Two-grid Methods for Characteristic Finite Volume Element Approximations of Semi-linear Sobolev Equations

Jin-liang Yan, and Zhi-yue Zhang

Abstract—**In this paper, two-grid methods for characteristic finite volume element solutions are presented for the semi-linear Sobolev equation. The method is based on the methods of characteristics, two-grid method and the finite volume element method. The nonsymmetric and nonlinear iterations are only executed on the coarse grid (with grid size H). And the fine-grid solution (with grid size h) can be obtained by a single symmetric and linear step. It is proved that the coarse grid can be much coarser than the fine grid. The two-grid methods achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = \mathcal{O}(H^3)$. As a result, solving such a large semi-linear Sobolev equations will not be much more difficult than solving one single linearized equation.**

Index Terms—**Characteristics, Two-grid method, Finite volume element method, Sobolev equation, Error estimates.**

I. Introduction

THIS paper consider the following semi-linear Sobolev equations:

$$\begin{cases} c(x)u_t + \mathbf{d}(x)\cdot\nabla u - \nabla\cdot(a(x)\nabla u_t) - \nabla\cdot(b(x)\nabla u) \\ = f(u,x,t),\ \ (x,t)\in\ \Omega\times I, \\ u(x,0) = u_0(x),\ \ x\in\Omega, \\ u(x,t) = 0,\ \ (x,t)\in\partial\Omega\times I \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain, with boundary $\partial\Omega$. In this paper, we consider the problem with periodic boundary. $d(x) = (d_1(x), d_2(x))^T$, $I = [0,T]$, $T > 0$ is some fixed final time. $f(u,x,t)$ is uniformly Lipschitz continuous with respect to u. On the other hand, the coefficients of (1) satisfy

$(a)\quad 0 < a_1 \le a(x) \le a_2, 0 < b_1 \le b(x) \le b_2, |\mathbf{d}| = \sqrt{d_1^2 + d_2^2} \le d^*, 0 < c_1 \le c(x) \le c_2;$

$(b)\quad \left|\frac{\mathbf{d}(x)}{c(x)}\right| + \left|\frac{\partial}{\partial x_i}\left(\frac{\mathbf{d}(x)}{c(x)}\right)\right| \le K_1;$

$(c)\quad \left|\frac{\partial f}{\partial x_i}\right| + \left|\frac{\partial f}{\partial u}\right| + \left|\frac{\partial^2 f}{\partial u^2}\right| \le K_2, i = 1,2;$

$(d)\quad u \in L^\infty\left(0,T; W^{q,2}(\Omega) \cap W^{q,p}(\Omega)\right);$

$(e)\quad \frac{\partial u}{\partial t} \in L^2\left(0,T; W^{q,2}(\Omega)\right) \cap L^\infty\left(0,T; W^{q,2}(\Omega)\right);$

$(f)\quad \frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T; L^2(\Omega) \cap H^1(\Omega)\right).$

where a_1, a_2, b_1, b_2, d^*, K_1 and K_2 are positive constants and $p > 1, q \ge 2$. Problems of the form (1) commonly arise in the flow of fluids through fissured rocks [1], thermodynamics [2], the migration of moisture in soil [3], and other applications. For a discussion of existence and uniqueness results, see [4]-[8]. Various numerical treatment of this problem can be found in [9]-[19] and their references.

The characteristic method was first introduced by Douglas and Russell in [20]. And then extended by Russell [21] to nonlinear coupled systems in two and three spatial dimensions. The main concept of characteristic method is to combine the time derivative and the convection term as a directional derivative along the characteristics, leading to a characteristic time-stepping procedure. Then, the standard method can be applied to the problem whose form is similar to heat equation. Comparing with standard methods, it can use larger time steps in numerical simulation, and can eliminate the excessive numerical diffusion and nonphysical oscillation.

Finite volume element (FVE) method, as a type of important numerical tool for solving differential equations, was widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. This method is also known as a box method [22], [23] or generalized difference method [24], [25] in China. Perhaps the most important property of FVE method is that it can preserve the conservation laws (mass, momentum and heat flux) on each computational element. This important property, combined with adequate accuracy and ease of implementation, has attracted many researchers to do research [26]-[32]. The theoretical framework and the basic tools for the analysis of FVE method have been developed in the last two decades [26]-[32].

Two-grid method was first introduced by Xu [33]-[35] as a discretization method for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial equations. The basic concept of this method is to solve a complicated problem (nonlinear, etc.) on a coarse grid (with mesh size H) and then solve an easier problem (linear, etc.) on a fine grid (with mesh size h and $h \ll H$) as correction. Later on, the two-grid method was further investigated by many authors, for instance, Dawson and Wheeler [36], [37] have applied two-grid mixed finite element method and two-grid finite difference method to a class of parabolic equations, respectively. Wu and Allen [38] have used the two-grid mixed

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finite element method to approximate the reaction-diffusion equations. Utnes [39] have applied this method to Navier-Stokes equations. Bi and Ginting [40] have studied the two-grid finite volume element method for linear and nonlinear elliptic equations. Chen [41], [42] and Chen, Bi [43] have applied the two-grid finite volume element method to a kind of nonlinear parabolic equations and convection diffusion equations, respectively.

In this paper, based on two linear conforming finite element spaces U_H and U_h, on one coarse grid (with grid size H) and one fine grid (with grid size $h \ll H$), we use the two-grid characteristic finite volume element methods to approximate the semi-linear Sobolev equations (1). We first solve a nonsymmetric and nonlinear problem on the coarse grid, then we use the known coarse grid solution and a Taylor expansion to get the solution of a symmetric and linear system on the fine grid. As shown in [33], the approach can use coarser mesh on the coarse grid without loss of accuracy. The outline of the paper is as follows. In Section 2, preliminaries and notations are introduced. In Section 3, the characteristic FVE method and two-grid characteristic FVE method are presented, respectively. The error estimates in the H^1- and L^2-norm of characteristic FVE method are demonstrated in Section 4. In Section 5, the error estimates in the H^1-norm for the two-grid characteristic FVE method are presented.

Throughout this paper, the letter C denotes a generic positive constant which independent of the mesh parameter and may be different at its different occurrences.

II. Preliminaries and notation

For a convex polygonal domain $\Omega \subset R^2$, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω and supply it with a norm $\|\cdot\|_{m,p}$

$$\|u\|_{m,p} = \begin{cases} \Big(\sum\limits_{0\le|\alpha|\le m} \|D^\alpha u\|_p^p\Big)^{1/p}, & 1 \le p < \infty, \\ \max\limits_{0\le|\alpha|\le m} \|D^\alpha u\|_\infty, & p = \infty. \end{cases} \tag{2}$$

Define $W_0^{m,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{m,p}$. In particular when $p = 2$ we write $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ as $H^m(\Omega)$ and $H_0^m(\Omega)$, respectively. Note that $H^0(\Omega) = L^2(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega); v = 0 \ \ on \ \ \partial\Omega\}$.

Let Ω is a polygonal region with boundary $\partial\Omega$. Divide $\overline{\Omega}$ into a sum of finite number of small triangles, each triangle is called an element and the vertexes of the triangle are called nodes. All the elements K constitute a triangulation of $\overline{\Omega}$, denoted by T_h, where h is the maximum length of all the sides.

Now we construct a dual decomposition T_h^* related to T_h. Let P_0 be a node of a triangle, $P_i (i = 1, 2, \cdots, 6)$ the adjacent nodes of P_0, and M_i the midpoint of $\overline{P_0P_i}$ (Figure 1). Choose a point Q_i in an element $\triangle P_0P_iP_{i+1} (P_7 = P_1)$ and connect successively $M_1, Q_1, \cdots, M_6, Q_6, M_1$ to form a polygonal region $K_{P_0^*}$, called a dual element. The modification of the definition is obvious when P_0 is on the boundary. All the dual elements constitute a new decomposition, called a dual decomposition. Q_i is called a node of the dual decomposition, Q_i is usually chosen as barycenter or circumcenter of the element $K \in T_h$.

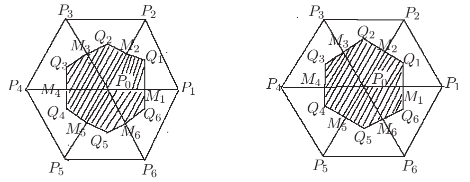

Fig. 1. The dual element of P_0.

