An Inverse Stefan Problem

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Abstract—**The moving solid/liquid interface of a melting solid in the one-dimensional case is identified from temperature and flux measurements performed solely on the solid part. An algorithm is used, based on the reduction of problem to an integral equation and using a Collocation method for solving of it.**

Keywords: ***Inverse Stefan problems, Solidification, Volterra integral equations***

1 Introduction

Problems with free boundaries concern a large number of physical phenomena of which many can be encountered in thermal industrial processes, such as casting, welding, purification by metal beams or laser machining by beams, gas production from hydrates in porous media. These phenomena are based on fusion and solidification. These transformation are accompanied by absorption (for fusion) or by release (solidification) of the latent heat on the solid/liquid interface, which characterizes the process of phase change. Optimization of the industrial process requires control of the moving interface, which makes its identification essential.
Many studies of inverse problems, in phase change, have been devoted to control which consist of searching for the boundary conditions in order to generate a prescribed interface [3,4,5,6,9,12,13,14,15].
In this paper we consider an identification problem of a particular moving boundary problem at the isothermal interface between the solid phase and the liquid phase (the Stefan problem). In [11] Gold'man has considered only solid part which is as follows.
The Solid part

$$I_s(t) = \{x | s(t) < x < 1\},$$

is governed by the heat equation

$$\theta_t - \theta_{xx} = 0 \qquad in \qquad \overline{Q} = I_s(t) \times (0, T), \tag{1}$$

satisfying the initial condition

$$\theta(x, 0) = 0 \qquad in \qquad (s(0), 1), \tag{2}$$

and the boundary condition

$$\theta_x(1, t) = u(t) \qquad in \qquad (0, T), \tag{3}$$

and the interface condition

$$\theta(s(t), t) = T_f \qquad in \qquad (0, T). \tag{4}$$

In [11] Gold'man has proved if the interface s is given and if $u \in C^{1+\gamma}[0, T]$, then the problem (1)-(4) admits a unique solution $\theta \in C^{2+\gamma,(2+\gamma)/2}(\overline{Q})$. One can calculate $\theta(1, t)$ and then define the observation operator

$$C(s) = \theta(1, t).$$

In this paper the only modeled part is the liquid one.

2 Statement of the problem

Let

$$\overline{\Omega} = \{(x, t) | 0 \le x \le s(t), 0 \le t \le T\}.$$

We consider the one-phase Stefan problem, in one-dimensional space, which is a particular moving boundary problem: the isothermal interface between the solid phase and the liquid phase is driven by the diffusive heat in two connected phases. The only modeled part is the liquid one. The interface solid/liquid is characterized by a positive function $s \in C^1[0, T]$.
The liquid part

$$I_l(t) = \{x | 0 < x < s(t)\},$$

is governed by the heat equation

$$u_t = u_{xx} \qquad in \qquad I_l(t) \times (0, T) \tag{5}$$

satisfying the initial Condition

$$u(x, 0) = \varphi(x) \qquad in \qquad (0, s(0)), \tag{6}$$

and the interface condition

$$u(s(t), t) = T_f \qquad in \qquad (0, T), \tag{7}$$

and the Stefan condition

$$u_x(s(t), t) = -\dot{s}(t) \qquad in \qquad (0, T), \tag{8}$$

where the function φ and the constant T_f, respectively the initial data and the melting point, are assumed to be known. Without loss of generality, we will suppose that $T_f = 0$.
The inverse problem we are concerned with, is as follows:

Inverse Problem. Given the interface s and $\varphi \in C^{1+\alpha}[0, \infty)$ from problem (5)-(8), can we calculate $u(0, t)$ and then define the observation operator?

