The Algorithm and Application of the Beta Function and Its Partial Derivatives

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***Abstract*—It is well-known that many generalized integrals can be expressed by the Beta function $B(x, y)$. In this paper, some relations between the generalized integrals and partial derivatives $B_{p,q}(x, y)$ of the Beta function $B(x, y)$ are given. Moreover, an algorithm for computing $B(x, y)$ and $B_{p,q}(x, y)$ has been developed. Finally, numerical examples show that the algorithm can be applied to compute some generalized integrals, which can improve the rate and precision of computing the generalized integrals.**

***Index Terms*—Riemann zeta function, Beta function, Digamma function, Hurwitz zeta function.**

I. Introduction

The Beta function $B(x, y)$ was defined by the following integral in [1]

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{1}$$

for $Rex > 0$ and $Rey > 0$. Moreover, many generalized integrals can be expressed by $B(x, y)$ in [2]. In fact, many mathematical softwares such as Mathematica, Maple and Matlab can also be applied to achieve the closed form of the generalized integrals. However, the calculation process is very time-consuming. Sometimes it is very difficult to derive the closed form for the generalized integrals. Meanwhile, some integrals can be analytically expressed by the Riemann zeta function by using the closed form. They can also be calculated by calling Integrate or NIntegrate in Mathematica.

Note that the values of x and y must be non-negative real numbers for $B(x, y)$ in Matlab, while they may be complex numbers in Mathematica and Maple. In Matlab, Mathematica and Maple, $B(x, y)$ is given as follows:

$$\begin{gathered} B(-n, y) = \infty, B(x, -m) = \infty, B(-n, -m) = \infty, \\ n, m = 0, 1, 2, \cdots. \end{gathered} \tag{2}$$

and

$$\begin{gathered} B(-1, \tfrac{1}{2}) = \infty, B(-\tfrac{3}{2}, \tfrac{1}{2}) = 0, \\ B(-1, \tfrac{5}{2}) = \infty, B(-\tfrac{3}{2}, \tfrac{5}{2}) = \pi. \end{gathered} \tag{3}$$

However, the above results are unreasonable. In order to remedy the unreasonable results, the additional definition of $B(x, y)$ was given in [3] by the neutrix calculus in [4-6]. Furthermore, some recurrence formulas of the partial derivatives $B_{p,q}(x, y)$ of the Beta function $B(x, y)$ were also obtained in [3], where $B_{p,q}(x, y) = \frac{\partial^{p+q}}{\partial x^p \partial y^q} B(x, y)(p, q = 0, 1, 2, \cdots)$.

The structrue of this paper is as follows. In Section 2, the additional definition of $B(x, y)$ and some recurrence formulas of $B_{p,q}(x, y)$ are obtained. In Section 3, an algorithm for calculating $B(x, y)$ and $B_{p,q}(x, y)$ is given. In Section 4, some relations between $B_{p,q}(x, y)$ and generalized integrals are obtained. Moreover, some numerical examples are given. The conclusion is given in last section.

Manuscript received February 6, 2015. This work was supported by National Natural Science Foundation of China under Grant No. 61379009.

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II. The additional definition of $B(x, y)$ and some recurrence formulas of $B_{p,q}(x, y)$

The following additional definition of $B(x, y)$ and recurrence formulas of $B_{p,q}(x, y)$ were obtained by Shang in [3].

Definition 2.1 Let m, n be integers and x, y be complex numbers. Then

1)

$$\begin{aligned} B(n, -m) &= B(-m, n) = \sum_{l=0, l\neq m}^{n-1} C_{n-1}^{l} \tfrac{(-1)^l}{l-m} \\ &= \begin{cases} t_1, & n = 1, 2, \cdots, m, \quad m = 1, 2, \cdots, \\ t_2, & n = m+1, m+2, \cdots, \quad m = 1, 2, \cdots, n. \end{cases} \end{aligned} \tag{4}$$

where

$$t_1 = \frac{(-1)^m (m-1)!(n-m)!}{n!},$$

and

$$t_2 = \frac{(-1)^n (m-1)! \,(H_n - H_{m-n-1})}{n!(m-n-1)!}.$$

2)

$$\begin{gathered} B(-n, y) \\ = (-1)^n C_{y-1}^{n} \left((y-n-1)B_{0,1}(y-n-1, 1) + H_n\right), \\ y \neq n+1, n, \cdots, 0, -1, -2, \cdots, n = 1, 2, \cdots. \end{gathered} \tag{5}$$

