Analysis of Systemic Risk: A Vine Copula-based ARMA-GARCH Model

Kuan-Heng Chen and Khaldoun Khashanah

Abstract— **In this paper, a model for analyzing each U.S. Equity sector's risk contribution (VaR ratio), the ratio of the Value-at-Risk of a sector to the Value-at-Risk of the system (S&P 500 Index), with vine Copula-based ARMA-GARCH (1, 1) modeling is presented. Vine copula modeling not only has the advantage of extending to higher dimensions easily, but also provides a more flexible measure to capture an asymmetric dependence among assets. We investigate systemic risk in 10 S&P 500 sector indices in the U.S. stock market by forecasting one-day ahead VaR and one-day ahead VaR ratio during the 2008 financial subprime crisis. Our evidence reveals vine Copula-based ARMA-GARCH (1, 1) is the appropriate model to forecast and analyze systemic risk.**

Index Terms—**Copula, Time Series, GARCH, Systemic Risk, Value-at-Risk**

I. Introduction

The definition of systemic risk from the Report to G20 Finance Ministers and Governors agreed upon among the International Monetary Fund (IMF), Bank for International Settlements (BIS) and Financial Stability Board (FSB) [3] that is *"(i) caused by an impairment of all or parts of the financial system and (ii) has the potential to have serious negative consequences for the real economy"*. Furthermore, *"G-20 members consider an institution, market or instrument as systemic if its failure or malfunction causes widespread distress, either as a direct impact or as a trigger for broader contagion."* A common factor in the various definitions of systemic risk is that a trigger event causes a chain of bad economic consequences, referred to as a *"domino effect"*. Given the definition of systemic risk quoted above, measuring systemic risk is done by estimating the probability of failure of an institute that is the cause of distress for the financial system. Therefore, we only consider the Value-at-Risk ratio of a sector to the system (S&P 500 Index), which interprets the sector risk adds to the entire system.

Girardi and Ergun [10] modify the CoVaR methodology, proposed by Adrian and Brunnermeier [1], by using the dynamic conditional correlation GARCH, while Hakwa *et al.* [11] also modified the methodology based on copula modeling. We extend their concepts and present vine Copula-based ARMA-GARCH (1, 1) VaR measure into a high dimensional analysis in systemic risk.

Sklar [22] introduced the copula, which describes the dependence structure between variables. Patton [18] defined the conditional version of Sklar's theorem, which extends the copula applications to the time series analysis. Otani and Imai [17] presented a basket CDSs pricing model with nested Archimedean copulas. However, multivariate Archimedean copulas are limited in that there are only one or two parameters to capture the dependence structure. Joe [12] introduced a construction of multivariate distribution based on pair-copula construction (PCC), while Aas *et al.* were the first to recognize that the pair-copula construction (PCC) principal can be used with arbitrary pair-copulas, referred to as the graphical structure of R-vines [14]. Furthermore, Dissmann *et al.* [7] developed an automated algorithm of jointly searching for an appropriate R-vines tree structures, the pair-copula families and their parameters. Accordingly, a high dimensional joint distribution can be decomposed to bivariate and conditional bivariate copulas arranged together according to the graphical structure of a regular vine. Besides, Rockinger and Jondeau [19] was the first to introduce the copula-based GARCH modeling. Afterwards, Lee and Long [15] concluded that copula-based GARCH models outperform the dynamic conditional correlation model, the varying correlation model and the BEKK model. In addition, Fang *et al.* [8] investigated that using Akaike Information Criterion (AIC) as a tool for choosing copula from a couple of candidates is more efficient and accurate than the multiplier goodness-of-fit test method.

The purpose of this paper is to present an application of the estimation of systemic risk in terms of the VaR/ES ratio by using vine copula-based ARMA-GARCH (1, 1) model, and the result provides the important conclusion that the method is a real-time and efficient tool to analyze systemic risk.

This paper has four sections. The first section briefly introduces existing research regarding systemic risk. The second section describes the definition of the VaR/ES ratio, and outlines the methodology of vine Copula-based GARCH (1, 1) modeling. The third section describes the data and explains the empirical results of VaR/ES ratio. The fourth section concludes our findings.