$$C(s) = u(0, t). \tag{9}$$

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3 Existence

In this section we reduce the problem (5)-(8) to an equivalent problem of solving a linear integral equation of Volterra type for $C(s)$. For this purpose we first consider following free boundary problem:

$$u_{xx} = u_t \qquad for \qquad 0 < x < s(t), \qquad t > 0, \tag{10}$$

$$u(0,t) = C(s) \qquad where \qquad C(s) \geq 0, \qquad t > 0, \tag{11}$$

$$u(x,0) = \varphi(x) \qquad where \qquad \varphi(x) \geq 0, 0 < x \leq b, \qquad and \qquad \varphi(b) = 0, b > 0, \tag{12}$$

$$u(s(t),t) = 0 \qquad for \qquad t > 0, and \qquad s(0) = b, \tag{13}$$

$$u_x(s(t),t) = -\frac{ds(t)}{dt} \qquad for \qquad t > 0, \tag{14}$$

$x = s(t)$ is the free boundary which is not given and is to be found together with $u(x,t)$.

Definition. We say that $u(x,t), s(t)$ form a solution of (10)-(14) for all $t < \sigma, (0 < \sigma \leq \infty)$ if (i)$\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial t}$ are continuous for $0 < x < s(t), 0 < t < \sigma$; (ii)$u$ and $\frac{\partial u}{\partial x}$ are continuous for $0 \leq x \leq s(t), 0 < t < \sigma$; (iii)$u(x,t)$ is continuous also for $t = 0, 0 < x \leq b$, and $0 \leq \liminf u(x,t) \leq \limsup u(x,t) < \infty$ as $t \longrightarrow 0, x \longrightarrow 0$ (if $\varphi(0) = f(0)$ then u is required to be continuous at $x = t = 0$); R (iv)$s(t)$ is continuously differentiable for $0 \leq t < \sigma$, and (v)the equations (10)-(14) are satisfied.

Theorem. Assume that $C(s)$ $(0 \leq t < \infty)$ and $\varphi(x)$ $(0 \leq x \leq b)$ are continuously differentiable functions. Then there exist a unique solution $u(x,t), s(t)$ of the system (2.0.10)-(2.0.14) for all $t < \infty$. Furthermore, the function $x = s(t)$ is monotone nondecreasing in t and the function $u(x,t)$ we find for following integral representation.

$$u(x,t) = \int_0^t u_\xi(s(t),t)G(x,t;s(\tau),\tau)d\tau \tag{15}$$

$$+\int_0^t C(s)G_\xi(x,t;0,\tau)d\tau + \int_0^b \varphi(\xi)G(x,t;\xi,0)d\xi$$

Where $G(x,t;\xi,\tau)$ is Green's function for the half-plan $x > 0$ and

$$G(x,t;\xi,\tau) = K(x,t;\xi,\tau) - K(-x,t;\xi,\tau),$$

where

$$K(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} exp\{-\frac{(x-\xi)^2}{4(t-\tau)}\}.$$

Proof. See [10].

We shall now reduce the problem of solving (10)-(14) to a problem of solving an integral equation. By introducing

$$\upsilon(t) = u_x(s(t),t), \tag{16}$$

and suppose that u, s form a solution of (10)-(14) and $\varphi(0) = C(s)|_{t=0}$, we can reduce the problem of solving (10)-(14) to a problem of solving an following integral equation [10],

$$\upsilon(t) = 2\int_0^b \varphi'(\xi)N(s(t),t;\xi,0)d\xi \tag{17}$$

$$-2\int_0^t C'(s)N(s(t),t;0,\tau)d\tau$$

$$+2\int_0^t \upsilon(\tau)G_x(s(t),t;s(\tau),\tau)d\tau,$$

and by (14),(16), we have

$$s(t) = b - \int_0^t \upsilon(\tau)d\tau. \tag{18}$$

Where

$$N(x,t;\xi,\tau) = K(x,t;\xi,\tau) + K(-x,t;\xi,\tau).$$

Thus for every solution u, s of the system (10)-(14) for all $t < \sigma$, the function $\upsilon(t)$ defined by (17) satisfies the nonlinear integral equation of Volterra type (17) (for $0 < t < \infty$), where $s(t)$ is given by (18) continuous for $0 \leq t \leq \sigma$.
Suppose conversely that for some $\sigma > 0$, $\upsilon(t)$ is a continuous solution of the integral equation (17) for $0 \leq t < \sigma$, with $s(t)$ given by (18). We prove that $u(x,t), s(t)$ then form a solution of (10)-(14) for all $t < \sigma$, where $u(x,t)$ is defined by (15) with $u_\xi(s(\tau),\tau)$ replaced by $\upsilon(\tau)$, [10].

