fig. 12. The solution for Case 3.3.1.

Because $\overrightarrow{S_4}$ can overtake J_2 at the point (x_2, t_2), we know that there are two possibilities. If $2\sqrt{\rho_4}p_3 + (\sqrt{\rho_2} - \sqrt{\rho_4})p_2 < 0$, we obtain $\overrightarrow{S_4}J_2 \rightarrow \overleftarrow{S_5}J_4\overrightarrow{S_6}$.

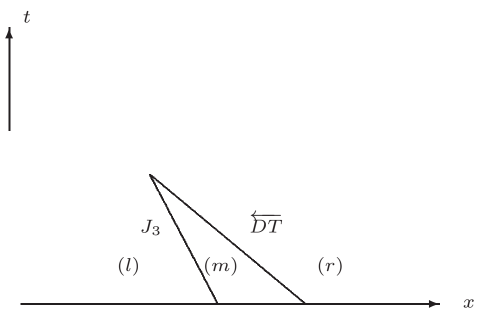

Fig. 13. The interaction of $\overleftarrow{DT}$ and J_3.

Notice that the new shock wave $\overleftarrow{S_5}$ can not overtake J_3 in the finite time. While the new shock wave $\overrightarrow{S_6}$ can overtake $\overrightarrow{DF}$ at (x_3, t_3). After the wave interaction process, we get that $\overrightarrow{S_6}\overrightarrow{DF} \rightarrow \overleftarrow{S}\, or\, \overleftarrow{R} + J + \overrightarrow{S}$. Notice that the new backward shock wave or backward rarefaction wave can not overtake J_4 in the finite time. The above discussions imply that $\overleftarrow{DF}$ can persist and $\overrightarrow{DF}$ may be extinguished by the perturbation.

If $2\sqrt{\rho_4}p_3 + (\sqrt{\rho_2} - \sqrt{\rho_4})p_2 \geq 0$, we know that $\overrightarrow{S_4}J_2 \rightarrow \overrightarrow{S}_{\delta_1}$. It follows that $\overrightarrow{S}_{\delta_1}\overrightarrow{DF} \rightarrow \overrightarrow{S}_{\delta_2}$ or $\overleftarrow{DT}J\overrightarrow{DT}$, and the forward deflagration wave $\overrightarrow{DF}$ may be extinguished or be transformed to the forward detonation wave $\overrightarrow{DT}$. Notice that $\overleftarrow{DT}$ can overtake J_3 in the finite time (Fig. 13.).

In (u,p) (Fig. 14.), there are at most two intersection points. From GEC, it yields that $J_3+\overleftarrow{DT} \to \overleftarrow{S}+J+\overrightarrow{S}$ or $\overleftarrow{S}+J+\overrightarrow{DT}$. The newly generated backward shock wave intersects with $\overleftarrow{S_3}$ and it follows that a new backward shock wave appears. The new backward shock can not overtake J_1, thus $\overleftarrow{DF}$ can persist after the small perturbation.

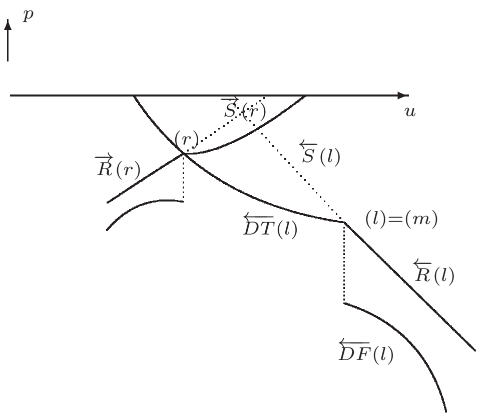

Fig. 14. Waves curves in the (u,p).

Case 3.3.2 When $p_1+p_2-\hat{p}\ge 0$, we obtain $\overrightarrow{S_1}\overleftarrow{S_2} \to \overrightarrow{S}_{\delta_1}$ (Fig. 15.).