In the sequel we denote by $\overline{\Omega}_h$ the set of the nodes of the decomposition T_h, $\dot{\Omega}_h = \overline{\Omega}_h \setminus \partial\Omega$ the set of the interior nodes, and Ω_h^* the set of the nodes of the dual decomposition T_h^*. For $Q \in \Omega_h^*$, K_Q denotes the triangular element containing Q. Let S_Q and $S_{P_0^*}$ be the areas of the triangular element K_Q and the dual element $K_{P_0^*}$, respectively. We call the mesh T_h and T_h^* are quasi-uniform if there exists constant $c_1, c_2, c_3 > 0$ independent of h such that

$$c_1 h^2 \le S_Q \le h^2, Q \in \Omega_h^*, \tag{3}$$

$$c_2 h^2 \le S_{P_0^*} \le c_3 h^2, P_0 \in \overline{\Omega}_h. \tag{4}$$

The trial function space U_h is chosen as the linear element space related to T_h,

$$U_h = \{u_h \in C(\overline{\Omega}) : u_h|_K \in P_1, \forall K \in T_h; u_h|_{\partial\Omega} = 0\},$$

and the test space V_h is chosen as the piecewise constant function space with respect to T_h^*, spanned by the following basis functions: For any point $P_0 \in \dot{\Omega}_h$

$$\Psi_{P_0}(P) = \begin{cases} 1, & P \in K_{P_0^*}, \\ 0, & \text{elsewhere}. \end{cases} \tag{5}$$

For any $v_h \in V_h$

$$v_h = \sum_{P_0 \in \dot{\Omega}_h} v_h(P_0)\Psi_{P_0}$$

Then we obtain U_h =span$\{\phi_i(x) : P_i \in \dot{\Omega}_h\}$ and V_h =span$\{\psi_i(x) : P_i \in \dot{\Omega}_h\}$ where $\phi_i(x)$ is the nodal basis function associated with the node P_i, and $\psi_i(x)$ is the characteristic function of $K_{P_0}^*$.

For any $u \in H_0^1(\Omega) \bigcap H^2(\Omega)$, we define an interpolation operator $\Pi_h : C(\overline{\Omega}) \to U_h$, such that

$$\Pi_h u = \sum_{P_0 \in \dot{\Omega}_h} u(P_0)\phi_i(x). \tag{6}$$

For any $u_h \in U_h$, we define another interpolation operator $\Pi_h^* : U_h \to V_h$, such that

$$\Pi_h^* u_h = \sum_{P_0 \in \dot{\Omega}_h} u_h(P_0)\psi_i(x). \tag{7}$$

By the interpolation theory we have

$$\|u_h - \Pi_h^* u_h\| \le Ch \|u_h\|_1, \tag{8}$$

and in [31] that

$$\|\Pi_h^* u_h\| \le C \|u_h\|. \tag{9}$$

III. The characteristic FVE method and two-grid FVE method

A. The characteristic FVE method

In the characteristic method, the time derivative and the convection term of (1) are combined as a directional derivative along the characteristics direction $\tau = \tau(x)$:

$$c(x)\frac{\partial u}{\partial t} + \mathbf{d}(x)\cdot\nabla u = \sqrt{c(x)^2 + |\mathbf{d}(x)|^2}\frac{\partial u}{\partial \tau}$$

where

$$\frac{\partial}{\partial \tau} = \frac{1}{\psi(x)}\left(c(x)\frac{\partial}{\partial t} + \mathbf{d}(x)\cdot\nabla\right),$$

$$\psi(x) = \sqrt{c(x)^2 + |\mathbf{d}(x)|^2}.$$

Then, (1) can be written as

$$\begin{cases} \psi(x)u_\tau - \nabla\cdot(a(x)\nabla u_t) - \nabla\cdot(b(x)\nabla u) \\ = f(u,x,t),\ (x,t)\in\ \Omega\times I, \\ u(x,0) = u_0(x),\ \ x\in\Omega, \\ u(x,t) = 0,\ \ (x,t)\in\partial\Omega\times I \end{cases} \tag{10}$$

We define a partition of the time interval $[0,T]$ by $t_n = n\Delta t, n = 0,1,2,\cdots,N$, with $\Delta t = T/N, u_h^n = u_h(t_n)$. In the standard characteristic method [20], the directional derivative along the characteristics is approximated by

$$\begin{aligned}\psi(x)\frac{\partial u^n}{\partial \tau} &\approx \psi(x)\frac{u(x,t^n) - u(\overline{x},t^{n-1})}{\sqrt{(x-\overline{x})^2 + \Delta t^2}} \\ &= c(x)\frac{u(x,t^n) - u(\overline{x},t^{n-1})}{\Delta t},\end{aligned} \tag{11}$$

where $\overline{x} = x - \frac{\mathbf{d}(x)}{c(x)}\Delta t$.
The variational problem related to (10) is: Find $u = u(\cdot,t)\in U$ such that

$$\begin{cases} (\psi(x)u_\tau, v) + a_h(u_t,v) + b_h(u,v) = (f(u),v), \forall v\in U, \\ u(x,0) = u_0(x), \quad x\in\Omega, \end{cases} \tag{12}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, and

$$a_h(u,v) = (a(x)\nabla u, \nabla v), \quad b_h(u,v) = (b(x)\nabla u, \nabla v). \tag{13}$$

Though the trial function space U_h satisfies $U_h\subset U$ like finite element methods, the test space $V_h\not\subset U_h$. As in the case of nonconforming finite element methods, this is due to the loss of continuity of the functions in V_h on the boundary of two neighboring elements. So the bilinear forms $a_h(u,v)$ and $b_h(u,v)$ must be revised accordingly. For nonconforming finite element methods, the idea is to write the integral on the whole region as a sum of the integrals on every element K, so $a_h(u,v)$ and $b_h(u,v)$ are rewritten as

$$a_h(u,v) = \sum_{K\in T_h}\int_K a(x)\nabla u\nabla v dx, \tag{14}$$

$$b_h(u,v) = \sum_{K\in T_h}\int_K b(x)\nabla u\nabla v dx. \tag{15}$$

Now $a_h(u,v)$ and $b_h(u,v)$ are well-defined on $U_h\times V_h$. For the FVE methods, i.e. generalized difference methods, we place a dual grid and interpret (14) in the sense of generalized function, i.e. δ functions on the boundary of neighboring dual elements. Or equivalently, we take $a_h(u,v)$ and $b_h(u,v)$ as the bilinear form resulting from the piecewise integration in parts on the dual elements $K_{P_0}^*$:

$$\begin{aligned}\int_\Omega v\Delta u dx = &- \sum_{K_{P_0}^*\in T_h^*}\int_{K_{P_0}^*}\nabla u\nabla v\,dx \\ &+ \sum_{K_{P_0}^*\in T_h^*}\int_{\partial K_{P_0}^*}(\nabla u)\cdot\mathbf{n}v\,ds.\end{aligned} \tag{16}$$

where $\int_{\partial K_{P_0^*}}$ denotes the line integrals, in the counterclockwise direction, on the boundary $\partial K_{P_0^*}$ of the dual element. So, we have

$$\begin{aligned}a_h(u,v) = &\sum_{K_{P_0}^*\in T_h^*}\int_{K_{P_0}^*}(a(x)\nabla u)\nabla v\,dx \\ &- \sum_{K_{P_0}^*\in T_h^*}\int_{\partial K_{P_0}^*}(a(x)\nabla u)\cdot\mathbf{n}v\,ds,\end{aligned} \tag{17}$$

$$\begin{aligned}b_h(u,v) = &\sum_{K_{P_0}^*\in T_h^*}\int_{K_{P_0}^*}(b(x)\nabla u)\nabla v\,dx \\ &- \sum_{K_{P_0}^*\in T_h^*}\int_{\partial K_{P_0}^*}(b(x)\nabla u)\cdot\mathbf{n}v\,ds,\end{aligned} \tag{18}$$

Since the test space V_h is chosen as the piecewise constant function space, so we have

$$a_h(u,v) = -\sum_{K_{P_0}^*\in T_h^*}\int_{\partial K_{P_0}^*}(a(x)\nabla u)\cdot\mathbf{n}v\,ds, \tag{19}$$

$$b_h(u,v) = -\sum_{K_{P_0}^*\in T_h^*}\int_{\partial K_{P_0}^*}(b(x)\nabla u)\cdot\mathbf{n}v\,ds, \tag{20}$$

Then, the semi-discrete FVE formulation of (1) is: Find $u_h = u_h(\cdot,t)\in U_h\ (0\le t\le T)$ such that

$$\begin{cases} (u_{h,\tau}, \Pi_h^* v_h) + a_h(u_{h,t}, \Pi_h^* v_h) + b_h(u_h, \Pi_h^* v_h) \\ = (f(u_h), \Pi_h^* v_h), \forall v_h\in U_h,\ t>0, \\ u_h(x,0) = u_{0h}(x). \quad x\in\Omega, \end{cases} \tag{21}$$

where $a_h(\cdot,\Pi_h^*\cdot)$ and $b_h(\cdot,\Pi_h^*\cdot)$ are defined by, for any $u_h, v_h\in U_h$, (19) and (20), respectively. u_{0h} is a certain approximation to u_0 on U_h. At time $t = t_n$, we use the backward difference quotient

$$\overline{\partial}_t u_h^n = \left(u_h^n - u_h^{n-1}\right)\Delta t \tag{22}$$

to approximate $u_{h,t}$, then we get the fully-discrete scheme of (1): Find $u_h^n\in U_h (n = 1,2,\cdots,N)$ such that

$$\begin{cases} \left(c(x)\frac{u_h^n - \overline{u}_h^n}{\Delta t}, \Pi_h^* v_h\right) + a_h\left(\overline{\partial}_t u_h^n, \Pi_h^* v_h\right) \\ + b_h(u_h^n, \Pi_h^* v_h) = (f(u_h^n), \Pi_h^* v_h), \forall v_h\in U_h,\ t>0, \\ u_h^0 = u_{0h}. \quad x\in\Omega, \end{cases} \tag{23}$$

On the other hand, from Lemma 2, we know that there exists a unique local solution for (12) and (23)(see e.g. [24]).