Now we consider the following Inverse problem . By above verifying, we note that, if for some $\sigma > 0$, $C(s)$ is a continuously differentiable solution of the linear Volterra integral equation of first kind (17) (for$C(s)$), where $s(t)$ is given, $u(x,t)$ then form a solution of Inverse problem. By (17), we can write

$$\int_0^t C'(s)N(s(t),t;0,\tau)d\tau = h(t), \tag{19}$$

where

$$h(t) = \int_0^b \varphi'(\xi)N(s(t),t;\xi,0)d\xi \tag{20}$$

$$+\int_0^t \upsilon(\tau)G_x(s(t),t;s(\tau),\tau)d\tau - 1/2\upsilon(t),$$

where $s(t)$, therefore $\upsilon(t)$ are given.
We want to solve the integral equation (19) where $h(t)$ is given by (20). For this purpose we write the equation (19) as following form,

$$\int_0^t u(\tau)K(t,\tau)d\tau = h(t) \qquad t \in [0,b], \tag{21}$$

where

$$u(\tau) = C'(s), \qquad K(t,\tau) = N(s(t),t;0,\tau),$$

and $h(t)$ is given by (20).

Proposition. Assume the following :
a) The function $K : [0, b] \times [0, b] \longrightarrow R$ is continuous.
b)We define,

$$L = \max_{t,\tau\in[0,b]} |K(t, \tau)|.$$

c) We set $X = C[0, b]$ and $M = \{u \in X : \|u\| < r\}$ for fixed $r > 0$.
Then, the original integral equation (21) has at least one solution $u \in M$.

Proof. Define the operator

$$(Tu)(t) = \frac{u(t)}{h(t)} \int_0^t u(\tau)K(t, \tau)d\tau \quad for \quad all \quad t \in [0, b].$$

Then, the integral equation (21) corresponds to the following fixed-point problem :

$$u = Tu, \qquad u \in M. \tag{22}$$

We claim that the equation (22) has a solution. For this purpose, we need to prove the following :
1) The set M is a bounded, closed, convex, nonempty subset of Banach space X.
2) The operator $T : M \longrightarrow M$ is compact.
Then, the Schauder fixed-point theorem tells us the equation (22) has a solution.

Lemma 1. The set M is a bounded, closed, convex and nonempty subset of Banach space X.

Proof. The bounded and nonempty property subset of Banach space M of X are clear.
We know that, the set M is convex iff $u, v \in M$ and $0 \le \alpha \le 1$ imply $\alpha u + (1 - \alpha)v \in M$.
Let $u, v \in M$ and $0 \le \alpha \le 1$, then

$$\begin{aligned}\|\alpha u + (1 - \alpha)v\| &\le \|\alpha u\| + \|(1 - \alpha)v\| \\ &= \alpha\|u\| + (1 - \alpha)\|v\| \\ &\le \alpha r + (1 - \alpha)r = r.\end{aligned}$$

Hence, $\alpha u + (1 - \alpha)v \in M$.

Now we want to show that the set M is closed. To this end, let u_n be a sequence in M such that

$$u_n \to u \qquad as \qquad n \to \infty.$$

By the definition of M, for each u_n, we have

$$\|u_n\| \le r, \qquad n = 1, 2, ...$$

Thus we can write

$$\begin{aligned}\|u\| &= \|u - u_n + u_n\| \\ &\le \|u - u_n\| + \|u_n\| \\ &\le \|u - u_n\| + r = r,\end{aligned}$$

as $n \to \infty$.
Hence, $u \in M$. Thus, the set M is closed.