3)

$$\begin{gathered} B(x, -m) = B(-m, x), \\ x \neq m+1, m, \cdots, 0, -1, -2, \cdots, m = 1, 2, \cdots. \end{gathered} \tag{6}$$

4)

$$\begin{gathered} B(-n, -m) \\ = -\sum_{i=0}^{m-1} \begin{pmatrix} n+i \\ i \end{pmatrix} \tfrac{1}{m-i} - \sum_{j=0}^{n-1} \begin{pmatrix} m+j \\ j \end{pmatrix} \tfrac{1}{n-j}, \\ n, m = 1, 2, \cdots. \end{gathered} \tag{7}$$

where $H_n = \sum_{l=1}^{n} \frac{1}{l}$.

For integers q, p, n, m satisfying $q, p \geq 1$ and $n, m \geq 0$, the following theorem is obtained.

Theorem 2.1 1) Let x and y be complex numbers satisfying $x, y, x+y \neq 0, -1, -2, \cdots$. Then

$$\begin{aligned} &B_{p,q}(x, y) \\ &= \sum_{j=0}^{q-1} C_{q-1}^{j} \left(\psi^{(q-1-j)}(y) - \psi^{(q-1-j)}(x+y)\right) B_{p,j}(x, y) \\ &\quad - \sum_{k=0}^{p-1} C_p^k \sum_{j=0}^{q-1} C_{q-1}^{j} \psi^{(p+q-1-k-j)}(x+y) B_{k,j}(x, y), \end{aligned} \tag{8}$$

where $\psi(x)$ is the Digamma function defined by

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=-\gamma-\frac{1}{x}+\sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{l+x}\right),$$

and

$$\psi^{(p)}(x)=\frac{d^p}{dx^p}\psi(x)(p=0,1,2,\cdots),$$

and γ denotes Euler-Mascheroni constant.

2) Let y be a complex number satisfying $y\neq 0,-1,-2,\cdots$. Then

$$\begin{aligned}B_{p,q}(-n,y)&=B_{q,p}(y,-n)\\&=\tfrac{1}{(p+1)a_{n+1,1}(-n)}\sum_{u=0}^{p+1}C_{p+1}^{u}\sum_{v=0}^{q}C_q^v\\&\cdot a_{n+1,p+q+1-u-v}(y-n)B_{u,v}(1,y)\\&-\tfrac{1}{(p+1)a_{n+1,1}(-n)}\sum_{u=0}^{p-1}C_{p+1}^{u}a_{n+1,p+1-u}(-n)\\&\cdot B_{u,q}(-n,y)\end{aligned}\tag{9}$$

and

$$\begin{aligned}&B_{p,q}(-n,-m)=B_{q,p}(-m,-n)\\&=\sum_{u=0}^{p+1}C_{p+1}^{u}\sum_{v=0}^{q+1}C_{q+1}^{v}\\&\cdot\tfrac{(-1)^{n+m}a_{n+m+2,p+q+2-u-v}(-n-m)B_{u,v}(1,1)}{(q+1)(p+1)n!m!}\\&-\tfrac{(-1)^n}{(p+1)n!}\sum_{u=0}^{p-1}C_{p+1}^{u}a_{n+1,p+1-u}(-n)B_{u,q}(-n,-m)\\&-\tfrac{(-1)^m}{(q+1)m!}\sum_{v=0}^{q-1}C_{q+1}^{v}a_{m+1,q+1-v}(-m)B_{u,v}(-n,-m)\\&-\sum_{u=0}^{p-1}C_{p+1}^{u}\sum_{v=0}^{q-1}C_{q+1}^{v}\\&\cdot\tfrac{(-1)^{n+m}a_{n+1,p+1-u}(-n)a_{m+1,q+1-v}(-m)B_{u,v}(-n,-m)}{(q+1)(p+1)n!m!},\end{aligned}\tag{10}$$

where

$$\begin{aligned}a_{n,i}(x)&=\tfrac{d^i}{dx^i}(x)_n\\&=i!\sum_{k=i}^{n}C_k^i(-1)^{n-k}s(n,k)x^{k-i},i=1,2,\cdots,\end{aligned}\tag{11}$$

$$(x)_n=x(x+1)\cdots(x+n-1)=\sum_{k=1}^{n}(-1)^{n-k}s(n,k)x^k,\tag{12}$$

and $s(n,k)$ is the Stirling number of the first kind.