II. Methodology

A. Risk Methodology

The definition of Value-at-Risk (VaR) is that the maximum loss at most is $(1-\alpha)$ probability given by a period [20].

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People usually determines α as 95%, 99%, or 99.9% to be their confidence level. In this study, we use the Copula-based ARMA-GARCH (1, 1) methodology to obtain the VaR from each sector. We denote $VaR_{1-\alpha}^{i\to j}$ ratio, the sector $i's$ risk contribution to the system j (S&P 500 index) at the confidence level α, by

$$VaR_{1-\alpha}^{i\to j}\ ratio = \frac{VaR_{1-\alpha}^{i}}{VaR_{1-\alpha}^{j}}$$

The higher $VaR_{1-\alpha}^{i\to j}$ ratio interprets the sector is the risk provider to the system. In addition, the methodology can be easily extended from VaR ratio to an expected shortfall (ES) ratio.

B. Univariate ARMA-GARCH Model

Engle is the first researcher to introduce the ARCH model, which deals with the volatility clustering, usually referred to as conditional heteroskedasticity. Bollerslev [4] extended the ARCH model to the generalized ARCH (GARCH) model. Chen and Khashanah [5] implemented ARMA (p, q)-GARCH (1, 1) with the Student's t distributed innovations for the marginal to account for the time-varying volatility, whereas the Student's t distributed innovations cannot explain the skewness. Therefore, we employ ARMA (p, q)-GARCH (1, 1) with the skewed Student's t distributed innovation can then be written as [9]

$$r_t = \mu_t + \sum_{i=1}^{p} \varphi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t,$$
$$\epsilon_t = \sigma_t z_t,$$
$$\sigma_t^2 = \gamma_t + \alpha_t \sigma_{t-1}^2 + \beta_t \epsilon_{t-1}^2$$

where r_t is the log return, μ_t is the drift term, ϵ_t is the error term, and the innovation term z_t is the skewed Student's t distribution. The skewed student's t density function can be expressed as

$$p(z|\xi,f) = \frac{2}{\xi+\frac{1}{\xi}}\left\{f\left(\frac{z}{\xi}\right)I_{[0,\infty)}(z) + f(\xi z)I_{(-\infty,0]}(z)\right\}$$

where ξ is the asymmetric parameter, and $\xi = 1$ for the symmetric Student's t distribution. f is a univariate pdf that is symmetric around 0, such that $f(s)$ is decreasing in $|s|$, and I_S is the indicator function on S. In addition, an overwhelming feature of Copula-based ARMA-GARCH model is the ease with which the correlated random variables can be flexible and easily estimated.

C. Sklar's theory

Sklar's Theorem [22] states that given random variables $X_1, X_2, \dots, X_n$ with continuous distribution functions $F_1, F_2, \dots, F_n$ and joint distribution function H, and there exists a unique copula C such that for all $x = (x_1, x_2, \dots, x_n) \in R^n$

$$H(x) = C\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big)$$

If the joint distribution function is n-times differentiable, then taking the n[th] cross-partial derivative of the equation:

$$f(x_1, x_2, \dots, x_n) = \frac{\partial^n}{\partial x_1 \dots \partial x_n} H(x)$$
$$= \frac{\partial^n}{\partial u_1 \dots \partial u_n} C\big(F_1(x_1), \dots, F_n(x_n)\big) \prod_{i=1}^{n} f_i(x_i)$$
$$= c(F_1(x_1), \dots, F_n(x_n)) \prod_{i=1}^{n} f_i(x_i)$$

where u_i is the probability integral transform of x_i.