Lemma 2. Let us consider the integral operator

$$(Tu)(t) = \frac{u(t)}{h(t)} \int_0^t u(\tau)K(t, \tau)d\tau \quad for \quad all \quad t \in [0, b].$$

Where

$$b \le \min\{\frac{1}{rL\|\frac{1}{h}\|}, \frac{\varepsilon - (r\|\frac{1}{h}\|\varepsilon + k_1(2r\|\frac{1}{h}\|))\delta}{r\|\frac{1}{h}\|\varepsilon + k_1(2r\|\frac{1}{h}\|)}\}.$$

Set

$$Q = \{(t, \tau, u) \in R^3 : (t, \tau) \in [0, \tau] \quad and \quad \|u\| \le r\}$$

for fixed $r > 0$.
Suppose that the function

$$F : Q \longrightarrow R$$

$$F(t, \tau, u) = u(\tau)K(t, \tau)$$

is continuous .
Set $X = C[0, b]$ and $M = \{u \in X : \|u\| < r\}$ for fixed $r > 0$.
Then, the operator $T : M \longrightarrow M$ is compact .

Proof. Since Q and R are normed spaces and $F : Q \longrightarrow R$ is a continuous operator on the compact set Q, hence, F is uniformly continuous on Q. This implies that, for each $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|F(t, \tau, u) - F(t', \tau, v)| < \varepsilon, \tag{23}$$

for all $(t, \tau, u), (t', \tau, v) \in Q$ with $|t - t'| + |u - v| < \delta$.
We first show that the operator $u : [0, b] \longrightarrow R$ is continuous. In fact, if $u \in M$, then the function $u : [0, b] \longrightarrow R$ is continuous, and $|u(\tau)| \le r$ for all $\tau \in [0, b]$ and $h : [0, b] \longrightarrow R$ is continuous and $h(\tau) \neq 0$ for all $\tau \in [0, b]$. Hence the function $Tu : [0, b] \longrightarrow R$ is also continuous.
Let

$$\|u - v\| = \max_{0\le\tau\le b} |u(\tau) - v(\tau)| < \delta$$

implies

$$\|Tu - Tv\| =$$

$$\max_{0\le t\le b} |\frac{u(t)}{h(t)} \int_0^t u(\tau)K(t, \tau)d\tau - \frac{v(t)}{h(t)} \int_0^t v(\tau)K(t, \tau)d\tau|$$

$$= \max_{0\le t\le b}(|\frac{1}{h(t)}|| \int_0^t (u(t)u(\tau) - v(t)v(\tau))K(t, \tau)d\tau|)$$

$$= \max_{0\le t\le b}(|\frac{1}{h(t)}|| \int_0^t (u(t)u(\tau) - u(t)v(\tau)$$

$$+u(t)v(\tau) - v(t)v(\tau))K(t, \tau)d\tau|)$$

$$= \max_{0\le t\le b}(|\frac{1}{h(t)}|| \int_0^t (u(t)(u(\tau) - v(\tau))$$

$$-v(\tau)(u(t)-v(t)))K(t,\tau)d\tau|)$$
$$\leq \|\frac{1}{h}\| \max_{0\leq t\leq b}\int_0^t (|u(t)||(u(\tau)-v(\tau))K(t,\tau)|$$
$$+|v(\tau)||(u(t)-v(t))K(t,\tau)|)d\tau$$
$$\leq \frac{1}{\|h\|}(r\varepsilon+r\varepsilon)b = 2rb\varepsilon\frac{1}{\|h\|}$$

by (23). Hence, $T: M \longrightarrow M$ is continuous.
We now show that $T(M)\subseteq M$.
If $u\in M$, then

$$\|Tu\| = \|\frac{u(t)}{h(t)}\int_0^t u(\tau)K(t,\tau)d\tau\|$$
$$\leq \|u\|\|\frac{1}{h}\| \max_{0\leq t\leq b}\int_0^t |u(\tau)||K(t,\tau)|d\tau$$
$$\leq r^2\|\frac{1}{h}\|Lb \leq r,$$

