Implementation of the Discontinuous Galerkin Method on a Multi-Story Seismically Excited Building Model

Helmi Temimi[†] Slimane Adjerid[‡] Mohamed Ayari[§]

Abstract—**In this paper, we implement the Discontinuous Galerkin (DG) method, proposed and well-analyzed for linear and nonlinear problems by Adjerid *et.al*, to study the response of a multi-story building seismically excited. In fact, we adopt the DG method to solve a system of second order ordinary differential equations. The unknowns are the time response of the inter-story displacement of each level of the building. Simulation results show the effectiveness and robustness of the DG method. A comparison of the results obtained through the DG method and through traditional numerical schemes is conducted. The results reveal the efficacy of the DG method, which lends it as an attractive alternative instead of currently used numerical techniques.**

Keywords: Discontinuous Galerkin method, Runge-Kutta method, earthquake, mechanical model

Introduction

The discontinuous Galerkin (DG) finite element method was studied for initial-value problems for first-order ordinary differential equations [2, 4, 7, 8, 10]. Cockburn and Shu [5] extended the DG method to solve first-order hyperbolic partial differential equations of conservation laws. Initial value problems for 1st-order ordinary differential systems were solved using standard DG methods [2, 6].

In this manuscript we present a new numerical scheme to solve the resulting system of ordinary differential equations from the force-analysis scope by [11], [1]. We apply the DG method developed by [3] on the resulting system. The DG method is very appealing regarding its efficiency proven for several types of problems in terms of its robustness, stability [9], higher order accuracy (pointwise error is h^{p+1}, where h is the step size and p is the degree of approximation) and approximation by polynomials of different degrees in different elements. We apply the method to a 100-story building and we derive the displacement and the acceleration of each floor.

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1 Structural Model

Consider an n-story 1-D building, subjected to earthquake ground acceleration. It is assumed that the structure understudy verifies the shear type representation: "structure with flexible massless columns and mass concentrated at rigid beams" as shown in Figure 1. Thus, our structure can be viewed as an n-degree of freedom structure, considering the horizontal displacement of each story. It is further assumed

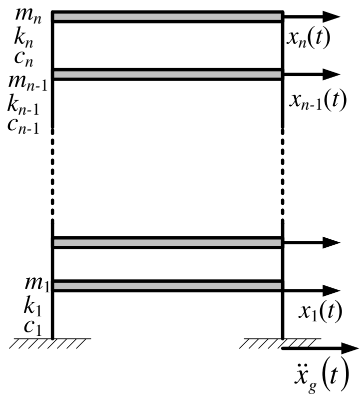

Figure 1: Modelisation of the 1-D structure.

that the structure is characterized by a stiffness and a damping coefficient in the $x-$direction. The formulation of the equations of motion will be presented in terms of the inter-story drift coordinates x_i, as shown in Figure 2. Thus, we have:

$$\begin{cases} \forall i > 1;\ x_i(t) = x_i^a(t) - x_{i-1}^a(t) \\ x_1(t) = x_1^a(t) - x_g(t) \end{cases} \tag{1}$$

where x_i^a is the absolute story displacement.

The damping in the structure is assumed to be linear viscous, i.e, the damping force is assumed to be proportional to the magnitude of the velocity and opposite to the direction of motion. In addition, we suppose that the restoring force is pro-

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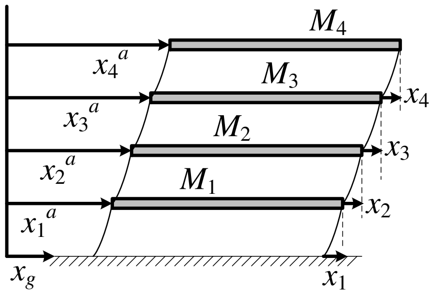


Figure 2: Absolute and inter-story drift coordinates of the structure.

portional to the magnitude of the inter-story drift and opposite to the direction of motion. The equivalent mechanical model for this structure is shown in Figure 3.

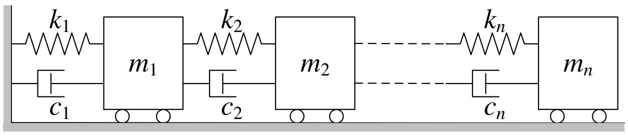


Figure 3: Mechanical model of the linear 1-D structure.

Thus, the total force (sum of damping and restoring forces) exerted by the $(i-1)^{th}$ story on the i^{th} story is given by:

$$F_i(t) = -k_i x_i(t) - c_i \dot{x}_i(t) \tag{2}$$

The governing differential equation of motion for an uncontrolled (unforced) linear structure under an earthquake excitation is given by:

$$M\ddot{X}(t) + C\dot{X}(t) + KX(t) = E\ddot{x}_g(t) \tag{3}$$

where

- $X(t) = [x_1(t), x_2(t), \cdots, x_{n-1}, x_n(t)]^T$ is the vector of inter-story drift displacements.
- T: denotes the transpose of the matrix.
- M is the mass matrix of the system:

$$M = \begin{pmatrix} m_1 & m_1 & \cdots & m_1 \\ 0 & m_2 & \cdots & m_2 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{pmatrix}_{n\times n}$$

- C is the damping matrix of the system:

$$C = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 \\ -c_1 & c_2 & 0 & 0 & 0 \\ 0 & -c_2 & c_3 & 0 & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix}_{n\times n}$$

- K is the stiffness matrix of the system:

$$K = \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 \\ -k_1 & k_2 & 0 & 0 & 0 \\ 0 & -k_2 & k_3 & 0 & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_n \end{pmatrix}_{n\times n}$$

- $E = -[m_1\, m_2\, \cdots m_n]^T$
- and $\ddot{x}_g$ is the ground acceleration.