B. The two-grid characteristic FVE method

In order to present two-grid FVE method for the semi-linear Sobolev equation (1), we introduce two quasi-uniform triangulations of Ω, T_H and T_h, with two different mesh sizes H and $h(H \gg h)$. We introduce the corresponding finite element spaces U_H and U_h which satisfy $U_H \subset U_h$. They will be called the coarse grid and fine grid spaces, respectively.

The basic idea of two-grid method is to use a coarse grid space to produce a rough approximation of the solution, and then use it as the initial guess for one Newton-like iteration on the fine grid. This method involves a nonlinear solve on the coarse grid and a linear solve on the fine grid space. We present the two-grid characteristic FVE method as two steps[40], [43]:

Algorithm 1.
Step 1: On the coarse grid T_H, find $u_H^n \in U_H(n=1,2,\cdots)$, such that

$$\begin{cases} \left(c(x)\dfrac{u_H^n-\overline{u}_H^n}{\Delta t}, \Pi_H^* v_H\right) + a_H\left(\dfrac{u_H^n-u_H^{n-1}}{\Delta t}, \Pi_H^* v_H\right) \\ +b_H\left(u_H^n, \Pi_H^* v_H\right) = \left(f(u_H^n), \Pi_H^* v_H\right), \\ \forall v_H \in U_H,\ t>0, \\ u_H^0 = u_{0H}. \quad x \in \Omega, \end{cases} \tag{24}$$

Step 2: On the fine grid T_h, find $u_h^n \in U_h(n=1,2,\cdots)$, such that

$$\begin{cases} \left(c(x)\dfrac{u_h^n-\overline{u}_h^{n-1}}{\Delta t}, \Pi_h^* v_h\right) + a_h\left(\dfrac{u_h^n-u_h^{n-1}}{\Delta t}, \Pi_h^* v_h\right) \\ +b_h\left(u_h^n, \Pi_h^* v_h\right) \\ = \left(f(u_H^n) + f'(u_H^n)\left(u_h^n - u_H^n\right), \Pi_h^* v_h\right), \\ \forall v_h \in U_h, t>0, \\ u_h^0 = u_{0h}. \quad x \in \Omega, \end{cases} \tag{25}$$

We note that the second step of Algorithm 1 is a linear problem but still nonsymmetric. In order to get a symmetric system, we introduce the following bilinear forms

$$a_c\left(u_h, v_h\right) = \int_\Omega \overline{a}\nabla u_h \cdot \nabla v_h\, dx, \forall u_h, v_h \in U_h, \tag{26}$$

$$b_c\left(u_h, v_h\right) = \int_\Omega \overline{b}\nabla u_h \cdot \nabla v_h\, dx, \forall u_h, v_h \in U_h, \tag{27}$$

$$a_{h,c}\left(u_h, \Pi_h^* v_h\right) = -\sum_{K_{P_0}^* \in T_h^*} \int_{\partial K_{P_0}^*} \left(\overline{a}\nabla u_h\right) \cdot \mathbf{n}\Pi_h^* v_h\, ds, \tag{28}$$

$$b_{h,c}\left(u_h, \Pi_h^* v_h\right) = -\sum_{K_{P_0}^* \in T_h^*} \int_{\partial K_{P_0}^*} \left(\overline{a}\nabla u_h\right) \cdot \mathbf{n}\Pi_h^* v_h\, ds, \tag{29}$$

where $\overline{a} = \overline{a}|_K = a_K, \overline{b} = \overline{b}|_K = b_K$ and

$$a_K = \frac{1}{meas(K)} \int_K a\left(x\right)\, dx$$

,

$$b_K = \frac{1}{meas(K)} \int_K b\left(x\right)\, dx, \forall K \in T_h$$

. Then from [29], [44], we have the following lemma.

Lemma 1: For any $u_h, v_h \in U_h$, we have

$$a_{h,c}\left(u_h, \Pi_h^* v_h\right) = a_c\left(u_h, v_h\right)$$

.

$$b_{h,c}\left(u_h, \Pi_h^* v_h\right) = b_c\left(u_h, v_h\right)$$

.

From this lemma we can see that $a_{h,c}(u_h, \Pi_h^* v_h)$ and $b_{h,c}(u_h, \Pi_h^* v_h)$ are symmetric. Then we obtain the second algorithm.

Algorithm 2.
Step 1: On the coarse grid T_H, find $u_H^n \in U_H(n=1,2,\cdots)$, such that

$$\begin{cases} \left(c(x)\dfrac{u_H^n-\overline{u}_H^n}{\Delta t}, \Pi_H^* v_H\right) + a_{H,c}\left(\dfrac{u_H^n-u_H^{n-1}}{\Delta t}, \Pi_H^* v_H\right) \\ +b_{H,c}\left(u_H^n, \Pi_H^* v_H\right) = \left(f(u_H^n), \Pi_H^* v_H\right), \\ \forall v_H \in U_H,\ t>0, \\ u_H^0 = u_{0H}. \quad x \in \Omega, \end{cases} \tag{30}$$

Step 2: On the fine grid T_h, find $u_h^n \in U_h(n=1,2,\cdots)$, such that

$$\begin{cases} \left(c(x)\dfrac{u_h^n-\overline{u}_h^{n-1}}{\Delta t}, \Pi_h^* v_h\right) + a_{h,c}\left(\dfrac{u_h^n-u_h^{n-1}}{\Delta t}, \Pi_h^* v_h\right) \\ +b_{h,c}\left(u_h^n, \Pi_h^* v_h\right) \\ = \left(f(u_H^n) + f'(u_H^n)\left(u_h^n - u_H^n\right), \Pi_h^* v_h\right), \\ \forall v_h \in U_h, t>0, \\ u_h^0 = u_{0h}. \quad x \in \Omega, \end{cases} \tag{31}$$

We note that the coefficient matrixes of the system in Algorithm 2 are symmetric. So the system is easier to solve (e.g. conjugate-gradient-like methods can be applied effectively). We call these algorithms as two-grid characteristic FVE methods.

IV. Error analysis for characteristic FVE method

A. Some lemmas

Lemma 2: ([29], [24]) There exist positive constants h_0, α and M such that when $0 < h \le h_0$, the coercive property

$$a_h\left(u_h, \Pi_h^* u_h\right) \ge \alpha \left\|u_h\right\|_1^2, \forall u_h \in U_h$$

and the boundness property

$$\left|a_h\left(u_h, \Pi_h^* v_h\right)\right| \le M\left\|u_h\right\|_1 \left\|v_h\right\|_1$$

hold true.