For the purpose of estimating the VaR or ES based on time series data, Patton [18] defined the conditional version of Sklar's theorem. Let $F_{1,t}$ and $F_{2,t}$ be the continuous conditional distriubtions of $X_1|\mathcal{F}_{t-1}$ and $X_2|\mathcal{F}_{t-1}$, given the conditioning set $\mathcal{F}_{t-1}$, and let H_t be the joint conditional bivariate distribution of $(X_1, X_2|\mathcal{F}_{t-1})$. Then, there exists a unique conditional copula C_t such that

$$H_t(x_1, x_2|\mathcal{F}_{t-1}) = C_t(F_{1,t}(x_1|\mathcal{F}_{t-1}), F_{2,t}(x_2|\mathcal{F}_{t-1})|\mathcal{F}_{t-1})$$

D. Parametric Copulas

Joe [13] and Nelsen [16] gave comprehensive copula definitions for each family.

(1) The bivariate Gaussian copula is defined as:

$$C(u_1, u_2; \rho) = \Phi_\rho\big(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\big)$$

where Φ_ρ is the bivariate joint normal distribution with linear correlation coefficient ρ and Φ is the standard normal marginal distribution.

(2) The bivariate student's t copula is defined by the following:

$$C(u_1, u_2; \rho, \nu) = t_{\rho,\nu}\big(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\big)$$

where ρ is the linear correlation coefficient and ν is the degree of freedom.

(3) The Clayton generator is given by $\varphi(u) = u^{-\theta} - 1$, its copula is defined by

$$C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}},$$
with $\theta \in (0, \infty)$

(4) The Gumbel generator is given by $\varphi(u) = (-\ln u)^\theta$, and the bivariate Gumbel copula is given by

$$C(u_1, u_2; \theta) = \exp(-[(-\ln u_1)^\theta + \big(-\ln u_2)^\theta]^{\frac{1}{\theta}}\big),$$
with $\theta \in [1, \infty)$

(5) The Frank generator is given by $\varphi(u) = ln(\frac{e^{-\theta u}-1}{e^{-\theta}-1})$, and the bivariate Frank copula is defined by

$$C(u_1, u_2; \theta) = -\frac{1}{\theta} log\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right),$$
with $\theta \in (-\infty, 0) \cup (0, \infty)$

(6) The Joe generator is $\varphi(u) = u^{-\theta} - 1$, and the Joe copula is given by

$$C(u_1, u_2) = 1 - (\overline{u_1}^\theta + \overline{u_2}^\theta - \overline{u_1}^\theta \overline{u_2}^\theta)^{\frac{1}{\theta}},$$
with $\theta \in [1, \infty)$

(7) The BB1 (Clayton-Gumbel) copula is given by

$$C(u_1, u_2; \theta, \delta) = (1 + [(u_1{}^{-\theta} - 1)^\delta + (u_2{}^{-\theta} - 1)^\delta]^{\frac{1}{\delta}})^{\frac{-1}{\theta}},$$
with $\theta \in (0, \infty) \cap \delta \in [1, \infty)$

(8) The BB6 (Joe-Gumbel) copula is

$$C(u_1, u_2; \theta, \delta) =$$
$$1 - (1 - \exp\{-[\big(-log(1 - \overline{u_1}^\theta)\big)^\delta + \big(-log(1 - \overline{u_2}^\theta)\big)^\delta]^{\frac{1}{\delta}}\})^{\frac{1}{\theta}},$$
with $\theta \in [1, \infty) \cap \delta \in [1, \infty)$

(9) The BB7 (Joe-Clayton) copula is given by

$$C(u_1,u_2;\theta,\delta) = 1-(1-[(1-\overline{u_1}^{\theta})^{-\delta}+(1-\overline{u_2}^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}})^{\frac{1}{\theta}},$$

with $\theta \in [1,\infty) \cap \delta \in [0,\infty)$

(10) The BB8 (Frank-Joe) copula is

$$C(u_1,u_2;\theta,\delta) = \frac{1}{\delta}(1-[1-\frac{1}{1-(1-\delta)^{\theta}}(1-(1-\delta u_1)^{\theta})\left(1-(1-\delta u_2)^{\theta}\right)]^{\frac{1}{\theta}}\Big),$$

with $\theta \in [1,\infty) \cap \delta \in (0,1]$

E. Vine Copulas

Even though it is simple to generate multivariate Archimedean copulas, they are limited in that there are only one or two parameters to capture the dependence structure. Vine copula method allows a joint distribution to be built from bivariate and conditional bivariate copulas arranged together according to the graphical structure of a regular vine, which is a more flexible measure to capture the dependence structure among assets. It is well known that any multivariate density function can be decomposed as

$$f(x_1,\dots,x_n) = f(x_1)\cdot f(x_2|x_1)\cdot f(x_3|x_1,x_2)\cdots f(x_n|x_1,\dots,x_{n-1})$$