2 Implementation of The DG Method

In order to find the response of a building to an earthquake excitation, we need to solve the following system of linear ordinary differential equations,

$$\begin{aligned} &M\ddot{X}(t) + C\dot{X}(t) + KX(t) = E\ddot{x}_g(t),\ 0 \le t \le T, \\ &X_i(0) = 0, \quad \dot{X}_i(0) = 0, \quad i = 1, \cdots, n \end{aligned} \tag{4}$$

In this work, we adopt the DG method to solve (4). Our solutions will be the response of each story as displacement and acceleration. For convenience, and without any loss of generality, we can rewrite the problem presented in (4) as

$$\begin{aligned} &\ddot{X}(t) + M^{-1}C\dot{X}(t) + M^{-1}KX(t) = M^{-1}E\ddot{x}_g(t), \\ &x_i(0) = 0, \quad \dot{x}_i(0) = 0, \quad i = 1, ..., n \end{aligned} \tag{5}$$

Let $\tilde{C} = M^{-1}C$, $\tilde{K} = M^{-1}K$ and $\tilde{E} = M^{-1}E$ so we obtain

$$\begin{aligned} &\ddot{X}(t) + \tilde{C}\dot{X}(t) + \tilde{K}X(t) = \tilde{E}\ddot{x}_g(t),\ 0 \le t \le T, \\ &x_i(0) = 0, \quad \dot{x}_i(0) = 0, \quad i = 1, ..., n \end{aligned} \tag{6}$$

Now let us write the i^{th} equation of (6),

$$\ddot{x}_i(t) + \sum_{j=1}^{n} \tilde{C}_{i,j}\dot{x}_j(t) + \sum_{j=1}^{n} \tilde{K}_{i,j}x_j(t) = \tilde{E}_i\ddot{x}_g(t) \tag{7a}$$

subject to the initial conditions

$$x_i(0) = 0, \quad \dot{x}_i(0) = 0, \quad i = 1, ..., n \tag{7b}$$

To start implementing the DG method, we first create a partition, $t_k = k\,\Delta t,\ k = 0, 1, 2, \cdots, n,\ \Delta t = T/n$ with $I_j = (t_j, t_j + 1)$ and define the piecewise polynomial spaces

$$S^{n,p} = \{U\ :\ U|_{I_j} \in \mathcal{P}_p\}, \tag{8}$$

$$S_0^{n,p} = \{U \in S^{n,p}\ :\ U(t_i^-) = U'(t_i^-) = 0,\ 1 \le i \le n\}, \tag{9}$$

where $\mathcal{P}_p$ denotes the space of Legendre polynomials of degree p. Each Legendre polynomial $P_n(x)$ is an n^{th} degree polynomial. It can be expressed using Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x-1)^2]$$

An important property of the Legendre polynomials is that they are orthogonal with respect to the L^2 inner product on the interval $-1 \leq x \leq 1$:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$$

where δ_{mn} denotes the Kronecker delta, equal t 1 if $m = n$ and to 0 otherwise.

Furthermore, $P_n(x)$ is an even function or an odd function whether n is even or n is odd, and we have

$$P_n(-x) = (-1)^n P_n(x)$$

the Legendre polynomials' definitions are "standardized" by being scaled so that

$$P_n(1) = 1$$

The derivative at the end point is given by

$$P_n'(1) = \frac{n(n+1)}{2}$$

Moreover, $Legendre$-polynomials can be constructed using the three term recurrence relations

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

And

$$\frac{x^2-1}{n}\frac{d}{dx}P_n(x) = xP_n(x) - P_{n-1}(x)$$

We define the weak discontinuous Galerkin (DG) formulation for (7) by multiplying (7a) by a test function, and then integrating over I_k. After integrating by parts, we obtain n equations of the form

$$\dot{x}_i v\big|_{t_k}^{t_{k+1}} - x_i\dot{v}\big|_{t_k}^{t_{k+1}} + \sum_{j=1}^{n}(\tilde{C}_{i,j}x_j v\big|_{t_k}^{t_{k+1}}) + \int_{I_k} x_i\ddot{v}dt - \sum_{j=1}^{n}\Big(\tilde{C}_{i,j}\int_{I_k} x_j\dot{v}dt - \tilde{K}_{i,j}\int_{I_k} x_j v dt\Big) = \int_{I_k}\tilde{E}_i\ddot{x}_g v dt. \quad (10)$$

Let's replace x_i by $X_{i,k}(t) = X_i|_{[t_k,t_{k+1}]} \in \mathcal{P}_p$ for $i = 1,\cdots,n$ and v by $V \in \mathcal{P}_p$ in (10). Moreover, let's choose the flux terms that will define the DG method to be the information that is propagating from the left since we are dealing with an initial value problem:

For $i = 1,\cdots,n$

$$X_{i,k}(t_{k+1}) = X_{i,k}(t_{k+1}^-), \quad X_{i,k}(t_k) = X_{i,k-1}'(t_k^-)$$

The discrete formulation consists of determining $X_{i,k}(t) = X_i|_{[t_k,t_{k+1}]} \in \mathcal{P}_p$, such that for $i = 1,\cdots,n$

$$\begin{aligned}
&\dot{X}_{i,k}(t_{k+1}^-)V(t_{k+1}^-) - \dot{X}_{i,k-1}(t_k^-)V(t_k^+) - \\
&X_{i,k}(t_{k+1}^-)\dot{V}(t_{k+1}^-) + X_{i,k-1}(t_k^-)\dot{V}(t_k^+) + \\
&\sum_{j=1}^{n}\tilde{C}_{i,j}\Big(X_{j,k}(t_{k+1}^-)V(t_{k+1}^-) - X_{j,k-1}(t_k^-)V(t_k^+)\Big) - \\
&\sum_{j=1}^{n}\Big(\tilde{C}_{i,j}\int_{I_k} X_{j,k}\dot{V}dt - \tilde{K}_{i,j}\int_{I_k} X_{j,k}V dt\Big) \\
&\int_{I_k} X_{i,k}\ddot{V}dt = \int_{I_k}\tilde{E}_i\ddot{x}_g V dt, \quad \forall\, V \in \mathcal{P}_p. \qquad (11)
\end{aligned}$$

On the initial step, $[t_0, t_1]$, we use $X_{i,-1}(t_0^-) = 0$ and $\dot{X}_{i,-1}(t_0^-) = 0$.