3) Let x and y be complex numbers satisfying $x+y=0,-1,-2,\cdots$ and $Rex\neq 0,-1,-2,\cdots$. Then

$$\begin{aligned}B_{p,q}(x,y)&=\tfrac{1}{(x)_n(y)_m}\sum_{u=0}^{p}C_p^u\sum_{v=0}^{q}C_q^v\\&\cdot a_{n+m,p+q-u-v}(x+y)B_{u,v}(x+n,y+m)\\&-\tfrac{1}{(y)_m}\sum_{v=0}^{q-1}C_q^v a_{m,q-v}(y)B_{p,v}(x,y)\\&-\tfrac{1}{(x)_n}\sum_{u=0}^{p-1}C_p^u a_{n,p-u}(x)B_{u,q}(x,y)\\&-\tfrac{1}{(x)_n(y)_m}\sum_{u=0}^{p-1}C_p^u\sum_{v=0}^{q-1}C_q^v a_{n,p-u}(x)\\&\cdot a_{m,q-v}(y)B_{u,v}(x,y),\end{aligned}\tag{13}$$

where n satisfies $0<Re(x+n)<1$ for $Rex\leq 0$ or $n=0$ for $Rex>0$ and n satisfies $0<Re(y+m)<1$ for $Rey\leq 0$ or $m=0$ for $Rey>0$.

Now some identities for the Digamma function $\psi(x)$ are obtained.

$$\begin{aligned}\psi(n+x)&=\psi(x)+\sum_{l=0}^{n-1}\tfrac{1}{(l+x)},\\\psi(x-n)&=\psi(x)+\sum_{l=1}^{n}\tfrac{1}{(l-x),}\end{aligned}\tag{14}$$

$$\psi^{(k)}(x)=k!(-1)^{k+1}\zeta(k+1,x),\qquad k=1,2,\cdots.\tag{15}$$

and

$$\begin{aligned}&\psi^{(k)}(n+x)=k!(-1)^{k+1}\zeta(k+1,x)\\&\qquad+(-1)^k k!\sum_{l=0}^{n-1}\tfrac{1}{(l+x)^{k+1}},\qquad k=1,2,\cdots.\\&\psi^{(k)}(x-n)=k!(-1)^{k+1}\zeta(k+1,x)\\&\qquad+k!\sum_{l=1}^{n}\tfrac{1}{(l-x)^{k+1}},\qquad k=1,2,\cdots.\\&\psi^{(k)}(n)=\begin{cases}-\gamma+H_{n-1}, & k=0,\\ \left(\zeta(k+1)-H_{n-1}^{(k+1)}\right)\\ \cdot k!(-1)^{k+1}, & k=1,2,\cdots.\end{cases}\\&\psi^{(k)}(-n)=\begin{cases}-\gamma+H_n, & k=0,\\ (-1)^{k+1}k!\zeta(k+1)\\ +k!H_n^{(k+1)}, & k=1,2,\cdots.\end{cases}\end{aligned}\tag{16}$$

where $H_n^{(s)}=\sum_{l=1}^{n}\frac{1}{l^s}(s=1,2,\cdots)$, $H_n=H_n^{(1)}$ and $\zeta(s)=\sum_{l=1}^{\infty}\frac{1}{l^s}(s=1,2,\cdots)$ is the Riemann zeta function and $\zeta(s,x)$ is the Hurwitz zeta function defined by

$$\zeta(s,x)=\sum_{l=0}^{\infty}\frac{1}{(l+x)^s}.$$

Moreover, the following identity of $\psi(x)$ was also given in [2].

$$\begin{aligned}\psi(\tfrac{p}{q})&=-\gamma-\ln(2q)-\tfrac{\pi}{2}\cot\tfrac{p\pi}{q}\\&+\sum_{k=1}^{[\frac{q+1}{2}]-1}\left[\cos\tfrac{2kp\pi}{q}\ln\sin\tfrac{k\pi}{q}\right].\end{aligned}\tag{17}$$

for $q=2,3,\cdots$ and $p=1,2,\cdots,q-1$.