Moreover, the conditional densities can be written as copula functions. For instance, the first and second conditional density can be decomposed as

$$f(x_2|x_1) = c_{1,2}\left(F_1(x_1),F_2(x_2)\right)\cdot f_2(x_2),$$

$$\begin{aligned} f(x_3|x_1,x_2) &= c_{2,3|1}\left(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1)\right)\cdot f_3(x_3|x_1) \\ &= c_{2,3|1}\left(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1)\right) \\ &\quad\cdot c_{1,3}(F_1(x_1),F_3(x_3))\cdot f_3(x_3) \end{aligned}$$

After rearranging the terms, the three dimensional joint density can be written as

$$\begin{aligned} f(x_1,x_2,x_3) = c_{2,3|1}&\left(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1)\right) \\ &\cdot c_{1,2}(F_1(x_1),F_2(x_2)) \\ &\cdot c_{1,3}(F_1(x_1),F_3(x_3))\cdot f_1(x_1)\cdot f_2(x_2) \\ &\cdot f_3(x_3) \end{aligned}$$

Bedford and Cooke [2] introduced canonical vine copulas, in which one variable plays a pivotal role. The summary of vine copulas is given by Kurowicka and Joe [14]. The general n-dimensional canonical vine copula can be written as

$$c(x_1,\dots,x_n) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-i} c_{i,i+j|1,\dots,i-1}\left(F(x_i|x_1,\dots,x_{i-1}),F\left(x_{i+j}|x_1,\dots,x_{i-1}\right)\right)$$

Similarly, D-vines are also constructed by choosing a specific order for the variables. The general n-dimensional D-vine copula can be written as

$$c(x_1,\dots,x_n) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-i} c_{j,j+i|j+1,\dots,j+i-1}\left(F(x_j|x_{j+1},\dots,x_{j+i-1}),F\left(x_{j+i}|x_{j+1},\dots,x_{j+i-1}\right)\right)$$

Dissmann *et al.* [7] proposed that the automated algorithm involves searching for an appropriate R-vine tree structure, the pair-copula families, and the parameter values of the chosen pair-copula families based on AIC, which is summarized in Table 1.

TABLE I

SEQUENTIAL METHOD TO SELECT AN R-VINE MODEL HE COEFFICIENTS OF TAIL DEPENDENCY

Algorithm. Sequential method to select an R-Vine model
1. Calculate the empirical Kendall's tau for all possible variable pairs.
2. Select the tree that maximizes the sum of absolute values of Kendall's taus.
3. Select a copula for each pair and fit the corresponding parameters based on AIC.
4. Transform the observations using the copula and parameters from Step 3. To obtain the transformed values.
5. Use transformed observations to calculate empirical Kendall's taus for all possible pairs.
6. Proceed with Step 2. Repeat until the R-Vine is fully specified.

F. Tail dependence

Tail dependence looks at the concordance and discordance in the tail, or extreme values of u_1 and u_2. It concentrates on the upper and lower quadrant tails of the joint distribution function. Given two random variables $u_1 \sim F_1$ and $u_2 \sim F_2$ with copula C, the coefficients of tail dependency are given by [6] [13] [16]

$$\lambda_L \equiv \lim_{u\to 0+} P[F_1(u_1)<u|F_2(u_2)<u] = \lim_{u\to 0+}\frac{C(u,u)}{u},$$

$$\lambda_U \equiv \lim_{u\to 1-} P[F_1(u_1)>u|F_2(u_2)>u] = \lim_{u\to 1-}\frac{1-2u+C(u,u)}{1-u}$$

where C is said to have lower (upper) tail dependency $iff\ \lambda_L \neq 0\ (\lambda_U \neq 0)$. The interpretation of the tail dependency is that it measures the probability of two random variables both taking extreme values shown as table 2 [6] [13] [16].