On another hand, we can rewrite (11) to obtain for $i = 1,\cdots,n$

$$\begin{aligned}
&\dot{X}_{i,k}(t_{k+1}^-)V(t_{k+1}^-) - X_{i,k}(t_{k+1}^-)\dot{V}(t_{k+1}^-) \\
&+\sum_{j=1}^{n}\tilde{C}_{i,j}X_{j,k}(t_{k+1}^-)V(t_{k+1}^-) + \int_{I_k} X_{i,k}\ddot{V}dt - \\
&\sum_{j=1}^{n}(\tilde{C}_{i,j}\int_{I_k} X_{j,k}\dot{V}dt - \tilde{K}_{i,j}\int_{I_k} X_{j,k}V dt) \\
&= \int_{I_k}\tilde{E}_i\ddot{x}_g V dt + \dot{X}_{i,k-1}(t_k^-)V(t_k^+) - X_{i,k-1}(t_k^-)\dot{V}(t_k^+) \\
&+\sum_{j=1}^{n}\tilde{C}_{i,j}X_{j,k-1}(t_k^-)V(t_k^+), \quad \forall\, V \in \mathcal{P}_p. \qquad (12)
\end{aligned}$$

A clear advantage of the DG method noticed from the previous formulation is that we are solving this problem locally on each step I_k and the inter-element continuity is weakly enforced, therefore the displacement of each story $X_{i,k-1}(t_k)$ will be an initial condition for I_k.

Let's call $D_{i,k}^0 = X_{i,k-1}(t_k)$ and $D_{i,k}^1 = \dot{X}_{i,k-1}(t_k)$ (representing the information coming from the previous element), so the discrete formulation comes to determining $X_{i,k}(t) = X_i|_{[t_k,t_{k+1}]} \in \mathcal{P}_p$, such that for $i = 1,\cdots,n$

$$\begin{aligned}
&\dot{X}_{i,k}(t_{k+1}^-)V(t_{k+1}^-) - X_{i,k}(t_{k+1}^-)\dot{V}(t_{k+1}^-) + \int_{I_k} X_{i,k}\ddot{V}dt \\
&+\sum_{j=1}^{n}(\tilde{C}_{i,j}X_{j,k}(t_{k+1}^-)V(t_{k+1}^-) - \tilde{C}_{i,j}\int_{I_k} X_{j,k}\dot{V}dt + \\
&\tilde{K}_{i,j}\int_{I_k} X_{j,k}V dt) = \int_{I_k}\tilde{E}_i\ddot{x}_g V dt + D_{i,k}^1 V(t_k^+) - \\
&D_{i,k}^0\dot{V}(t_k^+) + \sum_{j=1}^{n}\tilde{C}_{i,j}D_{j,k}^0 V(t_k^+), \quad \forall\, V \in \mathcal{P}_p. \qquad (13)
\end{aligned}$$

We note that the DG solutions on each element I_k is of the following form

$$X_{i,k}(t) = \sum_{j=0}^{p}\lambda_{i,k,j}\psi_j(t), \quad i = 1,\cdots,n \qquad (14)$$

where $\psi_j(t)$ are Legendre polynomial of degree j on the interval I_k. Now, we start our matrix formulation to show the computational process of this method. Let's choose our test function V to be ψ_l for $l = 0,...,p$ and substitute (14), therefore (13) becomes for $i = 1, ..n$ and $l = 0, ..., p$,

$$\sum_{j=0}^{p} \lambda_{i,k,j} (\dot{\psi}_j(t_{k+1}^-)\psi_l(t_{k+1}^-) - \psi_j(t_{k+1}^-)\dot{\psi}_l(t_{k+1}^-)) -$$
$$\sum_{j=1}^{n}\sum_{m=0}^{p} \lambda_{j,k,m}(\tilde{C}_{i,j}\int_{I_k}\psi_m\dot{\psi}_l dt - \tilde{K}_{i,j}\int_{I_k}\psi_m\psi_l dt +$$
$$\tilde{C}_{i,j}\psi_m(t_{k+1}^-)\psi_l(t_{k+1}^-)) + \sum_{j=0}^{p}(\lambda_{i,k,j}\int_{I_k}\psi_j \dddot{=}\psi_l dt)$$
$$\tilde{E}_i\int_{I_k}\ddot{x}_g\psi_l dt + D_{i,k}^1\psi_l(t_k^+) - D_{i,k}^0\dot{\psi}_l(t_k^+) +$$
$$\sum_{j=1}^{n}\tilde{C}_{i,j}D_{j,k}^0\psi_l(t_k^+), \quad \forall\psi_l \in \mathcal{P}_l. \tag{15}$$

Let's introduce the following matrices M_1, M_2, M_3, M_4 and M_5 such that for $i, j = 0, ..., p$

$$M_1(i,j) = [\psi_i(t_{k+1}^-)\dot{\psi}_j(t_{k+1}^-)]_{(p+1,p+1)} \tag{16}$$
$$M_2(i,j) = [\psi_i(t_{k+1}^-)\psi_j(t_{k+1}^-)]_{(p+1,p+1)} \tag{17}$$
$$M_3(i,j) = [\int_{t_k}^{t_{k+1}}\ddot{\psi}_i\psi_j dt]_{(p+1,p+1)} \tag{18}$$
$$M_4(i,j) = [\int_{t_k}^{t_{k+1}}\dot{\psi}_i\psi_j dt]_{(p+1,p+1)} \tag{19}$$
$$M_5(i,j) = [\int_{t_k}^{t_{k+1}}\psi_i\psi_j dt]_{(p+1,p+1)} \tag{20}$$

And let also

$$\Lambda_{i,k} = \begin{bmatrix} \lambda_{i,k,0} \\ \lambda_{i,k,1} \\ \vdots \\ \lambda_{i,k,p-1} \\ \lambda_{i,k,p} \end{bmatrix}_{(p+1)} \tag{21}$$

Therefore (15) becomes (for $i = 1, \cdots, n$),

$$M_1\Lambda_{i,k} - M_1^T\Lambda_{i,k} + \sum_{j=1}^{n}\tilde{C}_{i,j}M_2\Lambda_{j,k} + M_3\Lambda_{i,k} -$$
$$\sum_{j=1}^{n}\tilde{C}_{i,j}M_4\Lambda_{j,k} + \sum_{j=1}^{n}\tilde{K}_{i,j}M_5\Lambda_{j,k} = B_i \tag{22}$$

where

$$B_i(l) = \tilde{E}_i\int_{I_k}\ddot{x}_g\psi_l dt + D_{i,k}^1\psi_l(t_k^+) - D_{i,k}^0\dot{\psi}_l(t_k^+) + \sum_{j=1}^{n}\tilde{C}_{i,j}D_{j,k}^0\psi_l(t_k^+). \tag{23}$$