Lemma 3: ([24]) Set $|||u_h|||_0 = (u_h, \Pi_h^* u_h)^{1/2}$ and $|||u_h|||_1 = [b(u_h, \Pi_h^* u_h)]^{1/2}$. Then $|||\cdot|||_0$ and $|||\cdot|||_1$ are equivalent to $\|\cdot\|_0$ and $\|\cdot\|_1$ on U_h, respectively, that is, there exist positive constants c_1, c_2 and c_3, c_4 such that

$$c_1\left\|u_h\right\|_1 \le |||u_h|||_1 \le c_2\left\|u_h\right\|_1, \forall u_h \in U_h.$$

$$c_3\left\|u_h\right\|_0 \le |||u_h|||_0 \le c_4\left\|u_h\right\|_0, \forall u_h \in U_h.$$

Lemma 4: ([44]) For any $u_h, v_h \in U_h$, there are

$$\left|a_h\left(u_h, \Pi_h^* v_h\right) - a_h\left(v_h, \Pi_h^* u_h\right)\right| \le Ch\left\|u_h\right\|_1\left\|v_h\right\|_1$$

and

$$\left(u_h, \Pi_h^* v_h\right) = \left(v_h, \Pi_h^* u_h\right)$$

Lemma 5: ([24], [44]) Define an elliptic operator $P_h : C(\Omega) \to U_h$, such that

$$a_h\left(u - P_h u, \Pi_h^* v_h\right) = 0, \forall v_h \in U_h.$$

$$b_h\left(u-P_hu,\Pi_h^*v_h\right)=0,\forall v_h\in U_h.$$

Then we have

$$\|u-P_hu\|_1\le Ch\|u\|_2,$$

$$\|u-P_hu\|\le Ch^2\|u\|_{3,p},p>1,$$

$$\|u-P_hu\|_{0,\infty}\le Ch^2|lnh|\left(\|u\|_3+\|u\|_{2,\infty}\right).$$

Lemma 6: (Gronwall lemma) Let a_k, b_k, α_k, d_k, are non-negative sequence, for $k\ge 0$, and satisfy

$$a_J+\sum_{k=0}^{J}b_k\Delta t\le\sum_{k=0}^{J-1}a_kd_k\Delta t+\sum_{k=0}^{J}\alpha_k\Delta t, d_k\Delta t\le\frac{1}{2}$$

then

$$a_J+\sum_{k=0}^{J}b_k\Delta t\le exp\left[2\sum_{k=0}^{J-1}d_k\Delta t\right]\sum_{k=0}^{J}\alpha_k\Delta t.$$

Lemma 7: ([43]) Let $u\in L^\infty(0.T;H^1(\Omega))$ and $\overline{u}^{n-1}=u\left(\overline{x},T^{n-1}\right)$, where $\overline{x}$ is defined by (11), then we have

$$\left\|u_{n-1}-\overline{u}^{n-1}\right\|\le C\left\|u^{n-1}\right\|_1\Delta t.$$

B. Error analysis for characteristic FVE method

Now we consider the error estimate for the characteristic finite volume element method of (1). The error estimates in the H^1- and L^2-norm will be given in the following Theorem 4.1 and Theorem 4.2.

Theorem 4.1: Let u and u_h be the solutions of (12) and (23), respectively. Under assumption (a)-(f), for $\triangle t$ small enough, if $u_h^0=P_hu_0$ with P_h defined by Lemma 5, we have, for $t_n\le T$,

$$\max_{1\le n\le N}\|u^n-u_h^n\|_1\le C(\triangle t+h). \tag{32}$$

where $C=C(\|u\|_{L^\infty(H^2(\Omega))},\|u_{\tau\tau}\|_{L^2(L^2(\Omega))},\|u_{tt}\|_{L^2(H^1(\Omega))},\|u\|_{L^\infty(W^{3,p}(\Omega))},\|u_t\|_{L^\infty(W^{3,p}(\Omega))})$ is independent of h and Δt.

Proof: For convenience, let $u_h^n-u^n=(u_h^n-P_hu^n)-(u^n-P_hu^n)=:e^n-\rho^n$. Then from (12) and (23), we get the following error equation at t_n

$$\begin{aligned}&\left(c(x)\frac{e^n-\overline{e}^{n-1}}{\Delta t},\Pi_h^*v_h\right)+a_h\left(\partial_te^n,\Pi_h^*v_h\right)+\\&b_h\left(e^n,\Pi_h^*v_h\right)=\left(\psi(x)\frac{\partial u^n}{\partial\tau}-c(x)\frac{u^n-\overline{u}^{n-1}}{\Delta t},\Pi_h^*v_h\right)\\&+a_h\left(u_t^n-\partial_tu^n,\Pi_h^*v_h\right)+\left(c(x)\frac{\rho^n-\overline{\rho}^{n-1}}{\Delta t},\Pi_h^*v_h\right)\\&+\left(f(u_h^n)-f(u^n),\Pi_h^*v_h\right),\forall v_h\in U_h.\end{aligned}\tag{33}$$

where $\overline{e}^{n-1}=\overline{u}_h^{n-1}-P_hu^{n-1},\overline{\rho}^{n-1}=\overline{u}^{n-1}-P_h\overline{u}^{n-1}$. Letting $\partial_te^n=\frac{e^n-e^{n-1}}{\Delta t}$ and choosing $v_h=\partial_te^n$, we obtain

$$\begin{aligned}&\left(c(x)\frac{e^n-\overline{e}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)+a_h\left(\partial_te^n,\Pi_h^*\partial_te^n\right)\\&+b_h\left(e^n,\Pi_h^*\partial_te^n\right)\\&=\left(\psi(x)\frac{\partial u^n}{\partial\tau}-c(x)\frac{u^n-\overline{u}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+a_h\left(u_t^n-\partial_tu^n,\Pi_h^*\partial_te^n\right)+\left(f(u_h^n)-f(u^n),\Pi_h^*\partial_te^n\right)\\&+\left(c(x)\frac{\rho^n-\overline{\rho}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right).\end{aligned}$$

Rewritten (34) as

$$\begin{aligned}&(c(x)\partial_te^n,\Pi_h^*\partial_te^n)+a_h\left(\partial_te^n,\Pi_h^*\partial_te^n\right)\\&+b_h\left(e^n,\Pi_h^*\partial_te^n\right)\\&=\left(\psi(x)\frac{\partial u^n}{\partial\tau}-c(x)\frac{u^n-\overline{u}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+a_h\left(u_t^n-\partial_tu^n,\Pi_h^*\partial_te^n\right)+\left(f(u_h^n)-f(u^n),\Pi_h^*\partial_te^n\right)\\&+\left(c(x)\frac{\rho^{n-1}-\overline{\rho}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+\left(c(x)\frac{\rho^n-\rho^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+\left(c(x)\frac{\overline{e}^{n-1}-e^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right).\end{aligned}\tag{34}$$

Now we estimate (34). First

$$\begin{aligned}&b_h\left(e^n,\Pi_h^*\partial_te^n\right)\\&=\frac{1}{2\Delta t}[b_h\left(e^n+e^{n-1},\Pi_h^*(e^n-e^{n-1})\right)\\&+b_h\left(e^n-e^{n-1},\Pi_h^*(e^n-e^{n-1})\right)]\\&\ge\frac{1}{2\Delta t}b_h\left(e^n+e^{n-1},\Pi_h^*(e^n-e^{n-1})\right)\\&=\frac{1}{2\Delta t}\left[b_h\left(e^n,\Pi_h^*e^n\right)-b_h\left(e^{n-1},\Pi_h^*e^{n-1}\right)\right]\\&-\frac{1}{2}\left[b_h\left(\partial_te^n,\Pi_h^*e^n\right)-b_h\left(e^n,\Pi_h^*\partial_te^n\right)\right].\end{aligned}\tag{35}$$

By (34) and (35), we have

$$\begin{aligned}&(c(x)\partial_te^n,\Pi_h^*\partial_te^n)+a_h(\partial_te^n,\Pi_h^*\partial_te^n)\\&+\frac{1}{2\Delta t}\left[b_h\left(e^n,\Pi_h^*e^n\right)-b_h\left(e^{n-1},\Pi_h^*e^{n-1}\right)\right]\\&\le\left(\psi(x)\frac{\partial u^n}{\partial\tau}-c(x)\frac{u^n-\overline{u}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+a_h\left(u_t^n-\partial_tu^n,\Pi_h^*\partial_te^n\right)+\left(f(u_h^n)-f(u^n),\Pi_h^*\partial_te^n\right)\\&+\left(c(x)\frac{\rho^n-\rho^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+\left(c(x)\frac{\rho^{n-1}-\overline{\rho}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+\left(c(x)\frac{\overline{e}^{n-1}-e^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)\\&+\frac{1}{2}\left[b_h\left(\partial_te^n,\Pi_h^*e^n\right)-b_h\left(e^n,\Pi_h^*\partial_te^n\right)\right].\end{aligned}\tag{36}$$

Multiplying by Δt and summing over l from 1 to $l(1\le n\le N)$ at both sides of (36), by Lemma 3 and Lemma 4, since $e^0=0$ we have

$$\begin{aligned}&\frac{\alpha}{2}\|e^n\|_1^2+C\sum_{l=1}^{n}\left\|\partial_te^l\right\|^2\Delta t+C\sum_{l=1}^{n}\left\|\partial_te^l\right\|_1^2\Delta t\\&\le\sum_{l=1}^{n}\left(\psi(x)\frac{\partial u^l}{\partial\tau}-c(x)\frac{u^l-\overline{u}^{l-1}}{\Delta t},\Pi_h^*\partial_te^l\right)\Delta t\\&+\sum_{l=1}^{n}a_h\left(u_t^l-\partial_tu^l,\Pi_h^*\partial_te^l\right)\Delta t\\&+\sum_{l=1}^{n}\left(c(x)\frac{\rho^l-\rho^{l-1}}{\Delta t},\Pi_h^*\partial_te^l\right)\Delta t\\&+\sum_{l=1}^{n}\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},\Pi_h^*\partial_te^l\right)\Delta t\end{aligned}$$

$$+\sum_{l=1}^{n}\left(c(x)\frac{\overline{e}^{l-1}-e^{l-1}}{\Delta t},\Pi_h^*\partial_t e^l\right)\Delta t$$
$$+\frac{1}{2}\sum_{l=1}^{n}\left[b_h\left(\partial_t e^l,\Pi_h^* e^l\right)-b_h\left(e^l,\Pi_h^*\partial_t e^l\right)\right]\Delta t$$
$$+\sum_{l=1}^{n}\left(f(u_h^l)-f(u^l),\Pi_h^*\partial_t e^l\right)\Delta t\equiv\sum_{i=1}^{7}T_i \quad (37)$$