When $\rho_1 \ne \rho_2$,

$$\omega(t)=\sqrt{\rho_1\rho_2(u_2-u_1)^2-(\rho_2-\rho_1)(p_2-p_1)}\,(t-t_1)$$

$$u_\delta=\frac{\rho_2u_2-\rho_1u_1+\frac{\mathrm{d}\omega(t)}{\mathrm{d}t}}{\rho_2-\rho_1},$$

when $\rho_1=\rho_2$,

$$\omega(t)=(\rho_1u_1-\rho_2u_2)(t-t_1)$$

$$u_\delta=\frac{1}{2}(u_1+u_2).$$

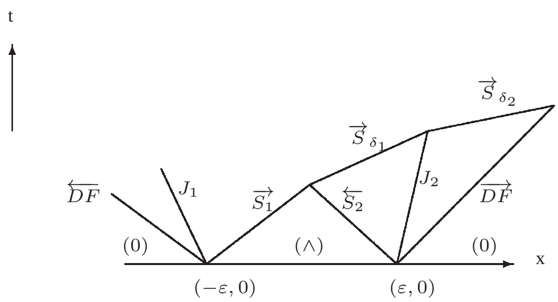

Fig. 15. The solution for Case 3.3.2.

Since for $\overrightarrow{S}_{\delta_1}$, it holds that $u_r+\sqrt{-\frac{p_r}{\rho_r}} < \frac{\mathrm{d}x(t)}{\mathrm{d}t} < u_l-\sqrt{-\frac{p_l}{\rho_l}}$, and for J_2 it holds that $\frac{dx}{dt}=u_l$, $\overrightarrow{S}_{\delta_1}$ will overtake J_2 at the finite time (x_3,t_3), It follows that there are two possibilities. If $p_*<0$, we know $\overrightarrow{S}_{\delta_1}J_2 \to \overleftarrow{S}\,or\,\overleftarrow{R}+\overrightarrow{S}\,or\,\overrightarrow{R}$, and the contact discontinuity may not appear. The next analysis are similar with Case 3.3.1 and it follows that $\overrightarrow{DF}$ may be extinguished after the perturbation and $\overleftarrow{DF}$ can preserve.

If $p_*\ge 0$, we have $\overrightarrow{S}_{\delta_1}J_2 \to \overrightarrow{S}_{\delta_2}$ and $\overrightarrow{S}_{\delta_2}\overrightarrow{DF} \to \overrightarrow{S}_{\delta_2}$ or $\overleftarrow{DT}+J+\overrightarrow{DT}$, and observe that $\overrightarrow{DF}$ may be extinguished or be transformed to $\overrightarrow{DT}$. Similar discussions with Case 3.3.1, the new $\overleftarrow{DT}$ intersects with J_1 and after the intersection process, we find that $\overleftarrow{DT}+J_1 \to \overleftarrow{S}+J+\overrightarrow{S}$ or $\overleftarrow{S}+J+\overrightarrow{DT}$. Since the new $\overleftarrow{S}$ can overtake $\overleftarrow{DF}$ and it follows that $\overleftarrow{DF}+\overleftarrow{S} \to \overleftarrow{S}+J+\overrightarrow{S}\,or\,\overrightarrow{R}$ which indicates that $\overleftarrow{DF}$ may be extinguished.

Theorem 3.3 For this case, we conclude that $\overleftarrow{DF}$ can persist or may be extinguished after the perturbation, while $\overrightarrow{DF}$ may be extinguished or be transformed to $\overrightarrow{DT}$.

Case 3.4. The Riemann problem at $(-\varepsilon,0)$ is $\overleftarrow{DF}+J+\overrightarrow{R}$, and the Riemann problem at $(\varepsilon,0)$ is $\overleftarrow{R}+J+\overrightarrow{DF}$.