Now define the following: $M = M_1 - M_1^T + M_3$ and $N = M_2 - M_4$. Finally, in order to find the response of each story, we need to solve the following linear system on each subinterval I_k

$$R\Lambda_k = B \tag{24}$$

where

$$R = \begin{bmatrix} M + L_{1,1} & L_{1,2} & ... & L_{1,n} \\ L_{2,1} & M + L_{2,2} & ... & L_{2,n} \\ \vdots & \vdots & & \vdots \\ L_{n,1} & L_{n,2} & ... & M + L_{n,n} \end{bmatrix} \tag{25}$$

with $L_{i,j} = \tilde{C}_{i,j}N + \tilde{K}_{i,j}M_5$ for $i, j = 1, ..., n$

$$\Lambda_k = \begin{bmatrix} \Lambda_{1,k} \\ \Lambda_{2,k} \\ \vdots \\ \Lambda_{n,k} \end{bmatrix}_{n(p+1)} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}_{n(p+1)} \tag{26}$$

3 Error Analysis and Computational Aspects of the Numerical Methods

3.1 DG Method

In this section, we start by restating the asymptotic behavior of the local DG error then *a priori* error bound in $\mathcal{L}^2$ and at the mesh points provided in [3].

In the analysis of the DG method, we will use Jacobi polynomials defined by the Rodrigues formula

$$P_k^{\alpha,\beta}(\tau) = \frac{(-1)^k}{2^k k!}(1-\tau)^{-\alpha}(1+\tau)^{-\beta}\frac{d^k}{d\tau^k}[(1-\tau)^{\alpha+k}(1+\tau)^{\beta+k}],$$
$$\alpha, \beta > -1, \ k = 0, 1, \cdots. \tag{27}$$

We note Jacobi polynomials satisfy the orthogonality condition

$$\int_{-1}^{1}(1-\tau)^{\alpha}(1+\tau)^{\beta}P_k^{\alpha,\beta}(\tau)P_l^{\alpha,\beta}(\tau)d\tau = c_k\delta_{kl}, \tag{28}$$

where $c_k > 0$ and δ_{kl} is the Kronecker symbol equal to 1 if $k = l$ and 0, otherwise. We further note that $P_k^{0,0} = P_k$, the k^{th}-degree Legendre polynomial.

In the following theorem, we state our first result on the asymptotic behavior of the local error.

Theorem 1. *Let $x_i \in C^{2p+2}$ and $X_{i,k} \in \mathcal{P}_p$, $p \geq 2$, for $i = 1, 2, ..., n$ be the solutions of (4) and (14), respectively. Then the local error satisfies*

$$\epsilon = \sum_{k=p+1}^{\infty} Q_k(\tau)\Delta t^k, \tag{29a}$$

where $Q_k(\tau) \in \mathcal{P}_k$ and

$$Q_{p+1}(\tau) = \alpha_{p+1}(1-\tau)^2 P_{p-1}^{2,0}(\tau). \tag{29b}$$

Next, we state a superconvergence result for the DG solution and its derivative at interior points.

Theorem 2. *Under the conditions of Theorem (1), the DG solution is superconvergent at interior points*

$$\epsilon^{(l)}(\bar{t}_j^l) = O(\Delta t^{p+2-l}),\ j = 1, \cdots, p-1-l,\ l = 0, 1,$$

where $0 < \bar{t}_j^l < \Delta t, j = 1, \cdots, p-1-l$, *are the shifted roots of* $((1-\tau)^2 P_{p-1}^{2,0})^{(l)},\ l = 0, 1.$

Let us recall the definition of Sobolev Spaces

$$\mathcal{H}^s = \{u\ :\ \int_0^T |u^{(l)}|^2 dt < \infty,\ 0 \le l \le s\}.$$

equipped with the norm

$$||u||_s^2 = \sum_{j=1}^{n} ||u||_{s,j}^2,$$

where

$$||u||_{s,j}^2 = \sum_{l=0}^{s} ||u^{(l)}||_j^2,\ \ ||u||_j^2 = \int_{t_{j-1}}^{t_j} |u|^2 dt.$$

The $\mathcal{L}^2$ norm is denoted by $||.||$.

Moreover, let us define the maximum norm at the downwind end-points by

$$||u^-||_{\infty,*} = \max_{1 \le i \le n} |u(t_i^-)|, \tag{30}$$

The following theorem provides an $\mathcal{L}^2$ error bound for the DG method.

Theorem 3. *If* $x_i \in \mathcal{H}^{p+1}$ *and* X_i *for* $i = 1, 2, ..., n$ *are respectively solutions of (4) and (14), we have*

$$||x_i - X_i|| \le C\Delta t^{p+1}. \tag{31}$$

Next, we state global superconvergence result for the DG method.

Theorem 4. *If* $x_i \in \mathcal{H}^{p+1}$ *and* X_i *for* $i = 1, 2, ..., n$ *are solutions of (4) and (14), respectively, Then, there exists a constant* $C, \tilde{C} > 0$ *independent of* Δt *such that*

$$||(x_i - X_i)^-||_{\infty,*} < C\Delta t^{2p} \tag{32}$$

and

$$||(x_i' - X_i')^-||_{\infty,*} < \tilde{C}\Delta t^{2p} \tag{33}$$

Next, in order to complete the presentation of the new DG numerical method, we present the following example where we expect the numerical results to be in full agreement with the stated theory. Let us consider now the following system of second order ordinary differential equations.

$$M\ddot{Y} + KY = F(t) \quad 0 \le t << 2, \tag{34}$$

where

$$M = \begin{bmatrix} 1 & 2 & ... & N_{sys}-1 & N_{sys} \\ 2 & 3 & ... & N_{sys} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ N_{sys}-1 & N_{sys} & ... & N_{sys}-3 & N_{sys}-2 \\ N_{sys} & 1 & ... & N_{sys}-2 & N_{sys}-1 \end{bmatrix}$$

$$K = \begin{bmatrix} N_{sys} & N_{sys}-1 & ... & 2 & 1 \\ N_{sys}-1 & N_{sys}-2 & ... & 1 & N_{sys} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 1 & ... & 4 & 3 \\ 1 & N_{sys} & ... & 3 & 2 \end{bmatrix}$$

$$F_i(t) = (N_{sys}+1)\frac{1-e}{1-e^{N_{sys}}} e^{t+\frac{1}{N_{sys}}}, \quad i = 1, 2, ..., N_{sys}.$$