We now estimate the right-hand terms of (37) For T_1, from the results given in [21], that

$$\left\|\psi\frac{\partial u^n}{\partial\tau}-c(x)\frac{u^n-\overline{u}^{n-1}}{\Delta t}\right\|^2\leq C\Delta t\int_{t_{n-1}}^{t_n}\int_\Omega\left|\frac{\partial^2 u}{\partial\tau^2}\right|^2dxdt$$
$$\leq C\Delta t\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(t_{n-1},t^n;L^2(\Omega))}. \quad (38)$$

by (9) and ε-inequality

$$|T_1|\leq\sum_{l=1}^{n}\left|\left(\psi(x)\frac{\partial u^l}{\partial\tau}-c(x)\frac{u^l-\overline{u}^{l-1}}{\Delta t},\Pi_h^*\partial_t e^l\right)\right|\Delta t$$
$$\leq C(\varepsilon)\sum_{l=1}^{n}\left\|\psi(x)\frac{\partial u^l}{\partial\tau}-c(x)\frac{u^l-\overline{u}^{l-1}}{\Delta t}\right\|^2\Delta t$$
$$+\varepsilon\sum_{l=1}^{n}\left\|\Pi_h^*\partial_t e^l\right\|^2\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|\frac{\partial^2 u^l}{\partial\tau^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}(\Delta t)^2$$
$$+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t$$
$$\leq C\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(0,T;L^2(\Omega))}(\Delta t)^2$$
$$+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t \quad (39)$$

For T_2, from the results given in [21],we get

$$|T_2|\leq\sum_{l=1}^{n}\left|a_h\left(\partial_t u^l-u_t^l,\Pi_h^*\partial_t e^l\right)\right|\Delta t$$
$$\leq M\sum_{l=1}^{n}\left\|\partial_t u^l-u_t^l\right\|_1\left\|\Pi_h^*\partial_t e^l\right\|_1\Delta t$$
$$\leq MC(\varepsilon)\sum_{l=1}^{n}\left\|\partial_t u^l-u_t^l\right\|_1^2\Delta t+\varepsilon\sum_{l=1}^{n}\left\|\Pi_h^*\partial_t e^l\right\|_1^2\Delta t$$
$$\leq MC(\varepsilon)\sum_{l=1}^{n}\left(\int_{t_{l-1}}^{t_l}\|u_{tt}\|_1\,dt\right)^2\Delta t$$
$$+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|_1^2\Delta t$$
$$\leq MC(\varepsilon)\left(\int_0^{t_n}\|u_{tt}\|_1^2\,dt\right)(\Delta t)^2$$
$$+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|_1^2\Delta t. \quad (40)$$

For T_3, from Lemma 5, we get

$$|T_3|\leq\sum_{l=1}^{n}\left|\left(c(x)\frac{\rho^l-\rho^{l-1}}{\Delta t},\Pi_h^*\partial_t e^l\right)\right|\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|\frac{\rho^l-\rho^{l-1}}{\Delta t}\right\|\left\|\Pi_h^*\partial_t e^l\right\|\Delta t$$
$$\leq C(\varepsilon)\sum_{l=1}^{n}\left\|\frac{\partial\rho^l}{\partial t}\right\|^2\Delta t+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t$$
$$\leq C(\varepsilon)\sum_{l=1}^{n}h^4\left\|\frac{\partial u^l}{\partial t}\right\|^2\Delta t+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t$$
$$\leq C(\varepsilon)h^4\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;W^{3,p}(\Omega))}$$
$$+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t. \quad (41)$$

By Lemma 5 and Lemma 7, for T_4, we obtain

$$|T_4|\leq\sum_{l=1}^{n}\left|\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},\Pi_h^*\partial_t e^l\right)\right|\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t}\right\|\left\|\Pi_h^*\partial_t e^l\right\|\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|\rho^{l-1}\right\|_1\left\|\Pi_h^*\partial_t e^l\right\|\Delta t$$
$$\leq C(\varepsilon)\sum_{l=1}^{n}\left\|\rho^{l-1}\right\|_1^2\Delta t+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t$$
$$\leq C(\varepsilon)h^2\|u\|^2_{L^\infty(0,T;H^2(\Omega))}$$
$$+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t \quad (42)$$

For T_5, we have the similar result,

$$|T_5|\leq\sum_{l=1}^{n}\left|\left(c(x)\frac{\overline{e}^{l-1}-e^{l-1}}{\Delta t},\Pi_h^*\partial_t e^l\right)\right|\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|\frac{\overline{e}^{l-1}-e^{l-1}}{\Delta t}\right\|\left\|\Pi_h^*\partial_t e^l\right\|\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|e^{l-1}\right\|_1\left\|\Pi_h^*\partial_t e^l\right\|\Delta t$$
$$\leq C(\varepsilon)\sum_{l=1}^{n}\left\|e^{l-1}\right\|_1^2\Delta t+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t. \quad (43)$$

For T_6, by Lemma 4 and the inverse estimate, we have

$$|T_6|\leq\frac{1}{2}\sum_{l=1}^{n}\left|\left[b_h\left(\partial_t e^l,\Pi_h^* e^l\right)-b_h\left(e^l,\Pi_h^*\partial_t e^l\right)\right]\right|\Delta t$$
$$\leq C\sum_{l=1}^{n}h\left\|\partial_t e^l\right\|_1\left\|e^l\right\|_1\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|\partial_t e^l\right\|\left\|e^l\right\|_1\Delta t$$
$$\leq C\sum_{l=1}^{n}\left\|e^l\right\|_1^2\Delta t+C\varepsilon\sum_{l=1}^{n}\left\|\partial_t e^l\right\|^2\Delta t. \quad (44)$$

For T_7, at any point $x \in \Omega$, by the Taylor expansion, we have

$$f(u_h^n) - f(u^n) = f'(\widetilde{u}^n)(u_h^n - u^n) = f'(\widetilde{u}^n)(e^n - \rho^n),$$

for some value $\widetilde{u}^n$ betweenu_h^n and u_n. By assumption (c) and Lemma 5, we have

$$\begin{aligned}|T_7| &\leq \sum_{l=1}^{n} \left| \left(f(u_h^l) - f(u^l), \Pi_h^* \partial_t e^l\right) \right| \Delta t \\ &\leq \sum_{l=1}^{n} \left\| f(u_h^l) - f(u^l) \right\| \left\| \Pi_h^* \partial_t e^l \right\| \\ &\leq C(\varepsilon) \sum_{l=1}^{n} \left\| f(u_h^l) - f(u^l) \right\|^2 \Delta t \\ &+C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|^2 \Delta t \\ &\leq C(\varepsilon) \sum_{l=1}^{n} (\left\| e^l \right\|^2 + \left\| \rho^l \right\|^2) \Delta t + C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|^2 \Delta t \\ &\leq C(\sum_{l=1}^{n} \left\| e^l \right\|^2 \Delta t + h^4 \left\| u \right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2) \\ &+C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|^2 \Delta t. \end{aligned} \tag{45}$$

Combing the error estimates of $T_i (1 \leq i \leq 7)$ with (37), we have

$$\begin{aligned}&\frac{\alpha}{2} \left\| e^n \right\|_1^2 + C \sum_{l=1}^{n} \left\| \partial_t e^l \right\|^2 \Delta t + C \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t \\ &\leq C \left(\left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_{L^2(0,T;L^2(\Omega))}^2 + \int_0^{t_n} \left\| u_{tt} \right\|_1^2 dt \right) (\Delta t)^2 \\ &+C \left(h^4 \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2 + h^2 \left\| u \right\|_{L^\infty(0,T;H^2(\Omega))}^2 \right. \\ &\left. +h^4 \left\| u \right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2 \right) \\ &+C \left(\sum_{l=1}^{n} \left\| e^l \right\|_1^2 \Delta t + \sum_{l=1}^{n} \left\| e^l \right\|^2 \Delta t \right) \\ &+C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t. \end{aligned} \tag{46}$$