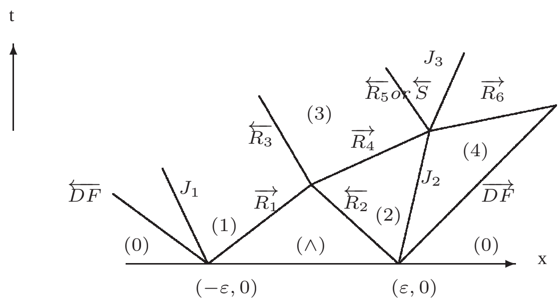

Fig. 16. The solution for Case 3.4.

Similar discuss with Case 3.2, we know that $\overrightarrow{R_1}\overleftarrow{R_2} \to \overleftarrow{R_3}\overrightarrow{R_4}$ (Fig. 16.). $\overleftarrow{R_3}$ can not overtake J_1 in the finite time. Due to $\frac{dx}{dt}=u_2+\sqrt{-\frac{p_2}{\rho_2}}$ for $\overrightarrow{R_4}$, and $\frac{dx}{dt}=u_2$ for J_2, we find that $\overrightarrow{R_4}$ can overtake J_2 at (x_2,t_2) and $\overrightarrow{R_4}J_2 \to \overleftarrow{S}\,or\,\overleftarrow{R_5}+J_3+\overrightarrow{R_6}$. Similar with Case 3.2, after the perturbation $\overleftarrow{DF}$ can persist.

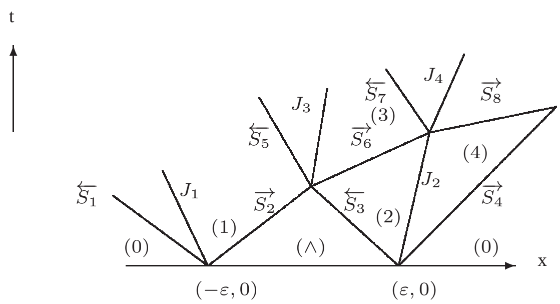

Fig. 17. The solution for Case 3.5.1.

Due to $\overrightarrow{R_6}$ can overtake $\overrightarrow{DF}$ at (x_3,t_3), after the interaction we obtain $\overleftarrow{R}+J+\overrightarrow{S}$ which implies that $\overrightarrow{DF}$ may be extinguished after the perturbation.

Theorem 3.4 In this case, we find that after perturbation, the backward deflagration wave $\overleftarrow{DF}$ can persist, while the forward deflagration wave $\overrightarrow{DF}$ may be extinguished.

Case 3.5. The Riemann problem at $(-\varepsilon,0)$ is $\overleftarrow{S}+J+\overrightarrow{S}$, and the Riemann problem at $(\varepsilon,0)$ is $\overleftarrow{S}+J+\overrightarrow{S}$.

Case 3.5.1 If $p_1+p_2-\hat{p}<0$, it follows that $\overrightarrow{S_2}\overleftarrow{S_3} \to \overleftarrow{S_5}J_3\overrightarrow{S_6}$ (Fig. 17.).

Because $\overrightarrow{S_6}$ can overtake J_2 at the point (x_2,t_2), we know that there are two possibilities. If $2\sqrt{\rho_4}p_3+(\sqrt{\rho_2}-\sqrt{\rho_4})p_2<0$, we obtain $\overrightarrow{S_6}J_2 \to \overleftarrow{S_7}J_4\overrightarrow{S_8}$. Notice that $\overleftarrow{S_5}$ can not overtake J_1 and the new shock wave $\overleftarrow{S_7}$ can not overtake J_3 in the finite time. While the new shock wave $\overrightarrow{S_8}$ can overtake $\overrightarrow{S_4}$ at (x_3, t_3) (Fig. 18.).