TABLE II

THE COEFFICIENTS OF TAIL DEPENDENCY

Family	Lower tail dependence	Upper tail dependence
Gaussian	*	*
Student's t	$2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{\frac{1-\theta}{1+\theta}})$	$2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{\frac{1-\theta}{1+\theta}})$
Clayton	$2^{-\frac{1}{\theta}}$	*
Gumbel	*	$2-2^{\frac{1}{\theta}}$
Frank	*	*
Joe	*	$2-2^{\frac{1}{\theta}}$
BB1 (Clayton-Gumbel)	$2^{-\frac{1}{\theta\delta}}$	$2-2^{\frac{1}{\delta}}$
BB6 (Joe-Gumbel)	*	$2-2^{\frac{1}{\theta\delta}}$
BB7 (Joe-Clayton)	$2^{-\frac{1}{\delta}}$	$2-2^{\frac{1}{\theta}}$
BB8 (Frank-Joe)	*	$2-2^{\frac{1}{\theta}}\ if\ \delta = 1,\ otherwise\ 0$

Note: * represents that there is no tail dependency.

G. Estimation method

Generally, the two-step separation procedure is called the inference functions for the margin method (IFM) [13]. It implies that the joint log-likelihood is simply the sum of univariate log-likelihoods and the copula log-likelihood shown is as below.

$$\log f(x) = \sum_{i=1}^{n} logf_i(x_i) + logc(F_1(x_1), \dots, F_n(x_n))$$

Therefore, it is convenient to use this two-step procedure to estimate the parameters by maximum log-likelihood, where marginal distributions and copulas are estimated separately.

III. Data and Empirical Findings

A. *Data Representation*

We use indices prices instead of other financial instruments or financial accounting numbers. One of the main reasons is that an index price could reflect a timely financial environment in contrast to financial accounting numbers that are published quarterly. Furthermore, indices can easily be constructed and tell us which sector contributes more risk to the entire market. Standard and Poor separates the 500 members in the S&P 500 index into 10 different sector indices based on the Global Industrial Classification Standard (GICS). All data is acquired from Bloomberg, sampled at daily frequency from January 1, 1995 to June 5, 2009. We separate the sample into two parts, the in-sample estimation period is from January 1, 1995 to December 31, 2007 (3271 observations) and the out-of-sample forecast validation period is from January 1, 2008 to June 5, 2009 (360 observations). The summary statistics of these indices is listed in table 3 as well as the statistical hypothesis testing. The statistical hypothesis testing for the unit-root based on Augmented Dickey-Fuller (ADF) test, and the result shows that the values of 1 in ADF test rejects the null hypothesis of a unit root in a univariate time series. The results of Jarque-Bera (J-B) test reject the distributions of returns are normality, and the results of Engle's ARCH test show that indices' returns present conditional heteroscedasticity at the 5% significance level. In addition, we assign the identify numbers to each sector.