Choosing proper ε and kicking the two terms into the left hand side of (46), and applying discrete Gronwall Lemma 6, we get

$$\begin{aligned}&\left\| e^n \right\|_1^2 + \sum_{l=1}^{n} \left\| \partial_t e^l \right\|^2 \Delta t + \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t \\ &\leq C \left(\left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_{L^2(0,T;L^2(\Omega))}^2 + \int_0^{t_n} \left\| u_{tt} \right\|_1^2 dt \right) (\Delta t)^2 \\ &+C \left(\left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2 + \left\| u \right\|_{L^\infty(0,T;H^2(\Omega))}^2 \right. \\ &\left. + \left\| u \right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2 \right) h^2. \end{aligned} \tag{47}$$

Then, we have

$$\left\| e^n \right\|_1 \leq C \left(h + \Delta t \right), \tag{48}$$

Together Lemma 5, we have the estimate (32). ■

Theorem 4.2: Let u and u_h be the solutions of (12) and (23), respectively. Under assumption (a)-(f), for $\triangle t$ small enough, if $u_h^0 = P_h u_0$ with P_h defined by Lemma 5, we have, for $t_n \leq T$,

$$\max_{1 \leq n \leq N} \left\| u^n - u_h^n \right\| \leq C(\Delta t + h^2). \tag{49}$$

where $C = C(\|u\|_{L^\infty(H^2(\Omega))}, \|u_{tt}\|_{L^2(H^1(\Omega))}, \|u_{\tau\tau}\|_{L^2(L^2(\Omega))}, \|u_t\|_{L^\infty(W^{3,p}(\Omega))}, \|u\|_{L^\infty(W^{3,p}(\Omega))})$ is independent of h and Δt.

Proof: In (33), choosing $v_h = e^n$,we obtain

$$\begin{aligned}&(c(x)\partial_t e^n, \Pi_h^* e^n) + a_h \left(\partial_t e^n, \Pi_h^* e^n \right) + b_h \left(e^n, \Pi_h^* e^n \right) \\ &= \left(\psi(x) \frac{\partial u^n}{\partial \tau} - c(x) \frac{u^n - \overline{u}^{n-1}}{\Delta t}, \Pi_h^* e^n \right) \\ &+a_h \left(u_t^n - \partial_t u^n, \Pi_h^* e^n \right) + \left(f(u_h^n) - f(u^n), \Pi_h^* e^n \right) \\ &+ \left(c(x) \frac{\rho^n - \rho^{n-1}}{\Delta t}, \Pi_h^* e^n \right) \\ &+ \left(c(x) \frac{\rho^{n-1} - \overline{\rho}^{n-1}}{\Delta t}, \Pi_h^* e^n \right) \\ &+ \left(c(x) \frac{\overline{e}^{n-1} - e^{n-1}}{\Delta t}, \Pi_h^* e^n \right). \end{aligned} \tag{50}$$

For the first term of the left-hand side of (50), we have

$$\begin{aligned}&(c(x)\partial_t e^n, \Pi_h^* e^n) \\ &= \frac{1}{2\Delta t} [c(x) \left(e^n - e^{n-1}, \Pi_h^* \left(e^n + e^{n-1} \right) \right) \\ &+c(x) \left(e^n - e^{n-1}, \Pi_h^* \left(e^n - e^{n-1} \right) \right)] \\ &\geq \frac{1}{2\Delta t} c(x) \left(e^n - e^{n-1}, \Pi_h^* \left(e^n + e^{n-1} \right) \right) \\ &= \frac{1}{2\Delta t} c(x) \left[\left(e^n, \Pi_h^* e^n \right) - \left(e^{n-1}, \Pi_h^* e^{n-1} \right) \right] \\ &= \frac{1}{2\Delta t} c(x) \left(|||e^n|||_0^2 - |||e^{n-1}|||_0^2 \right). \end{aligned} \tag{51}$$

For the second term of the left-hand side of (50), we have

$$\begin{aligned}&a_h \left(\partial_t e^n, \Pi_h^* e^n \right) \\ &= \frac{1}{2\Delta t} [a_h \left(e^n - e^{n-1}, \Pi_h^* \left(e^n + e^{n-1} \right) \right) \\ &+a_h \left(e^n - e^{n-1}, \Pi_h^* \left(e^n - e^{n-1} \right) \right)] \\ &\geq \frac{1}{2\Delta t} a_h \left(e^n - e^{n-1}, \Pi_h^* \left(e^n + e^{n-1} \right) \right) \\ &= \frac{1}{2\Delta t} \left[a_h \left(e^n, \Pi_h^* e^n \right) - a_h \left(e^{n-1}, \Pi_h^* e^{n-1} \right) \right] \\ &+ \frac{1}{2} \left[a_h \left(\partial_t e^n, \Pi_h^* e^n \right) - a_h \left(e^n, \Pi_h^* \partial_t e^n \right) \right]. \end{aligned} \tag{52}$$

By (50)-(52), we have

$$\begin{aligned}&\frac{1}{2\Delta t} c(x) [\left(|||e^n|||_0^2 - |||e^{n-1}|||_0^2 \right) \\ &+ \left(|||e^n|||_1^2 - |||e^{n-1}|||_1^2 \right)] + b_h \left(e^n, \Pi_h^* e^n \right) \\ &\leq \left(\psi(x) \frac{\partial u^n}{\partial \tau} - c(x) \frac{u^n - \overline{u}^{n-1}}{\Delta t}, \Pi_h^* e^n \right) \\ &+a_h \left(u_t^n - \partial_t u^n, \Pi_h^* e^n \right) + \left(c(x) \frac{\rho^n - \rho^{n-1}}{\Delta t}, \Pi_h^* e^n \right) \\ &+ \left(c(x) \frac{\rho^{n-1} - \overline{\rho}^{n-1}}{\Delta t}, \Pi_h^* e^n \right) \\ &+ \left(c(x) \frac{\overline{e}^{n-1} - e^{n-1}}{\Delta t}, \Pi_h^* e^n \right) \end{aligned}$$

$$+\frac{1}{2}\left[a_h\left(e^n,\Pi_h^*\partial_t e^n\right)-a_h\left(\partial_t e^n,\Pi_h^* e^n\right)\right]$$
$$+\left(f(u_h^n)-f(u^n),\Pi_h^* e^n\right). \quad (53)$$

By Lemma 2 and (50)-(52) multiplying $\triangle t$ and summing over l from 1 to $n(1\le n\le N)$ at both sides of (53), since $e^0=0$ we have

$$C\left\|e^n\right\|^2+\left\|e^n\right\|_1^2+C\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t$$
$$\le\sum_{l=1}^n\left(\psi(x)\frac{\partial u^l}{\partial\tau}-c(x)\frac{u^l-\overline{u}^{l-1}}{\Delta t},\Pi_h^* e^l\right)\Delta t$$
$$+\sum_{l=1}^n a_h\left(u_t^l-\partial_t u^l,\Pi_h^* e^l\right)\Delta t$$
$$+\sum_{l=1}^n\left(c(x)\frac{\rho^l-\rho^{l-1}}{\Delta t},\Pi_h^* e^l\right)\Delta t$$
$$+\sum_{l=1}^n\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},\Pi_h^* e^l\right)\Delta t$$
$$+\sum_{l=1}^n\left(c(x)\frac{\overline{e}^{l-1}-e^{l-1}}{\Delta t},\Pi_h^* e^l\right)\Delta t$$
$$+\frac{1}{2}\sum_{l=1}^n\left[a_h\left(e^l,\Pi_h^*\partial_t e^l\right)-a_h\left(\partial_t e^l,\Pi_h^* e^l\right)\right]\Delta t$$
$$+\sum_{l=1}^n\left(f(u_h^l)-f(u^l),\Pi_h^* e^n\right)\Delta t\equiv\sum_{i=1}^7 Q_i. \quad (54)$$

For Q_1, similar as the estimate of T_1, we have

$$|Q_1|\le C\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|_{L^2(0,T;L^2(\Omega))}^2(\Delta t)^2+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|^2\Delta t. \quad (55)$$

Similarly, for Q_2, we have

$$|Q_2|\le MC(\varepsilon)\left(\int_0^{t_n}\left\|u_{tt}\right\|_1^2\,dt\right)(\Delta t)^2+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t. \quad (56)$$

For Q_3, we have

$$|Q_3|\le C(\varepsilon)h^4\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|^2\Delta t. \quad (57)$$