for $b\leq \frac{1}{rL\|\frac{1}{h}\|}$.
Hence, $Tu\in M$. Thus $T(M)\subseteq M$ for $b\leq \frac{1}{rL\|\frac{1}{h}\|}$.
We now show that $T: M\longrightarrow M$ is compact.
Since the set M is bounded, it suffices to show that the set $T(M)$ is relatively compact. By the Arzela-Ascolli theorem it remains to show that $T(M)$ is equicontinuous. Let $|t-t'|<\delta$ and $t,t'\in[0,b]$. Then by (23)

$$|(Tu)(t)-(Tu)(t')| =$$
$$|\frac{u(t)}{h(t)}\int_0^t u(\tau)K(t,\tau)d\tau - \frac{u(t')}{h(t')}\int_0^{t'} u(\tau)K(t',\tau)d\tau|$$

$$= |\int_0^t [\frac{u(t)}{h(t)}F(t,\tau,u) - \frac{u(t')}{h(t')}F(t',\tau,u)]d\tau$$
$$+\int_t^{t'} [\frac{u(t)}{h(t)}F(t,\tau,u) - \frac{u(t')}{h(t')}F(t',\tau,u)]d\tau|$$

$$= |\int_0^t [\frac{u(t)}{h(t)}(F(t,\tau,u)-F(t',\tau,u))$$
$$+F(t',\tau,u)(\frac{u(t)}{h(t)}-\frac{u(t')}{h(t')})]d\tau$$

$$+\int_t^{t'} [\frac{u(t)}{h(t)}(F(t,\tau,u)-F(t',\tau,u))$$
$$+F(t',\tau,u)(\frac{u(t)}{h(t)}-\frac{u(t')}{h(t')})]d\tau|$$

$$\leq (r\|\frac{1}{h}\|\varepsilon + k_1(2r\|\frac{1}{h}\|))b + (r\|\frac{1}{h}\|\varepsilon + k_1(2r\|\frac{1}{h}\|))\delta \leq \varepsilon$$

for

$$b \leq \frac{\varepsilon - (r\|\frac{1}{h}\|\varepsilon + k_1(2r\|\frac{1}{h}\|))\delta}{r\|\frac{1}{h}\|\varepsilon + k_1(2r\|\frac{1}{h}\|)}.$$

Where

$$k_1 = \max_{t,\tau\in[0,b]}|F(t,\tau,u)|.$$

4 Numerical Results

In this section we apply the Collocation method to some examples in order to compare numerical solution with exact solution.

Example 1. In Inverse problem, suppose that $s(t)=t$. Then we obtain the following integral equation :

$$\int_0^t C'(s)\frac{\exp\{-\frac{t^2}{4(t-\tau)}\}}{\sqrt{t-\tau}}d\tau = \frac{\sqrt{\pi}}{2}$$
$$+\frac{1}{4}\int_0^t \frac{1}{\sqrt{t-\tau}}[\exp\{\frac{\tau-t}{4}\} - \frac{t+\tau}{t-\tau}\exp\{-\frac{(t+\tau)^2}{4(t-\tau)}\}]d\tau$$

with exact solution $C(s)=e^t-1$. Suppose that $t\in(0,1)$. The result of applying Collocation method with 12 nodes of interval (0,1) and with 12 base functions $\varphi_j(t)=t^j, j=0,1,2,...,11$ for above integral equation is in the following form:

$$C_{collo}(s) = \frac{417660}{12}t^{12} - \frac{441640}{11}t^{11} - \frac{77870}{10}t^{10}$$
$$+\frac{145570}{9}t^9 - \frac{5320}{8}t^8 + \frac{12360}{7}t^7 - \frac{18730}{6}t^6$$
$$+850t^5 + \frac{170}{4}t^4 - \frac{130}{3}t^3 + 10t^2,$$

which the right hand integral is approximated by Gaussian three points rule.
Comparing of numerical solution and exact solution is given in figure1.