Similarly, we have the following identities of the Hurwitz zeta function $\zeta(s,x)$:

$$\begin{aligned}\zeta(s,n+x)&=\zeta(s,x)-\sum_{l=0}^{n-1}\tfrac{1}{(l+x)^s},\\\zeta(s,-n+x)&=\zeta(s,x)+\sum_{l=1}^{n}\tfrac{1}{(x-l)^s},\\\zeta(s,\tfrac{1}{2})&=(2^s-1)\zeta(s).\end{aligned}\tag{18}$$

In particular, the following results were given in [7],

$$\begin{aligned}\zeta(k,0)&=\begin{cases}\gamma, & k=1,\\ \zeta(k), & k>1.\end{cases}\\\zeta(k,\tfrac{1}{2})&=\begin{cases}\gamma+2\ln 2, & k=1,\\ \left(2^k-1\right)\zeta(k), & k>1.\end{cases}\end{aligned}\tag{19}$$

$$\left.\begin{matrix}\zeta(2n+1,\frac{1}{3})\\ \zeta(2n+1,\frac{2}{3})\end{matrix}\right\}=\tfrac{3^{2n+1}-1}{2}\zeta(2n+1)\pm\tfrac{\sqrt{3}}{2\pi}I_1,\tag{20}$$

$$\left.\begin{matrix}\zeta(2n+1,\frac{1}{4})\\ \zeta(2n+1,\frac{3}{4})\end{matrix}\right\}=2^{2n}(2^{2n+1}-1)\zeta(2n+1)\pm\tfrac{1}{2\pi}I_2,\tag{21}$$

and

$$\left.\begin{array}{c}\zeta(2n+1,\frac{1}{6})\\ \zeta(2n+1,\frac{5}{6})\end{array}\right\} = \frac{6^{2n+1}-3^{2n+1}-2^{2n+1}+1}{2}\zeta(2n+1) \pm \frac{1}{2\sqrt{3}\pi}I_3, \tag{22}$$

where

$$\begin{aligned} I_1 &= \left(2n+2+3^{2n+2}\right)\zeta(2n+2) \\ &\quad -2\sum_{l=0}^{n-1} 3^{2n-2l}\zeta(2n-2l)\zeta(2l+2), \\ I_2 &= \left(2n+2+4^{2n+2}\right)\zeta(2n+2) \\ &\quad -2\sum_{l=0}^{n-1} 4^{2n-2l}\zeta(2n-2l)\zeta(2l+2), \\ I_3 &= \left(6^{2n+2}-3^{2n+2}\right)\zeta(2n+2) \\ &\quad -2\sum_{l=0}^{n-1}\left(6^{2n-2l}-3^{2n-2l}\right)\zeta(2n-2l)\zeta(2l+2), \end{aligned}$$

and $\zeta(1) = \gamma$.

Remark 2.1 It follows from the above results that $B_{p,q}(x,y)$ certainly has a closed form for $x, y = \pm n, \frac{1}{2} \pm n$ and $n = 0, 1, 2, \cdots$. If $x+y$ and y are rational numbers, then $B_{p,q}(x,y)$ has a closed form expressed by (8) and (17). Moreover, if $x, y = \frac{1}{3} \pm n, \frac{1}{4} \pm n, \frac{1}{6} \pm n$ and $n = 0, 1, 2, \cdots$, then $B_{p,q}(x,y)$ may have the closed form. Otherwise, $B_{p,q}(x,y)$ does not seem to have the closed form for non-negative integers p and q. However, $B_{p,q}(x,y)$ can always be expressed by the Hurwitz zeta function $\zeta(s,x)$.

III. The algorithm for calculating $B(x,y)$ and $B_{p,q}(x,y)$

The program BetaD$[x,y,p,q,all]$ can be used to calculate $B(x,y)$ and $B_{p,q}(x,y)$, where BetaD$[x,y,p,q,all]$ includes the following five key subprograms: PolyGammaAmend$[k,x]$, BetaAll$[x,y]$, PochhammerD$[k,x]$, BetaD1$[x,y,p,q,all]$ and BetaD2$[x,y,p,q,all]$. In the following, we will give the function of five key subprograms, respectively.

1) PolyGammaAmend$[k,x]$ performs the calculation of (14)$\sim$(22) for $x = a \pm n$, $a = 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}, \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}, \cdots$ and $n = 0, 1, 2, 3, \cdots$. Otherwise, PolyGamma from Mathematica replaces PolyGammaAmend.

2) BetaAll$[x,y]$ performs the calculation of (4)$\sim$(7) when x or $y = 0, -1, -2, \cdots$. Otherwise, Beta from Mathematica replaces BetaAll.

3) PochhammerD$[k,x]$ performs the calculation of (11).