For Q_4, we have

$$|Q_4|=\left|\sum_{l=1}^n\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},\Pi_h^* e^l\right)\Delta t\right|$$
$$\le\left|\sum_{l=1}^n\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},\Pi_h^* e^l-e^l\right)\Delta t\right|$$
$$+\left|\sum_{l=1}^n\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},e^l\right)\Delta t\right|$$
$$\equiv Q_{41}+Q_{42}. \quad (58)$$

For Q_{41}, by (8), Lemma 5 and Lemma 7, we obtain

$$Q_{41}=\left|\sum_{l=1}^n\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},\Pi_h^* e^l-e^l\right)\Delta t\right|$$
$$\le C\sum_{l=1}^n\left\|\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t}\right\|\left\|\Pi_h^* e^l-e^l\right\|\Delta t$$
$$\le C\sum_{l=1}^n\left\|\rho^{l-1}\right\|_1 h\left\|e^l\right\|_1\Delta t$$
$$\le C\sum_{l=1}^n h^2\left\|\rho^{l-1}\right\|_1^2\Delta t+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t$$
$$\le Ch^4\left\|u\right\|_{L^\infty(0,T;H^2(\Omega))}^2+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t \quad (59)$$

In [20] Douglas and Russell have proved that

$$\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},e^l\right)\le C\left\|\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t}\right\|_{-1}\left\|e^l\right\|_1$$
$$\le C\left\|\rho^{l-1}\right\|^2+\varepsilon\left\|e^l\right\|_1^2,$$

using this inequality and Lemma 5, we have

$$Q_{42}\le C\sum_{l=1}^n\left\|\rho^{l-1}\right\|^2\Delta t+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t$$
$$\le Ch^4\left\|u\right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t. \quad (60)$$

Then, there is

$$|Q_4|\le Ch^4\left\|u\right\|_{L^\infty(0,T;H^2(\Omega))}^2+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t$$
$$+Ch^4\left\|u\right\|_{L^\infty(0,T;W^{3,p}(\Omega))}^2. \quad (61)$$

For Q_5, by Lemma 7 and (9), we have

$$|Q_5|\le\sum_{l=1}^n\left|\left(c(x)\frac{\overline{e}^{l-1}-e^{l-1}}{\Delta t},\Pi_h^* e^l\right)\right|\Delta t$$
$$\le C\sum_{l=1}^n\left\|\frac{\overline{e}^{l-1}-e^{l-1}}{\Delta t}\right\|\left\|\Pi_h^* e^l\right\|\Delta t$$
$$\le C\sum_{l=1}^n\left\|e^{l-1}\right\|_1\left\|\Pi_h^* e^l\right\|\Delta t$$
$$\le C\varepsilon\sum_{l=1}^n\left\|e^{l-1}\right\|_1^2\Delta t+C\sum_{l=1}^n\left\|e^l\right\|^2\Delta t \quad (62)$$

For Q_6, we have

$$|Q_6|\le\frac{1}{2}\sum_{l=1}^n\left|\left[a_h\left(e^l,\Pi_h^*\partial_t e^l\right)-a_h\left(\partial_t e^l,\Pi_h^* e^l\right)\right]\right|\Delta t$$
$$\le C\sum_{l=1}^n h\left\|\partial_t e^l\right\|_1\left\|e^l\right\|_1\Delta t\le C\sum_{l=1}^n\left\|\partial_t e^l\right\|\left\|e^l\right\|_1\Delta t$$
$$\le C\sum_{l=1}^n\left\|e^l\right\|_1^2\Delta t+C\varepsilon\sum_{l=1}^n\left\|\partial_t e^l\right\|^2\Delta t. \quad (63)$$

For Q_7, we have

$$|Q_7|\le\sum_{l=1}^n\left|\left(f(u_h^l)-f(u^l),\Pi_h^* e^l\right)\right|\Delta t$$
$$\le\sum_{l=1}^n\left\|f(u_h^l)-f(u^l)\right\|\left\|\Pi_h^* e^l\right\|$$
$$\le C(\varepsilon)\sum_{l=1}^n\left\|f(u_h^l)-f(u^l)\right\|^2\Delta t+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|^2\Delta t$$
$$\le C(\varepsilon)\sum_{l=1}^n(\left\|e^l\right\|^2+\left\|\rho^l\right\|^2)\Delta t+C\varepsilon\sum_{l=1}^n\left\|e^l\right\|^2\Delta t$$

$$\leq C(\sum_{l=1}^{n}\left\|e^l\right\|^2\Delta t+h^4\left\|u\right\|^2_{L^\infty(0,T;W^{3,p}(\Omega))})$$
$$+C\varepsilon\sum_{l=1}^{n}\left\|e^l\right\|^2\Delta t. \quad (64)$$

Combing the error estimates of $Q_i(1\leq i\leq 7)$ with (54), we have

$$C\left\|e^n\right\|^2+\left\|e^n\right\|_1^2+C\sum_{l=1}^{n}\left\|e^l\right\|_1^2\Delta t$$
$$\leq\sum_{l=1}^{n}\left(\psi(x)\frac{\partial u^l}{\partial\tau}-c(x)\frac{u^l-\overline{u}^{l-1}}{\Delta t},\Pi_h^*e^l\right)\Delta t$$
$$\leq C\left(\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2_{L^2(0,T;L^2(\Omega))}+\int_0^{t_n}\left\|u_{tt}\right\|_1^2\,dt\right)(\Delta t)^2$$
$$+C(\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;W^{3,p}(\Omega))}+\left\|u\right\|^2_{L^\infty(0,T;H^2(\Omega))}$$
$$+\left\|u\right\|^2_{L^\infty(0,T;W^{3,p}(\Omega))})h^4+C\varepsilon\sum_{l=1}^{n}\left\|e^l\right\|_1^2\Delta t$$
$$+C\left(\sum_{l=1}^{n}\left\|e^l\right\|^2\Delta t+C\varepsilon\sum_{l=1}^{n}\left\|\partial_te^l\right\|^2\Delta t\right). \quad (65)$$

Choosing proper ε and kicking the two terms into the left hand side of (65), and applying discrete Gronwall Lemma 6, we get

$$\left\|e^n\right\|^2+\left\|e^n\right\|_1^2+\sum_{l=1}^{n}\left\|e^l\right\|_1^2\Delta t$$
$$\leq\sum_{l=1}^{n}\left(\psi(x)\frac{\partial u^l}{\partial\tau}-c(x)\frac{u^l-\overline{u}^{l-1}}{\Delta t},\Pi_h^*e^l\right)\Delta t$$
$$\leq C\left(\left\|\frac{\partial^2u}{\partial\tau^2}\right\|^2_{L^2(0,T;L^2(\Omega))}+\int_0^{t_n}\left\|u_{tt}\right\|_1^2\,dt\right)(\Delta t)^2$$
$$+C(\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;W^{3,p}(\Omega))}+\left\|u\right\|^2_{L^\infty(0,T;H^2(\Omega))}$$
$$+\left\|u\right\|^2_{L^\infty(0,T;W^{3,p}(\Omega))})h^4 \quad (66)$$

Then we get, for all

$$\left\|e^n\right\|\leq C\left(\Delta t+h^2\right), \quad (67)$$

Together Lemma 5, we have the estimate (49). ■

V. Error analysis for two-grid characteristic FVE method

In this section, we consider the error estimates in the H^1-norm for the two-grid characteristic FVE method. For the two-grid characteristic FVE method Algorithm 1, we have:

Theorem 5.1: Let u and u_h are the solutions of (12) and the two-grid FVE method Algorithm 1, respectively. Under assumption(a)-(f), and the coarse grid size H and the time step Δt satisfies $H^{-1}\Delta t<C$. For Δt small enough, if $u_h^0=P_hu_0$ with P_h defined by Lemma 5, we have, for $t_n\leq T$, the following estimate

$$\max_{1\leq n\leq N}\left\|u^n-u_h^n\right\|_1\leq C\left(\Delta t+h+H^3\right). \quad (68)$$

Proof: Once again, let $u_h^n-u^n=(u_h^n-P_hu^n)-(u^n-P_hu^n)=:e^n-\rho^n$. Then from (12) and (30), we get the following error equation at t_n

$$\left(c(x)\frac{e^n-\overline{e}^{n-1}}{\Delta t},\Pi_h^*v_h\right)+a_h\left(\partial_te^n,\Pi_h^*v_h\right)+$$
$$b_h\left(e^n,\Pi_h^*v_h\right)=\left(\psi(x)\frac{\partial u^n}{\partial\tau}-c(x)\frac{u^n-\overline{u}^{n-1}}{\Delta t},\Pi_h^*v_h\right)$$
$$+a_h\left(u_t^n-\partial_tu^n,\Pi_h^*v_h\right)+\left(c(x)\frac{\rho^n-\overline{\rho}^{n-1}}{\Delta t},\Pi_h^*v_h\right)$$
$$-\left(f(u^n)-f(u_H^n)-f'(u_H^n)(u_h^n-u_H^n),\Pi_h^*v_h\right),$$
$$\forall v_h\in U_h. \quad (69)$$