Subject to the following initial conditions

$$y_i(0) = e^{\frac{i}{N_{sys}}}, \quad \dot{y}_i(0) = e^{\frac{i}{N_{sys}}}, \quad i = 1, 2, ..., N_{sys}, \tag{35}$$

for that specific choice of $F(t)$ the exact solution is

$$y_i(t) = e^{t+\frac{i}{N_{sys}}}, \quad i = 1, 2, ..., N_{sys}.$$

We solve the problem (34) for $N_{sys} = 100$, on uniform meshes having Nstep $= 5, 6, .., 25$ steps and $p = 1, 2, ..., 5$. In Figure 4 and 5, we plot the maximum error at the Jacobi points versus Nstep to observe an $O(\Delta t^{p+2})$ superconvergence rate at the roots of Jacobi polynomial $P_{p-1}^{2,0}$.
The global $\mathcal{L}^2$ norm of error $u - U$ presented in Figure 6 versus the number of steps Nstep converges algebraically at the expected optimal rate of $p + 1$ for $i = 7, 39, 72$.
Moreover, the maximum error at the downwind end-points $||e^-||_{\infty,*}$ and the maximum of the derivative of the error at the downwind end-points $||e'^-||_{\infty,*}$ versus the number of steps Nstep shown in Figure 7 and 8 confirm the $O(\Delta t^{2p})$ superconvergence rate at the end points.

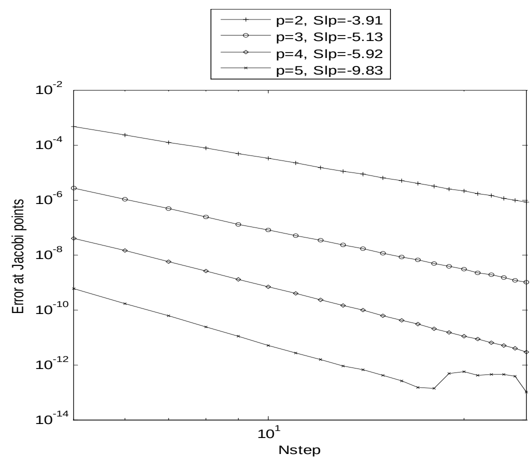


Figure 4: the 7^{th} component of maximum error $u - U$ at Jacobi points on all steps versus Nstep in log-log scale.

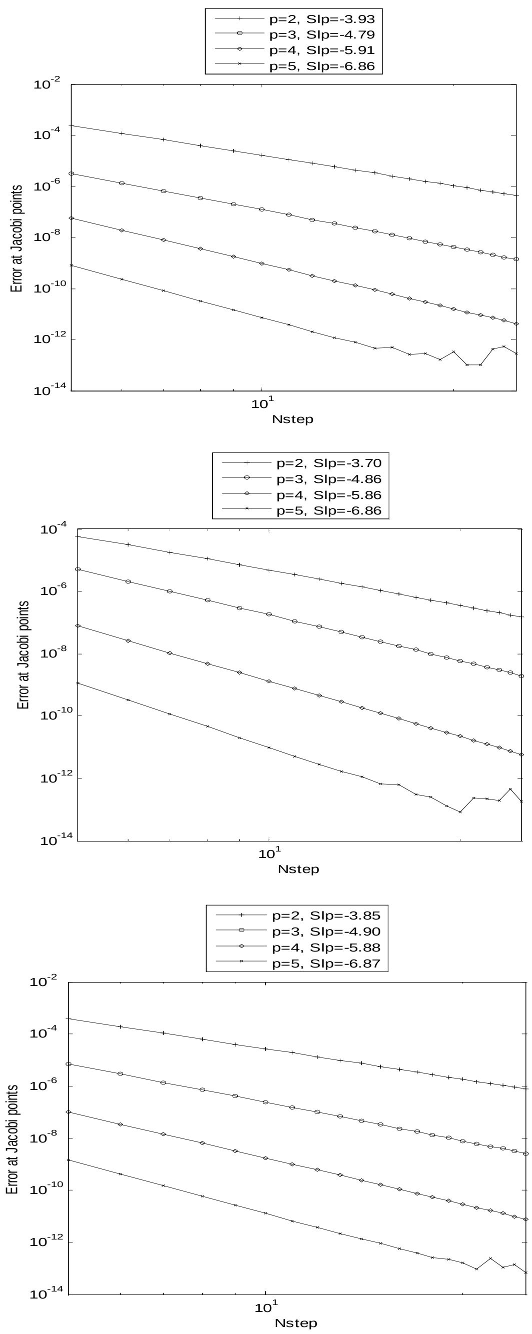


Figure 5: the 39^{th}, 72^{th} and 98^{th} (top to bottom) component of maximum error $u - U$ at Jacobi points on all steps versus Nstep in log-log scale.

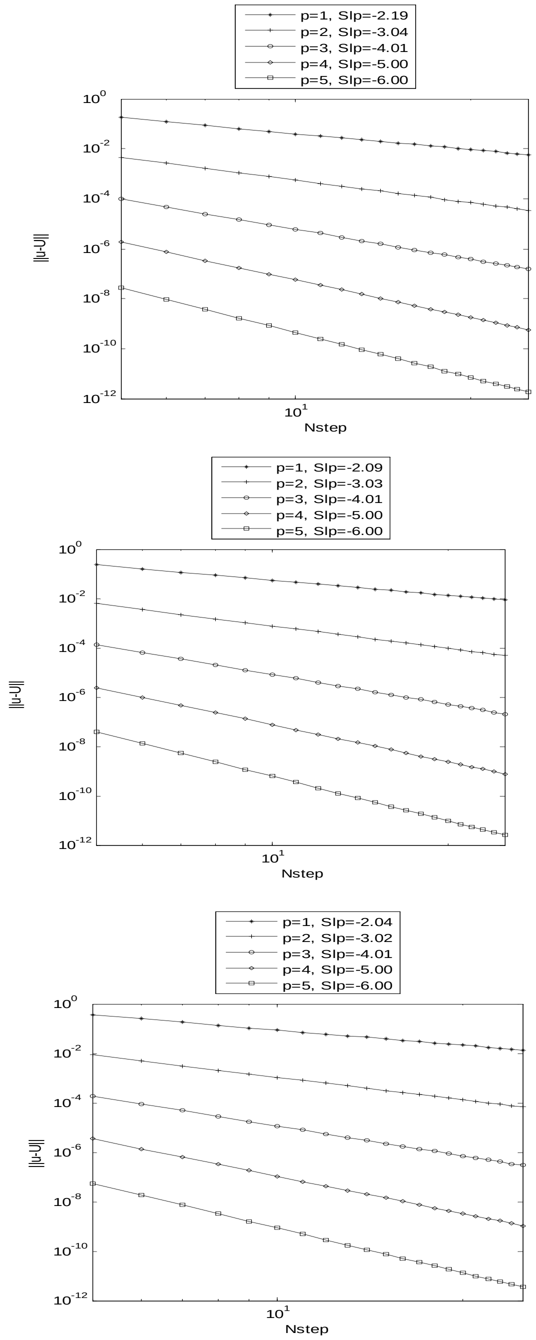


Figure 6: The $\mathcal{L}^2$ of the 7^{th}, 39^{th} and 72^{th} (top to bottom) component of the error $u - U$ versus Nstep in log-log scale.