Example 2. For $s(t)=t^{3/2}$, we obtain the following integral equation :

$$\int_0^t C'(s)\frac{\exp\{-\frac{t^3}{4(t-\tau)}\}}{\sqrt{t-\tau}}d\tau = 3/4\sqrt{\pi t}$$
$$+3/8\sqrt{t}\int_0^t \frac{1}{\sqrt{t-\tau}}[\frac{t^{3/2}-\tau^{3/2}}{t-\tau}\exp\{-\frac{(t^{3/2}-\tau^{3/2})^2}{4(t-\tau)}\}$$
$$-\frac{t^{3/2}+\tau^{3/2}}{t-\tau}\exp\{-\frac{(t^{3/2}-\tau^{3/2})^2}{4(t-\tau)}\}]d\tau.$$

With exact solution

$$C(s) = -\frac{5103}{6800}t^{20} - \frac{716607}{409600}t^{18} + \frac{380593737}{11468800}t^{16}$$
$$+\frac{449377571}{5734400}t^{14} + \frac{38271457}{5160960}t^{12} + \frac{2221627}{1290240}t^{10}$$
$$+\frac{4251}{2240}t^8 + \frac{7}{10}t^6 + \frac{5}{4}t^4 + \frac{3}{2}t^2.$$

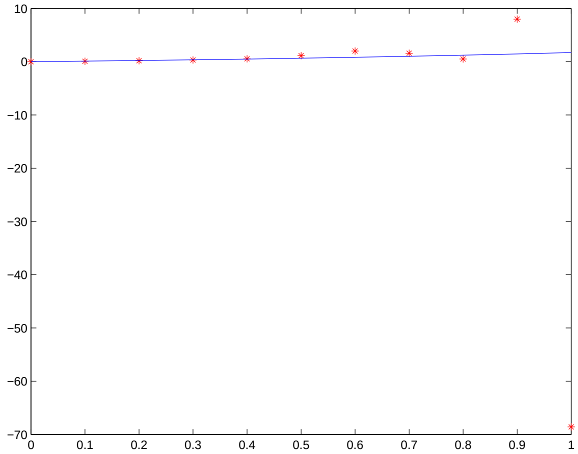

Figure 1: (—) exact solution, (*) approximated points

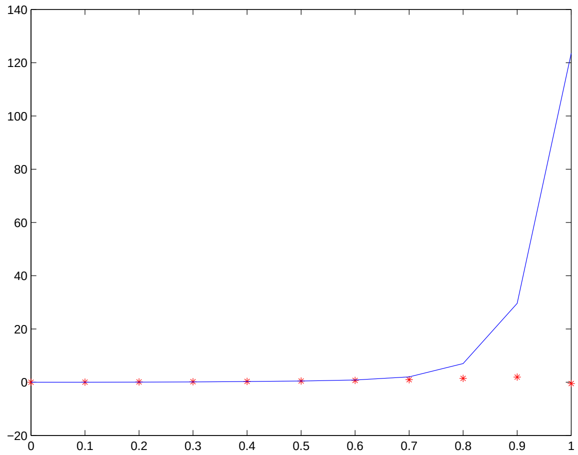

Figure 2: (—) exact solution, (*) approximated points

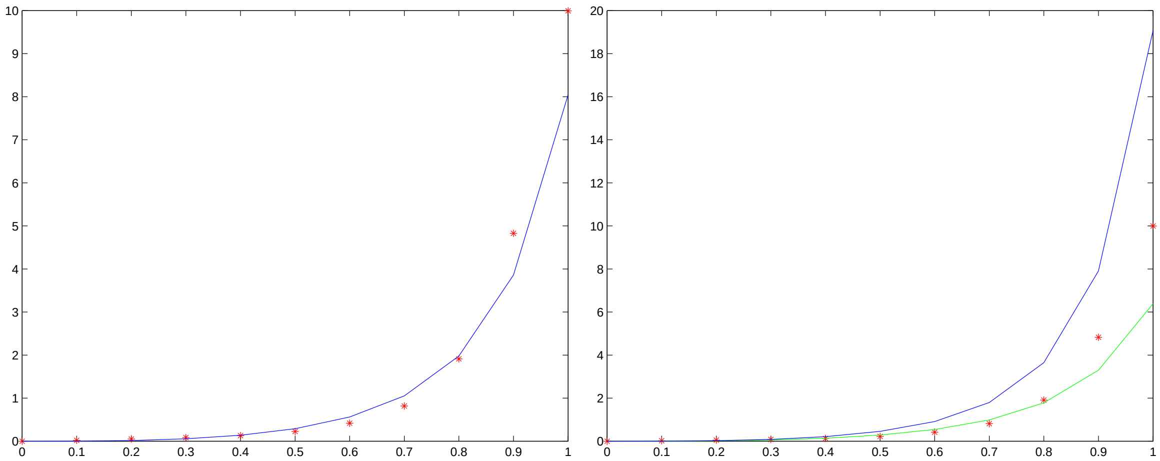

Figure 3: (—) exact solution, (*) approximated points

Suppose that $t \in (0,1)$. The result of applying Collocation method with 12 nodes of interval (0,1) and with 12 base functions $\varphi_j(t) = t^j, j = 0,1,2,...,11$ for above integral equation is in the following form:

$$C_{collo}(s) = \frac{537.3}{12}t^{12} + \frac{34.5}{11}t^{11} - \frac{200.8}{10}t^{10}$$
$$+\frac{1621}{9}t^{9} - \frac{769.6}{8}t^{8} + \frac{586.3}{7}t^{7} + \frac{410.9}{6}t^{6}$$
$$-\frac{39.5}{5}t^{5} + \frac{7.6}{4}t^{4} + \frac{7.1}{3}t^{3} + \frac{1.3}{2}t^{2} + 0.7t,$$

which the right hand integral is approximated by Gaussian three points rule.
Comparing of numerical solution and exact solution is given in figure2.