Choosing $v_h=\partial_te^n$ and (69) can be written as

$$\left(c(x)\partial_te^n,\Pi_h^*\partial_te^n\right)+a_h\left(\partial_te^n,\Pi_h^*\partial_te^n\right)$$
$$+b_h\left(e^n,\Pi_h^*\partial_te^n\right)$$
$$=\left(\psi(x)\frac{\partial u^n}{\partial\tau}-c(x)\frac{u^n-\overline{u}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)$$
$$+a_h\left(u_t^n-\partial_tu^n,\Pi_h^*\partial_te^n\right)$$
$$+\left(c(x)\frac{\rho^n-\rho^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)$$
$$+\left(c(x)\frac{\rho^{n-1}-\overline{\rho}^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)$$
$$+\left(c(x)\frac{\overline{e}^{n-1}-e^{n-1}}{\Delta t},\Pi_h^*\partial_te^n\right)$$
$$-(f(u^n)-f(u_H^n)-f'(u_H^n)(u_h^n-u_H^n),$$
$$\Pi_h^*\partial_te^n). \quad (70)$$

By Lemma 2 and (37), we have

$$\frac{\alpha}{2}\left\|e^n\right\|_1^2+C\sum_{l=1}^{n}\left\|\partial_te^l\right\|^2\Delta t+C\sum_{l=1}^{n}\left\|\partial_te^l\right\|_1^2\Delta t$$
$$\leq\sum_{l=1}^{n}\left(\psi(x)\frac{\partial u^l}{\partial\tau}-c(x)\frac{u^l-\overline{u}^{l-1}}{\Delta t},\Pi_h^*\partial_te^l\right)\Delta t$$
$$+\sum_{l=1}^{n}a_h\left(u_t^l-\partial_tu^l,\Pi_h^*\partial_te^l\right)\Delta t$$
$$+\sum_{l=1}^{n}\left(c(x)\frac{\rho^l-\rho^{l-1}}{\Delta t},\Pi_h^*\partial_te^l\right)\Delta t$$
$$+\sum_{l=1}^{n}\left(c(x)\frac{\rho^{l-1}-\overline{\rho}^{l-1}}{\Delta t},\Pi_h^*\partial_te^l\right)\Delta t$$
$$+\sum_{l=1}^{n}\left(c(x)\frac{\overline{e}^{l-1}-e^{l-1}}{\Delta t},\Pi_h^*\partial_te^l\right)\Delta t$$
$$+\frac{1}{2}\sum_{l=1}^{n}\left[b_h\left(\partial_te^l,\Pi_h^*e^l\right)-b_h\left(e^l,\Pi_h^*\partial_te^l\right)\right]\Delta t$$
$$-\sum_{l=1}^{n}(f(u^n)-f(u_H^n)-f'(u_H^n)(u_h^n-u_H^n),$$
$$\Pi_h^*\partial_te^l)\Delta t$$
$$\equiv\sum_{i=1}^{6}T_i+T_7' \quad (71)$$

For T_1-T_6, we can estimate them similarly as in Theorem 4.1. For the last term of the right-hand side of (71), a Taylor expansion about u_H^n yields

$$f(u^n) = f(u_H^n) + f'(u_H^n)(u^n - u_H^n) + \frac{1}{2}f''(\widetilde{u})(u^n - u_H^n)^2,$$

for some function $\widetilde{u}$. Then

$$\begin{aligned} & f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n) \\ &= f'(u_H^n)(u^n - u_h^n) + \frac{1}{2}f''(\widetilde{u})(u^n - u_H^n)^2 \\ &= f'(u_H^n)(\rho^n + e^n) + \frac{1}{2}f''(\widetilde{u})(u^n - u_H^n)^2. \end{aligned}$$

So by assumption (c) and Lemma 5, we have

$$\begin{aligned} & |T_7'| \leq \sum_{l=1}^{n} |(f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n), \\ & \Pi_h^* \partial_t e^l)|\Delta t \\ & \leq \sum_{l=1}^{n} \|f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n)\| \, \|\Pi_h^* \partial_t e^l\| \\ & \leq C(\varepsilon) \sum_{l=1}^{n} \|f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n)\|^2 \Delta t \\ & + C\varepsilon \sum_{l=1}^{n} \left\|\partial_t e^l\right\|^2 \Delta t \\ & \leq C(\varepsilon) \sum_{l=1}^{n} (\left\|e^l\right\|^2 + \left\|\rho^l\right\|^2)\Delta t \\ & + C(\varepsilon) \sum_{l=1}^{n} \left\|\left(u^l - u_H^l\right)^2\right\|^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\|\partial_t e^l\right\|^2 \Delta t \\ & \leq C\left(\sum_{l=1}^{n} \left\|e^l\right\|^2 \Delta t + h^4 \|u\|^2_{L^\infty(0,T;H^2(\Omega))}\right) \\ & + C(\varepsilon) \sum_{l=1}^{n} \left\|\left(u^l - u_H^l\right)^2\right\|^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\|\partial_t e^l\right\|^2 \Delta t \quad (72) \end{aligned}$$

So we have

$$\begin{aligned} & \|e^n\|_1^2 + \sum_{l=1}^{n} \left\|\partial_t e^l\right\|^2 \Delta t + \sum_{l=1}^{n} \left\|\partial_t e^l\right\|_1^2 \Delta t \\ & \leq C\left(\left\|\frac{\partial^2 u}{\partial \tau^2}\right\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^{t_n} \|u_{tt}\|_1^2 \, dt\right)(\Delta t)^2 \\ & + C\left(\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;H^2(\Omega))} + \|u\|^2_{L^\infty(0,T;H^2(\Omega))} \right. \\ & \left. + \|u\|^2_{L^\infty(0,T;H^2(\Omega))}\right) h^2 \\ & + C(\varepsilon) \sum_{l=1}^{n} \left\|\left(u^l - u_H^l\right)^2\right\|^2 \Delta t. \quad (73) \end{aligned}$$

For the last term of (73), we have

$$\begin{aligned} & \left\|(u^n - u_H^n)^2\right\|^2 \leq \|u^n - u_H^n\|^2_{0,\infty} \|u^n - u_H^n\|^2 \\ & \leq \left(\|u^n - P_H u^n\|_{0,\infty} + \|P_H u^n - u_H^n\|_{0,\infty}\right)^2 \\ & \|u^n - u_H^n\|^2, \quad (74) \end{aligned}$$

where P_H is defined in the same way as P_h is defined by Lemma 5. By Theorem 4.2, and the inverse estimate, we get

$$\begin{aligned} & \left\|(u^n - u_H^n)^2\right\|^2 \\ & \leq C\left(H^2 |lnH| + H^{-1} \|P_H u^n - u_H^n\|\right)^2 \left(\Delta t + H^2\right)^2 \\ & \leq C\left(H^2 |lnH| + H^{-1}\left(\Delta t + H^2\right)\right)^2 \left(\Delta t + H^2\right)^2 \\ & \leq C(H^2 |lnH| \Delta t + H^4 |lnH| \\ & + H^{-1}(\Delta t)^2 + 2H\Delta t + H^3)^2. \quad (75) \end{aligned}$$

We can choose H and Δt such that $H^{-1}\Delta t < C$, then we have

$$\left\|(u^n - u_H^n)^2\right\|^2 \leq C\left(\Delta t + H^3\right)^2, \quad (76)$$

with (73), we get

$$\|e^l\|_1 \leq C(\Delta t + h + H^3). \quad (77)$$

■

For the two-grid characteristic FVE method Algorithm 2, we can have a similar result.

Theorem 5.2: Let u and u_h are the solutions of (12) and the two-grid FVE method Algorithm 2, respectively. Under assumption (a)-(f), and the coarse grid size H and the time step Δt satisfied $H^{-1}\Delta t < C$. For Δt small enough, if $u_h^0 = P_h u_0$ with P_h defined by Lemma 5, we have, for $t_n \leq T$,

$$\max_{1 \leq n \leq N} \|u^n - u_h^n\|_1 \leq C\left(\Delta t + h + H^3\right). \quad (78)$$

VI. Conclusion

In this paper, we have presented the error estimates for the two-grid FVE method and the two-grid characteristic FVE method for a semi-linear Sobolev equation. The theorems above demonstrate a remarkable fact about two-grid characteristic FVE method: we can iterate on a very coarse grid T_H and still get good approximations by taking one iteration on the fine grid T_h . It is proved that the coarse grid can be much coarser than the fine grid ($h \ll H$). We can achieve optimal approximation in H^1-norm error estimate as long as the mesh sizes satisfy $h = O(H^3)$.

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