TABLE III
THE SUMMARY STATISTICS OF THE IN-SAMPLE AND STATISTICAL HYPOTHESIS TESTINGS

ID	Sector	Mean	Sigma	Skew	Kurt	ADF test	J-B test	ARCH test	min / Max
1	S5FINL Index Financials	0.04%	1.41%	0.073	6.078	1	1	1	-8.04%/8.39%
2	S5INFT Index Technology	0.04%	1.99%	0.183	6.775	1	1	1	-10.01%/16.08%
3	S5COND Index Consumer Discretionary	0.03%	1.24%	-0.147	8.231	1	1	1	-10.33%/8.47%
4	S5ENRS Index Energy	0.06%	1.39%	-0.089	4.648	1	1	1	-7.21%/7.94%
5	S5HLTH Index Health Care	0.04%	1.21%	-0.180	7.097	1	1	1	-9.17%/7.66%
6	S5INDU Index Industrials	0.04%	1.18%	-0.227	7.410	1	1	1	-9.60%/7.21%
7	S5UTIL Index Utilities	0.02%	1.12%	-0.409	9.608	1	1	1	-9.00%/8.48%
8	S5CONS Index Consumer Staples	0.03%	0.97%	-0.233	9.905	1	1	1	-9.30%/7.59%
9	S5MATR Index Materials	0.03%	1.31%	0.036	5.929	1	1	1	-9.12%/6.98%
10	S5TELS Index Telecommunication Services	0.02%	1.44%	-0.100	6.674	1	1	1	-10.32%/8.03%
11	S&P 500 Index	0.04%	1.07%	-0.136	6.438	1	1	1	-7.11%/5.57%

B. *Results for the marginal models*

We estimate the parameters of p and q by minimizing Akaike information criterion (AIC) values for possible values ranging from zero to five. Table 4 lists the parameters which are estimated by minimum AIC values, and the statistical hypothesis testings for residuals are based on the Jarque-Bera (J-B) test and the Engle's ARCH test. The result shows that using the skewed Student's t innovation distribution for the residual term is appropriately fitted to the return data because the degree of freedom is usually smaller than 15 and the result of Jarque-Bera test rejects the null hypothesis of normality. In addition, the asymmetric parameter ξ is around one. Using GARCH (1, 1) model is appropriate because the result of the Engle's ARCH test of residuals shows no conditional heteroscedasticity, and parameter β is usually larger than 0.9, which indicates the conditional volatility is time-dependent.

TABLE IV
THE ESTIMATION OF THE IN-SAMPLE PARAMETERS AND STATISTICAL HYPOTHESIS TESTINGS FOR EACH MARGINAL

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
p	2	4	3	4	5	4	3	5	4	5	5
q	2	2	4	3	4	5	4	4	4	5	5
φ_1	1.6256	1.7689	-0.4612	1.6198	-0.7028	1.2286	1.0204	0.1458	1.4902	0.3616	0.6979
φ_2	-0.7665	-0.9974	-0.3371	-1.5609	-0.5651	-0.0331	-0.6161	0.6418	-0.9572	1.0907	1.2852
φ_3	*	0.0734	0.6153	0.6348	-0.1623	-1.1187	-0.2893	-0.5944	-0.0752	-0.9842	-1.0651
φ_4	*	-0.0482	*	-0.0689	-0.5042	0.6068	*	-0.4466	0.5080	-0.6221	-0.6927
φ_5	*	*	*	*	-0.0843	*	*	-0.0693	*	0.5777	0.5499
θ_1	-1.6273	-1.753	0.4946	-1.6470	0.7178	-1.2266	-1.0108	-0.1831	-1.4455	-0.3846	-0.7344
θ_2	0.7492	0.9615	0.2787	1.5377	0.5228	-0.0115	0.5790	-0.6619	0.8624	-1.0992	-1.3215
θ_3	*	*	-0.6871	-0.5938	0.0663	1.1439	0.3364	0.6157	0.1714	0.9956	1.1208
θ_4	*	*	-0.0937	*	-0.6008	-0.5825	-0.0258	0.4600	-0.5652	0.6238	0.7330
θ_5	*	*	*	*	*	-0.0552	*	*	*	-0.6223	-0.6187
μ	0.0007	0.0007	0.0005	0.0007	0.0005	0.0007	0.0004	0.0005	0.0006	0.0005	0.0006
γ	9.3e-07	8.4e-07	1.1e-06	1.4e-06	8.1e-07	1.2e-06	1.5e-06	6.1e-07	1.5e-06	8.0e-07	6.3e-07
α	0.0741	0.0570	0.0723	0.0577	0.0602	0.0689	0.0934	0.0582	0.0677	0.0471	0.0688
β	0.9240	0.9420	0.9214	0.9214	0.9357	0.9231	0.8953	0.9359	0.9268	0.9497	0.9278
ξ	0.9644	0.9118	0.9099	0.9099	0.9441	0.9353	0.8907	0.9672	1.0048	1.0009	0.8932
ν	8.5060	12.6200	9.2451	9.2451	7.6553	9.5213	9.2638	7.5744	8.9202	8.2249	8.1142
LLH	9863	8789	10259	9597	10273	10365	10662	11020	9901	9725	10686
AIC	-19737	-17554	-20492	-19169	-20516	-20700	-21298	-22010	-19774	-19419	-21339
J-B test	1	1	1	1	1	1	1	1	1	1	1
ARCH test	0	0	0	0	0	0	0	0	0	0	0