4) BetaD1$[x,y,p,q,all]$ can be used to calculate $B_{p,q}(x,y)$ by using (8) for complex numbers x and y satisfying $x, y, x+y \neq 0, -1, -2, \cdots$. All the values of $B_{i,j}(x,y)$ for $i = 0, 1, 2, \cdots, p$ and $j = 0, 1, 2, \cdots, q$ can be displayed when the parameter all is a positive number. However, the value of $B_{i,j}(x,y)$ for $i = p$ and $j = q$ can only be displayed when the parameter all is zero.

5) BetaD2$[x,y,p,q,all]$ can be used to calculate $B_{p,q}(x,y)$ by using (9), (10) and (13), where two subprograms BetaAll$[x,y,all]$ and PochhammerD$[k,x]$ can also be used.

Note that the above algorithms can run for symbolic computation in Mathematica. However, we need add a letter "N" in the demand for numerical integration. For example, change BetaAll to NBetaAll, then the above algorithm can run in the $Prec$, where $Prec$ denotes a public constant and calculation precision.

IV. The partial derivatives of the Beta function and related generalized integrals

Some scholars have shown that some generalized integrals can be expressed by $B(x,y)$. For example, the following relations between $B(x,y)$ and generalized integrals were given in [2].

$$\int_0^\infty \frac{\cosh 2yt}{\cosh^{2x} zt}dt = \frac{4^{x-1}}{z}B(x+\frac{y}{z}, x-\frac{y}{z}), \quad \left[Re\left(x+\frac{y}{z}\right) > 0, Re\left(x-\frac{y}{z}\right) > 0.\right] \tag{23}$$

$$\int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}}dt = \frac{B(x,y)}{z^y(1+z)^x}, \quad [Rex, Rey > 0, Re(x+y) < 1, -1 < z < 0.] \tag{24}$$

$$\int_{-\infty}^\infty e^{2yt}\cosh^{-2x}(t-a)dt = \frac{e^{2ay}}{2^{1-2x}}B(x+y, x-y), \quad [Rex > 0, y, a \ are \ real.] \tag{25}$$

and

$$\int_0^\infty \cosh^{-2x} t \cosh 2yt dt = 4^{x-1}B(x+y, x-y), \quad [Rex > |Rey|, Rex > 0.] \tag{26}$$

By (1) and (23)$\sim$(26), the following generalized integrals can also be expressed by $B_{p,q}(x,y)$.

1) Let p and q be non-negative integers and x, y be complex numbers satisfying $q + Rex > 0$ and $p + Rey > 0$. Then

$$\int_0^1 t^{x-1}(1-t)^{y-1}\ln^p t \ln^q(1-t)dt = B_{p,q}(x,y). \tag{27}$$

For example, the left integral of (27) can be calculated by calling Integrate in Mathematica (I_1) and the algorithm given in Section 3 (B_1), respectively. Now the time-consumption of computing the left integral of (27) is shown in Table I by different methods in Mathematica.

Table I The time − consumption of computing the left integral of (27) (Prec = 64)

x, y	p, q			$Time(second)$
$-\frac{3}{2}, \frac{1}{2}$	4, 6	I_1 B_1	closed form	52.859375 0.062500
$-\frac{3}{2}, \frac{1}{2}$	6, 6	I_1 B_1	closed form	118.203125, Unfinished 0.125000
3, −2	4, 6	I_1 B_1	closed form	109.403501 0.093601
3, −2	6, 6	I_1 B_1	closed form	275.372965, Unfinished 0.171601
$\frac{5}{2}, -\frac{9}{2}$	5, 5	I_1 B_1	closed form	130.354436 0.124801
$\frac{5}{2}, -\frac{9}{2}$	6, 6	I_1 B_1	closed form	130.026834, Unfinished 0.280802

Here "Unfinished" denotes that we can not obtain the results in running the algorithm for a long time.

From Table I, we can see that B_1 method is much better than I_1 in calculating the closed form.

Moreover, the running time *(unit: second)* and the relative error of computing the left integral of (27) can also be obtained with different precision for numerical integration. The comparison of the two numerical methods in Mathematica are listed in Table II and Table III.