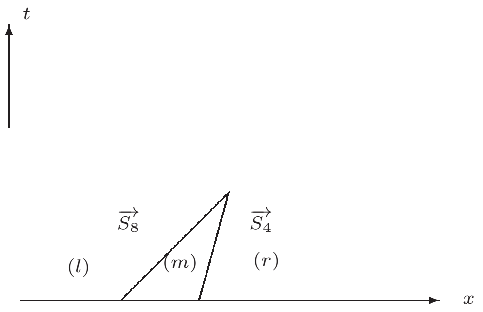

Fig. 18. The interaction of $\overrightarrow{S_8}$ and $\overrightarrow{S_4}$.

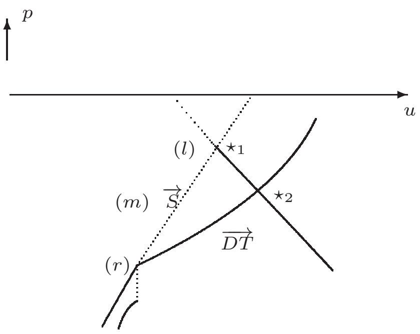

Fig. 19. The wave curves in (u, p).

In the (u, p) plane (Fig. 19.), we observe there are at most two intersection points. From GEC, we get $\overrightarrow{S_8}\overrightarrow{S_4} \to \overrightarrow{S}$ or $\overleftarrow{S} + J + \overrightarrow{DT}$. The newly generated backward shock wave can not overtake J_4, thus $\overleftarrow{S_1}$ can persist after the small perturbation. Although there is no combustion in the forward direction, the combustion wave occurs.

If $2\sqrt{\rho_4}p_3 + (\sqrt{\rho_2} - \sqrt{\rho_4})p_2 \geq 0$, we know that $\overrightarrow{S_6}J_2 \to \overrightarrow{S}_{\delta_1}$. It follows that $\overrightarrow{S}_{\delta_1}\overrightarrow{S_4} \to \overrightarrow{S}_{\delta_2}$ or $\overleftarrow{S}\,J\,\overrightarrow{S}$, and there is no combustion after perturbation.

Case 3.5.2 When $p_1 + p_2 - \hat{p} \geq 0$, we obtain that $\overrightarrow{S_2}\overleftarrow{S_3} \to \overrightarrow{S}_{\delta_1}$ (Fig. 20.).

When $\rho_1 \neq \rho_2$,

$$\omega(t) = \sqrt{\rho_1\rho_2(u_2 - u_1)^2 - (\rho_2 - \rho_1)(p_2 - p_1)}\ (t - t_1)$$

$$u_\delta = \frac{\rho_2 u_2 - \rho_1 u_1 + \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}}{\rho_2 - \rho_1},$$

when $\rho_1 = \rho_2$,

$$\omega(t) = (\rho_1 u_1 - \rho_2 u_2)(t - t_1)$$

$$u_\delta = \frac{1}{2}(u_1 + u_2).$$

Since $\overrightarrow{S}_{\delta_1}$ will overtake J_2 at (x_2, t_2), there are two possibilities. If $p_* < 0$, we know $\overrightarrow{S}_{\delta_1}J_2 \to \overleftarrow{S}\, or\, \overleftarrow{R} + \overrightarrow{S}\, or\, \overrightarrow{R}$, and the contact discontinuity may not appear. Similar discussions with Case 3.5.1, we conclude that $\overrightarrow{S}\, or\, \overrightarrow{R} + \overrightarrow{S_4} \to \overleftarrow{S} + J + \overrightarrow{S}$ or $\overleftarrow{S} + J + \overrightarrow{DT}\, or\, \overrightarrow{DF}$ and the combustion wave occurs after the small perturbation.

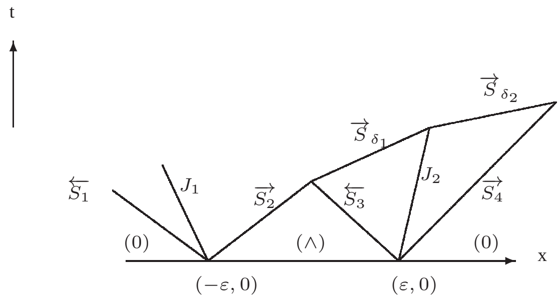

Fig. 20. The solution for Case 3.5.2.