3.2 Runge-Kutta Method

In order to have a better evaluation of the DG method, we present a short reminder for the numerical integrator used in the MATLAB Simulink™ tool. Runge-Kutta methods are very popular because of their good efficiency and are used in most computer programs for differential equations. They are single-step methods, as the Euler methods. Higher order differential equations can be treated as if they were a set of first-order equations. Let us recall the following equation

$$x_i' = f(t, x_i)$$

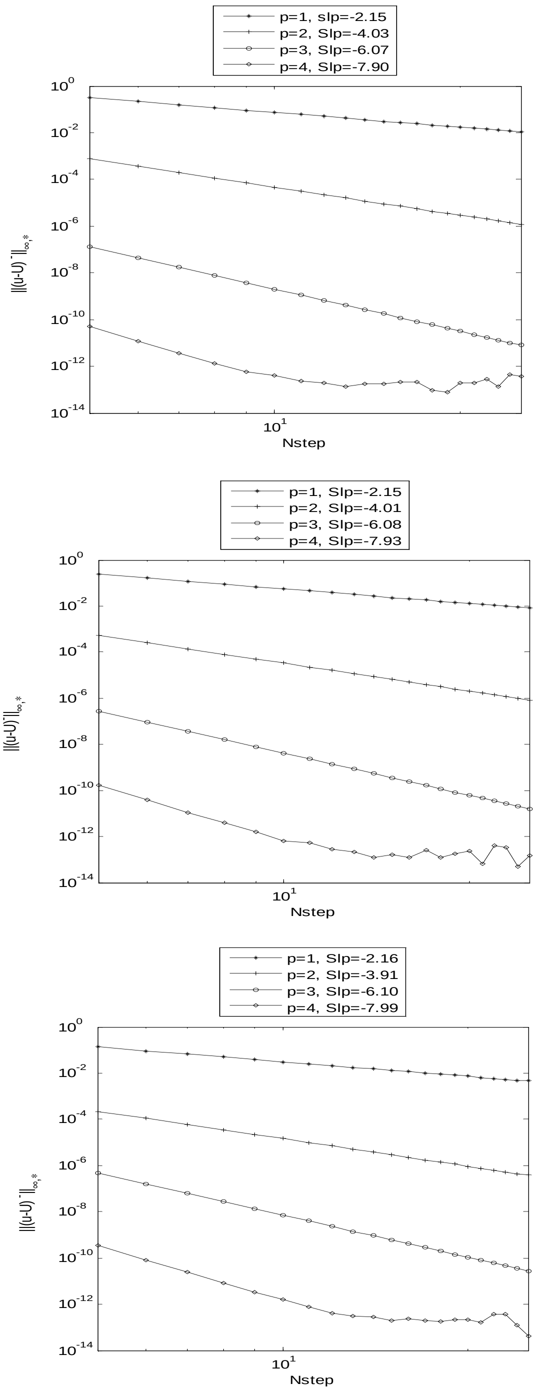


Figure 7: The 7^{th}, 39^{th} and 72^{th} (top to bottom) component of the maximum error $||e^{-}||_{\infty,*}$ at the end-points versus Nstep in log-log scale.

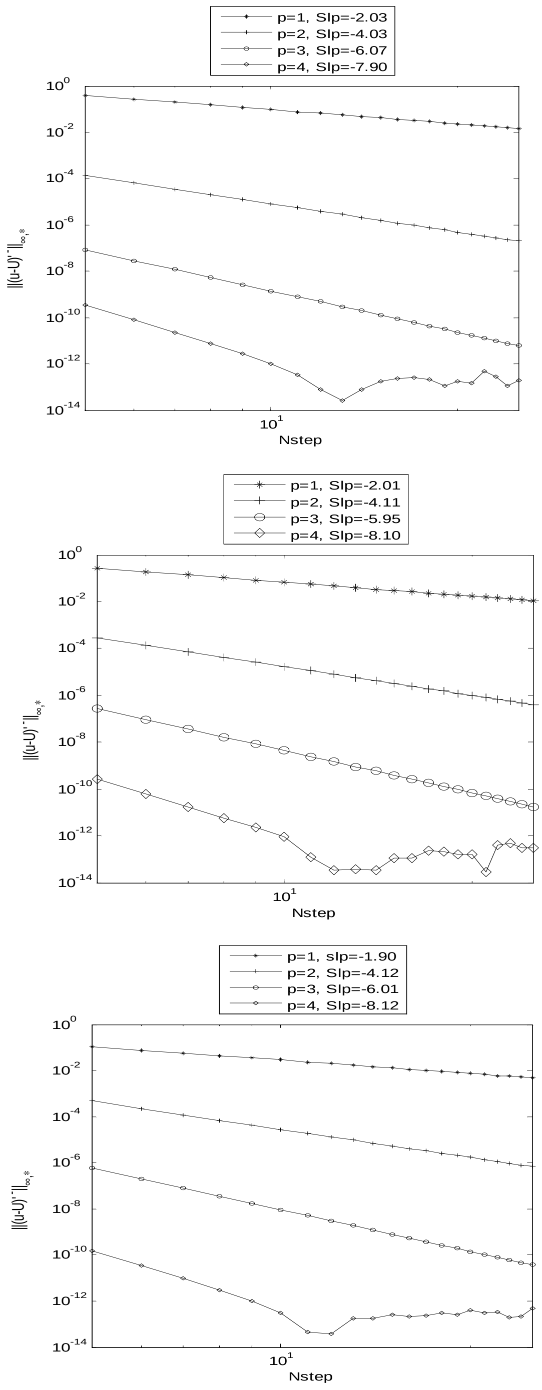


Figure 8: The 7^{th}, 39^{th} and 72^{th} (top to bottom) component of the maximum of the derivative of the error $||e'^{-}||_{\infty,*}$ at the end-points versus Nstep in log-log scale.