Table II Comparison of computing the left integral of (27) for two numerical methods with different precision (Prec = 32 and Prec = 64)

x, y, p, q		T_{32}, r_{32}	T_{64}, r_{64}
1, 1, 6, 6	NI NB	$0.046800, 10^{-32}$ $0.015600, 10^{-32}$	$0.109201, 10^{-65}$ $0., 10^{-72}$
$2, \frac{5}{3}, 4, 5$	NI NB	$0.046800, 10^{-32}$ $0.015600, 10^{-36}$	$0.156001, 10^{-65}$ $0.015600, 10^{-76}$
$-\frac{8}{3}, -\frac{4}{5}, 1, 3$	NI NB	$0.078001, 10^{-15}$ $0.015600, 10^{-39}$	$0.171601, 10^{-21}$ $0.015600, 10^{-79}$
$\frac{1}{4}, -\frac{7}{2}, 4, 2$	NI NB	$0.078001, 10^{-32}$ $0., 10^{-35}$	$0.312002, 10^{-22}$ $0., 10^{-75}$

Table III Comparison of computing the left integral of (27) for two numerical methods with different precision($Prec = 128$ and $Prec = 256$)

x, y, p, q		T_{128}, r_{128}	T_{256}, r_{256}
$1, 1, 6, 6$	NI	$0.421203, 10^{-129}$	$1.825212, 10^{-257}$
	NB	$0.015600, 10^{-152}$	$0.015600, 10^{-312}$
$2, \frac{5}{3}, 4, 5$	NI	$0.546003, 10^{-129}$	$2.199614, 10^{-257}$
	NB	$0.015600, 10^{-156}$	$0., 10^{-316}$
$-\frac{8}{3}, -\frac{4}{5}, 1, 3$	NI	$0.358802, 10^{-24}$	$1.014007, 10^{-39}$
	NB	$0.015600, 10^{-159}$	$0.062400, 10^{-319}$
$\frac{1}{4}, -\frac{7}{2}, 4, 2$	NI	$0.592804, 10^{-27}$	$1.918812, 10^{-34}$
	NB	$0., 10^{-155}$	$0., 10^{-315}$

Here NI and NB represent the calling NIntegrate in Mathematica and the algorithm given in the Section 3 of this paper, respectively. Moreover, T_P and r_P represent the running time*(unit: second)* and the relative error with the precision P, respectively. Comparing to NI method, Table II and Table III show that NB method does not only improve the accuracy up to the specified precision, but also reduces the time-consuming effectively.

2) Let p be non-negative integers and x, y, z be complex numbers satisfying $Re\left(x + \frac{y}{z}\right) > 0$ and $Re\left(x - \frac{y}{z}\right) > 0$. Then

$$\int_0^\infty \frac{\cosh(2yt)\ln^p\cosh(zt)}{\cosh^{2x}(zt)}dt = \frac{(-1)^p 2^{2x-2}}{z}\sum_{l=0}^{p} C_p^l \frac{\ln^{p-l}2}{2^l}\sum_{u=0}^{l} C_l^u B_{u,l-u}(x + \tfrac{y}{z}, x - \tfrac{y}{z}). \tag{28}$$

For example, the left integral of (28) can be calculated by calling Integrate or NIntegrate in Mathematica (I_2) and the algorithm given in Section 3 (B_2), respectively. Now the time-consumption of computing the left integral of (28) is shown in Table IV by different methods in Mathematica.

Table IV The time − consumption of computing the left integral of (28)($Prec = 64$)

x, y, z	p			$Time(second)$
$\frac{9}{2}, -5, 2$	1	I_2	*closed form*	0.624004
		B_2		0.
$\frac{9}{2}, -5, 2$	2	I_2	*closed form*	85.691349
		B_2		0.
$\frac{9}{2}, -5, 2$	1	I_2	*numerical integral*	0.093601
		B_2		0.
$\frac{9}{2}, -5, 2$	2	I_2	*numerical integral*	0.140401
		B_2		0.
$\frac{9}{2}, -4, 2$	1	I_2	*closed form*	48.968714
		B_2		0.
$\frac{9}{2}, -4, 2$	2	I_2	*closed form*	*3 hours Unfinished*
		B_2		0.
$\frac{9}{2}, -4, 2$	1	I_2	*numerical integral*	0.093601
		B_2		0.
$\frac{9}{2}, -4, 2$	2	I_2	*numerical integral*	0.093601
		B_2		0.

Similarly, "Unfinished" is the same notation that shown in Table I. From Table IV, we can see that B_2 method reduces the time-consuming effectively for computing the left integral of (28).