If $p_* \geq 0$, we have $\overrightarrow{S}_{\delta_1}J_2 \to \overrightarrow{S}_{\delta_2}$ and $\overrightarrow{S}_{\delta_2}\overrightarrow{S_4} \to \overrightarrow{S}_{\delta_3}$ or $\overleftarrow{S}\,J\,\overrightarrow{S}$, and there is no combustion after perturbation.

Theorem 3.5 For this case, we conclude that after the perturbation the combustion wave may occur.

Case 3.6. Solution at $(-\varepsilon, 0)$ is $\overleftarrow{DF} + J + \overrightarrow{R}$, and at $(\varepsilon, 0)$ is $\overleftarrow{R} + J + \overrightarrow{DF}$.

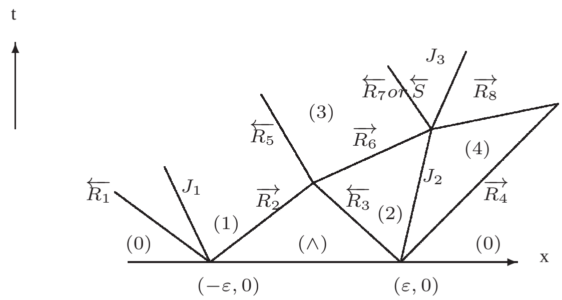

Fig. 21. The solution for Case 3.6.

Similar discuss with Case 3.2, we know that $\overrightarrow{R_2}\overleftarrow{R_3} \to \overleftarrow{R_5}\overrightarrow{R_6}$ (Fig. 21.). $\overleftarrow{R_5}$ can not overtake J_1 in the finite time. Because $\overrightarrow{R_6}$ can overtake J_2 at (x_2, t_2) and the solution of the new Riemann problem is $\overrightarrow{R_6}J_2 \to \overleftarrow{S}\, or\, \overleftarrow{R_7} + J_3 + \overrightarrow{R_8}$. Similar with Case 3.5, after the perturbation there is no combustion wave occur in the backward direction.

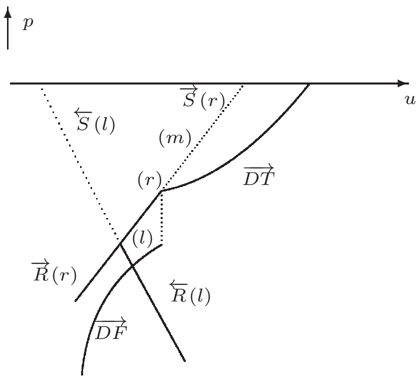

Fig. 22. The wave curves in (u, p).

Due to $\overrightarrow{R_8}$ can overtake $\overrightarrow{R_4}$ at (x_3, t_3) (Fig. 22.). After the interaction we obtain $\overrightarrow{R}$ or $\overleftarrow{R} + J + \overrightarrow{DF}$.

Theorem 3.6 In this case, we find that after perturbation the combustion wave may occur.

IV. Conclusion

Here we give our main conclusions.

We find that after the perturbation, $\overleftarrow{DF}$ can persist or may be extinguished, while $\overleftarrow{DT}$ can persist. On the other hand, $\overrightarrow{DT}$ may be extinguished, and $\overrightarrow{DF}$ may be extinguished or be transformed to $\overrightarrow{DT}$ after the perturbation. Although there is no burning phenomenon, the combustion wave may occur. The above conclusions indicate the instability of the unburnt gas of (1) which revealed deeply the internal mechanism of combustion phenomenon.

The ignition problem for the Chalygin gas dynamic system (1) plays the important role in Chalygin system not only the mathematical theory of the Chalygin system but also the exploration of the internal mechanism of combustion.

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