Then, the numerical solution obeys

$$
\begin{aligned}
&\Delta x = \tfrac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \\
&k_1 = \Delta t\left[f\left(t, x_i\right)\right] \\
&k_2 = \Delta t\left[f\left(t + \tfrac{1}{2}\Delta t, x_i + \tfrac{1}{2}k_1\right)\right] \\
&k_3 = \Delta t\left[f\left(t + \tfrac{1}{2}\Delta t, x_i + \tfrac{1}{2}k_2\right)\right] \\
&k_4 = \Delta t\left[f\left(t + \Delta t, x_i + k_3\right)\right]
\end{aligned}
$$

$$X_i(t_{n+1}) = X_i(t_n) + \Delta x \qquad (36)$$

This method results in a local error of $O(\Delta t^5)$ and a global error of $O(\Delta t^4)$.

4 Response of a Multi-Story Seismically Excited Building

In the following section we are investigating the response of a hundred-story building excited by to the S00E component of El-Centro, Imperial Valley Earthquake, 1940, normalized to a peak ground acceleration of $0.4g$ ($g = 9.81\text{m/s}^2$). In order to apply our DG method, we transform the data recorder by the seismograph to a piecewise linear functions that will generate a quasi-real linear signal as shown in Figure 9.

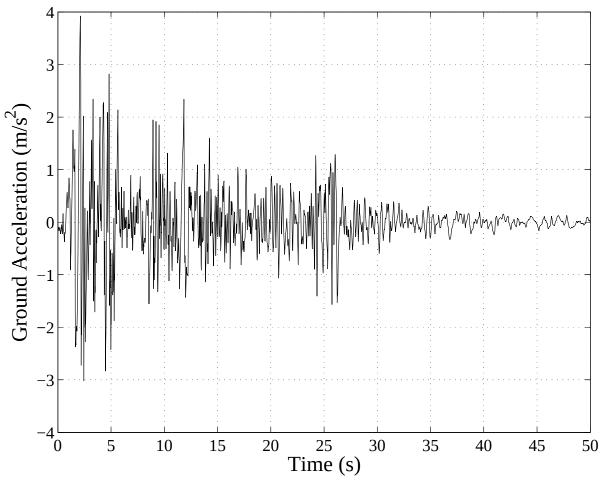


Figure 9: El-Centro Earthquake scaled to $0.4g$ (NS components).

The hundred-story building considered in this paper has the following features for each floor:

- 1^{st} story: m_1=6.00 10^5 Kg, k_1=10.180 10^6 KN/m and c_1=6.50 10^7 kN.sec/m.
- 2^{nd} story: m_2=6.00 10^5 Kg, k_2=10.080 10^6 KN/m and c_2=6.435 10^7 kN.sec/m.
- ...
- i^{th} story: m_i=6.00 10^5 Kg, k_i=(10.180-0.1(i-1)) 10^6 KN/m and c_i=(6.50-0.065(i-1)) 10^7 kN.sec/m.
- ...
- 100^{th} story: m_{100}=6.00 10^5 Kg, k_{100}=5.28 10^6 KN/m and c_{100}=3.315 10^7 kN.sec/m.

We are solving the problem posed for (4) between $t = 0$ and $t = 50$ seconds which is the duration of earthquake signal in order to investigate the behavior of the structure which the worst during the excitation, with a number of steps $Nstep = 1000$ (Δt=0.05 sec) and using a degree of approximation $p = 3$.
Since we are getting the information from the downwind end point where the error of the DG method is $O(\Delta t^{2p})$ superconvergent at the end of each step (Theorem 4). Then, the accuracy of the method is
Accuracy=Δt^{2p}=0.05^6=1.5 10^{-8}.

To check the efficiency of the proposed method, we adopt the MATLAB Simulink™ tool to compare it with our method using as the *ODE45* Runge-Kutta numerical integrator with Tolerence=10^{-8}. The following diagram exhibits the Simulink™ block diagram use to obtain the numerical results presented in this work.

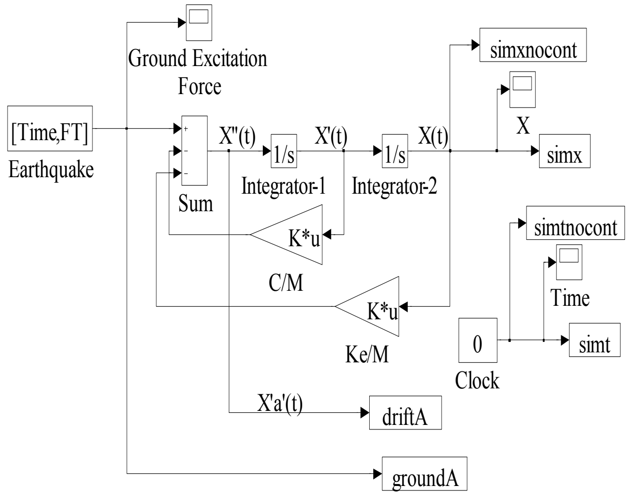


Figure 10: MATLAB Simulink™ block diagram.

In order to show the result of the earthquake excitation on the behavior of the structure, we plot in Figure 11 and 12, the inter-story displacement of the highest levels where we expect the major effect to happen using the DG method and the MATLAB Simulink™ tools.
Moreover, the acceleration of each floor is also an efficient indicator of the behavior of the building to the proposed excitation. In Figure 13, we exhibit the acceleration of the 99th and 100th stories.