3) Let p and q be non-negative integers and x, y, z be complex numbers satisfying $Rex > 0, Rey > 0$, $Re(x+y) < 1$ and $-1 < z < 0$. Then

$$\begin{aligned}&\int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}}\left(\ln t - \ln(t+z)\right)^p\left(\ln(1-t) - \ln(t+z)\right)^q dt\\ &= z^{-y}(1+z)^{-x}\sum_{j=0}^{p}(-1)^{p-j}C_p^j \ln^{p-j}(1+z)\\ &\cdot\sum_{k=0}^{q}(-1)^{q-k}C_q^k \ln^{q-k} z B_{j,k}(x,y).\end{aligned} \tag{29}$$

Moreover, we notice that if $Imz \neq 0$ or $z > 0$ and $Rex > 0, Rey > 0$, then (29) also holds. When the values of $2x$ and $2y$ are integers, the left integral of (29) must have the closed form.

For example, using the algorithm given in Section 3 , we can obtain the following result

$$\begin{aligned}&\int_0^1 \frac{\sqrt{t}(1-t)^2}{(t+z)^4\sqrt{t+z}}\left(\ln t - \ln(t+z)\right)^2\left(\ln(1-t) - \ln(t+z)\right)^2 dt\\ &= \frac{1}{z^3\sqrt{(1+z)^3}}\\ &\cdot\left(\begin{array}{l}\frac{202905805568}{4254271875} - \frac{164384\pi^2}{70875} - \frac{8\pi^4}{105} - \frac{842151424\ln 2}{40516875}\\ +\frac{9088\pi^2\ln 2}{11025} + \frac{1774592\ln^2 2}{1157625} + \frac{443648\ln^2 z}{1157625} - \frac{21536\zeta(3)}{1575}\\ +\frac{128\zeta(3)\ln 2}{15} + \frac{18176\ln(1+z)\ln^2 2}{11025} + \frac{64\ln(1+z)\pi^2\ln 2}{105}\\ +\frac{16\ln^2(1+z)\ln^2 z}{105} + 2\ln(1+z)\left(\frac{2272\ln^2 z}{11025} - \frac{32\zeta(3)}{15}\right)\\ +\left(\frac{1211408}{1157625} - \frac{16\pi^2}{315} - \frac{12448\ln 2}{11025} + \frac{64\ln^2 2}{105}\right)\ln^2(1+z)\\ -2\ln z\left(\frac{2272\pi^2}{11025} - \frac{210537856}{40516875} + \frac{887296\ln 2}{1157625} + \frac{32\zeta(3)}{15}\right)\\ -2\ln(1+z)\left(\frac{11608\pi^2}{33075} - \frac{9927648}{1500625} + \frac{5074112\ln 2}{1157625}\right)\\ -4\left(\frac{1268528}{1157625} - \frac{8\pi^2}{105} - \frac{4544\ln 2}{11025}\right)\ln z\ln(1+z)\\ -2\left(\frac{32\ln 2}{105} - \frac{3112}{11025}\right)\ln z\ln^2(1+z)\end{array}\right).\end{aligned} \tag{30}$$

Thus,

$$\begin{aligned}&\int_0^1 \frac{\sqrt{t}(1-t)^2}{(t+z)^4\sqrt{t+z}}\left(\ln t - \ln(t+z)\right)^2\left(\ln(1-t) - \ln(t+z)\right)^2 dt\Big|_{z=\frac{1}{2}}\\ &= 0.69596709848089147579477126216622906549122278378 6\cdots.\end{aligned} \tag{31}$$

However, the closed form of (30) can not be obtained even if z is a constant by calling Integrate in Mathematica. In particular, for the left integral of (31), the error of numerical integration always exists regardless of the calculation precision for $-1 < z < 0, Rex > 0, Rey > 0$ and $Re(x+y) < 1$.

4) Let p, q be non-negative integers and x, y be complex numbers satisfying $Rex > |y|$. Then

$$\begin{aligned}&\int_{-\infty}^{\infty} t^q e^{2yt}\cosh^{-2x}(t-a)\ln^p\cosh(t-a)dt\\ &= \frac{(-1)^p e^{2ay}}{2^{1-2x}}\sum_{j=0}^{p} C_p^j \frac{\ln^{p-j}2}{2^j}\sum_{u=0}^{j} C_j^u \sum_{k=0}^{q} C_q^k \frac{a^{q-k}}{2^k}\\ &\cdot\sum_{v=0}^{k}(-1)^{k-v}C_k^v B_{u+v,j+k-u-v}(x+y, x-y).\end{aligned} \tag{32}$$

Note that $2x + 2y$ and $2x - 2y$ are integers, then the left integral of (32) must exist the closed form.