Example 3. Suppose that $s(t) = t^2$, then we obtain the following integral equation :

$$\int_0^t C'(s)\frac{\exp\{-\frac{t^4}{4(t-\tau)}\}}{\sqrt{t-\tau}}d\tau =$$
$$t\sqrt{\pi} + 1/2\int_0^t \frac{t}{\sqrt{t-\tau}}[(t+\tau)\exp\{-\frac{(t+\tau)(t^2-\tau^2)}{4}\}$$
$$-\frac{t^2+\tau^2}{t-\tau}\exp\{-\frac{(t^2+\tau^2)^2}{4(t-\tau)}\}]d\tau,$$

with exact solution

$$C(s) = \frac{4}{14175}t^{30} + \frac{301}{113400}t^{27} + \frac{1157}{90720}t^{24}$$
$$+\frac{131}{2016}t^{21} + \frac{3059}{15120}t^{18} + \frac{323}{630}t^{15} + \frac{13}{12}t^{12} + \frac{11}{6}t^{9}$$
$$+\frac{7}{3}t^{6} + 2t^{3}.$$

Suppose that $t \in (0,1)$. The result of applying Collocation method with 12 nodes of interval (0,1) and with 12 base functions $\varphi_j(t) = t^j, j = 0,1,2,...,11$ for above integral equation is in the following form:

$$C_{collo}(s) = -\frac{2176.4}{12}t^{12} + \frac{2686.2}{11}t^{11} + \frac{233.3}{10}t^{10}$$
$$-\frac{51.1}{9}t^{9} - \frac{688.1}{8}t^{8} - \frac{590.3}{7}t^{7} + \frac{836.2}{6}t^{6} - \frac{193.5}{5}t^{5}$$
$$-\frac{7.2}{4}t^{4} - \frac{0.2}{3}t^{3} + \frac{1.9}{2}t^{2} + 0.1t,$$

which the right hand integral is approximated by Gaussian three points rule.
Comparing of numerical solution and exact solution is given in figure 3 (left).
Also for $s(t) = t^2$ by asymptotic approximation given in [2], we can obtain upper and lower bounds for $C(s)$ in the following form:

$$\exp\{2t^3\} - 1 \le C(s) \le \exp\{3t^3\} - 1.$$

Figure 3 (right) shows that numerical approximation lies between upper and lower bounds.

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