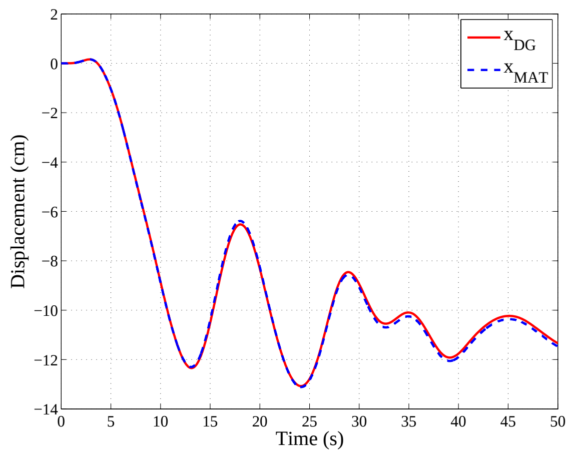
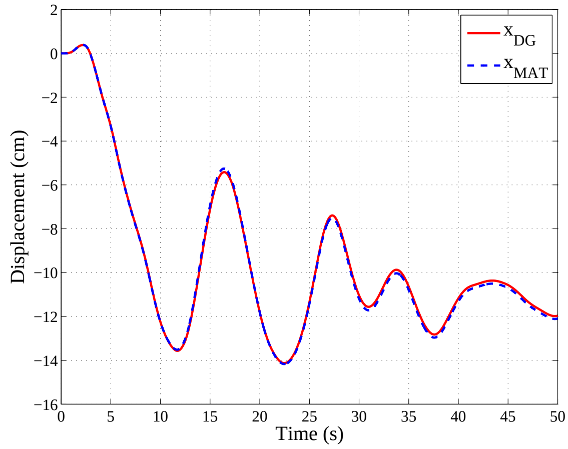
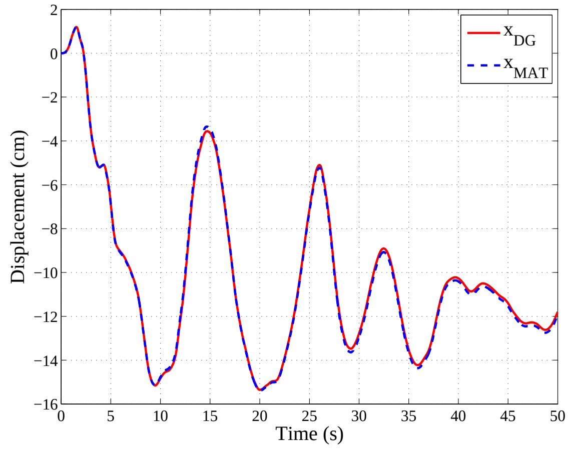


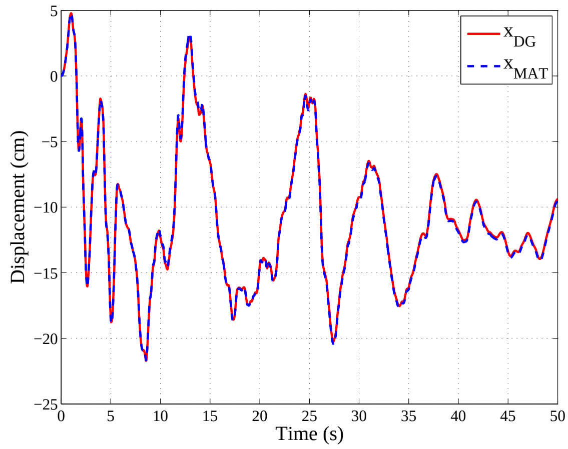
Figure 11: The inter-story displacement of 97th story.



(a) Displacement of 98th story.

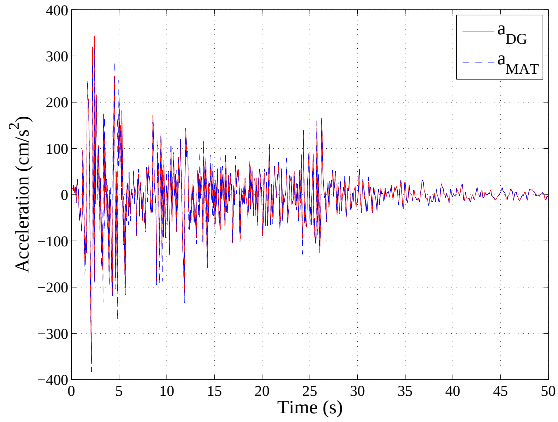


(b) Displacement of 99th story.

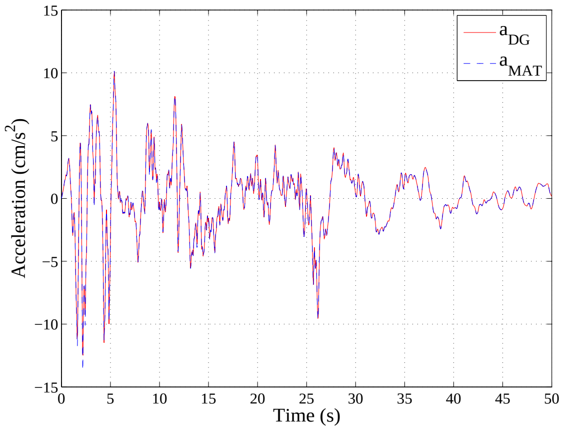


(c) Displacement of 100th story.

Figure 12: The inter-story displacement.



(a) Acceleration of 99th story.



(b) Acceleration of 100th story.

Figure 13: The inter-story acceleration

Table 1: Comparison between the two methods.

Story	Time_{DG} (s)	Time_{MATLAB}(s)
10	1.2418	2.3133
20	2.5236	3.4950
30	4.0959	5.5880
40	5.6782	8.2318
50	7.4006	11.1661
60	9.2433	14.2305
70	11.1360	17.3550
80	13.2290	20.9601
90	14.9415	24.7456
100	17.2748	28.9116
130	24.2449	41.1291
160	32.0160	60.0664
200	42.7815	90.2698

The following Table 1 presents the efficiency of the DG method compared to the MATLAB Simulink™ tool in term of Time consumed to solve the problem computed on a Pentium M processor 1.6 GHz with a RAM of 1GHz. In addition to the stability, robustness and approximation flexibility, the elapsed time for the DG solver is smaller than the efficient MATLAB™ tools. Therefore the DG method remain very attractive to the engineers, however, it is not as known as the classical continuous finite element methods.

5 Conclusion

In this paper we adopted the Discontinuous Galerkin (DG) method to solve for the response of a seismically excited building model. This is a new scheme to solve a large system of ordinary differential equations as the proposed technique exhibits high efficiency and stability when compared with traditional numerical techniques. The efficiency of the method proposed is demonstrated through numerical simulations. In this paper, the DG method was used to solve for the response of a linear, one dimensional structure. However, this work can be extended to nonlinear systems as well as three dimensional structural models. The DG method can also be used to solve nano-scale problems where high accuracy and stability of the numerical solution are highly demanded.

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