For example, we can obtain the following result by (32).

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{t^3 e^{\frac{7t}{2}}\ln^2\cosh(t-a)}{\cosh^{\frac{9}{2}}(t-a)}dt = \frac{8\sqrt{2}e^{\frac{7a}{2}}}{1157625}\cdot\\ &\left(\begin{array}{l}189\pi^4(3113 - 1820\ln 2) - 396900\zeta(3)) + 36102372a^2\\ -144a^3(-75547 + 2450\pi^2 + 210(341 - 140\ln 2)\ln 2)\\ +48(43892 + 434304\ln 2 + 1070394\zeta(3) + 496125\zeta(5))\\ -12a^2 m_1 + 4\pi^2 m_2 + 12\ln 2 m_3 + 12a m_4\end{array}\right),\end{aligned} \tag{33}$$

where

$$\begin{aligned}m_1 &= 12\ln 2(299 + 420(319 - 210\ln 2)\ln 2\\ &\quad +105\pi^2(-253 + 1680\ln 2),\end{aligned}$$

$$\begin{aligned}m_2 &= 3(138142 + 105\ln 2(-3839 + 560\ln 2))\ln 2\\ &\quad +1698463 + +330750\zeta(3),\end{aligned}$$

$$\begin{aligned}m_3 &= 1410851\ln 2 + 630(121 - 1890\ln 2)\zeta(3)\\ &\quad +12\ln^2 2(-107293 + 840\ln 2(11 + 35\ln 2)),\end{aligned}$$

and

$$\begin{aligned}m_4 &= 6615\pi^4 + \pi^2(636122 - 210\ln 2(2563 + 420\ln 2))\\ &\quad +24\ln^2 2(-183139 + 210\ln 2(-253 + 420\ln 2))\\ &\quad +2\ln 2(2187641 - 661500\zeta(3)) + 3305610\zeta(3)\\ &\quad +1885056.\end{aligned}$$

However, the left integral of (33) can not be calculated by symbolic integration and numerical integration.

5) Let p, q be non-negative integers and x, y be complex numbers satisfying $Rex > |Rey|$. Then

$$\begin{aligned}&\int_0^\infty t^{2q}\cosh^{-2x} t\cosh 2yt\ln^p\cosh t dt\\&=2^{2x-2q-2}\sum_{j=0}^{p}2^{-j}C_p^j\ln^{p-j}2\sum_{u=0}^{j}C_j^u\sum_{k=0}^{2q}(-1)^kC_{2q}^k\\&\cdot B_{u+2q-k,j+k-u}(x+y,x-y).\end{aligned} \quad (34)$$

Note that if the values of $2(x+y)$ and $2(x-y)$ are integers, then the integral (34) must exist be closed form.

Moreover, the following integral can also be transformed into the partial derivatives of $B(x, y)$.

$$\begin{aligned}\int_0^1\frac{t^m(1-t)}{1-t^n}dt &= \frac{1}{n}\int_0^1\frac{t^{m+1-n}(1-t)}{1-t^n}dt^n\\&=\frac{1}{n}\int_0^1\frac{t^{\frac{m+1}{n}-1}}{1-t}dt-\frac{1}{n}\int_0^1\frac{t^{\frac{m+2}{n}-1}}{1-t}dt\\&=\frac{m+1-n}{n^2}\int_0^1 t^{\frac{m+1-n}{n}-1}\ln(1-t)dt\\&-\frac{m+2-n}{n^2}\int_0^1 t^{\frac{m+2-n}{n}-1}\ln(1-t)dt\\&=\frac{m+1-n}{n^2}B_{0,1}(\frac{m+1-n}{n},1)\\&-\frac{m+2-n}{n^2}B_{0,1}(\frac{m+2-n}{n},1).\end{aligned} \quad (35)$$

By (8) and (17), the left integral of (35) also has the closed form for positive integers n and m.

V. Conclusion

Based on the additional definition of $B(x, y)$ and some recurrence formulas of $B_{p,q}(x, y)$ given in [3], we develop an algorithm for computing $B(x, y)$ and its partial derivatives $B_{p,q}(x, y)$. Furthermore, numerical examples show that the algorithm can be applied to compute some special generalized integrals, which improve the rate and precision of computing the corresponding generalized integrals.

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