C. *Results for the copula models*

After the estimation of each marginal, we consider the set of standardized residuals from the ARMA-GARCH (1, 1) model and transform them to the set of uniform variables. Table 5 provides the correlation matrix of the transformed residuals and the result of the Kolmogorov-Smirnov (KS) test. The result of the Kolmogorov-Smirnov test is 0, and it fails to reject the null hypothesis that the distribution of transformed residuals and the uniform distribution are from the same continuous distribution at the 5% significance level.

TABLE V
THE PEARSON CORRELATION MATRIX AND KOLMOGOROV-SMIRNOV (KS) TEST FROM THE IN-SAMPLE DATA

ID / Correlation	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	1	0.5571	0.7549	0.4038	0.5928	0.7836	0.4678	0.5911	0.6099	0.5626	0.8564
2	0.5571	1	0.6483	0.2574	0.3833	0.6529	0.2455	0.2849	0.4191	0.5132	0.8089
3	0.7549	0.6483	1	0.3974	0.5692	0.8167	0.3997	0.5564	0.6474	0.5786	0.8699
4	0.4038	0.2574	0.3974	1	0.3945	0.4631	0.4699	0.3987	0.5007	0.3306	0.5255
5	0.5928	0.3833	0.5692	0.3945	1	0.6056	0.4011	0.6595	0.4448	0.437	0.6954
6	0.7836	0.6529	0.8167	0.4631	0.6056	1	0.4579	0.6008	0.7216	0.5716	0.8934
7	0.4678	0.2455	0.3997	0.4699	0.4011	0.4579	1	0.433	0.4022	0.3785	0.5048
8	0.5911	0.2849	0.5564	0.3987	0.6595	0.6008	0.433	1	0.5085	0.4254	0.6388
9	0.6099	0.4191	0.6474	0.5007	0.4448	0.7216	0.4022	0.5085	1	0.422	0.6771
10	0.5626	0.5132	0.5786	0.3306	0.437	0.5716	0.3785	0.4254	0.422	1	0.6902
11	0.8564	0.8089	0.8699	0.5255	0.6954	0.8934	0.5048	0.6388	0.6771	0.6902	1
KS test	0	0	0	0	0	0	0	0	0	0	0

Due to our benchmark using the Student's t copula, the parameters are the correlation matrix shown in table 5 and the degree of freedom 9.1334. Table 6 shows that using vine copula-based model has a better performance than using the Student's t copula-based model based on AIC values, and the evidence supports that vine copula-based model is an appropriate method to apply to high-dimensional modeling.

TABLE VI
THE ESTIMATION FOR THE COPULA MODELS FROM THE IN-SAMPLE DATA

	Number of parameters	Log-likelihood	AIC
Gaussian copula	55	15662	-31213
t copula	56	16578	-33044
Vine copula	99	16762	-33326

The catalogue of pair-copula families includes elliptical copulas such as Gaussian and Student's t, single parameter

Archimedean copulas such as Clayton, Frank, and Gumbel, as well as two parameter families such as BB1, BB6, BB7, and BB8. All various copulas we implement are in the *VineCopula* library in R [21].

D. Results for the Copula VaR/ES and Copula VaR/ES ratio

We empirically examine which sector dominates more risk contributions on systemic risk with 10,000 Monte Carlo simulations using vine Copula-based ARMA-GARCH (1, 1) modeling. The results of residuals, fitted by ARMA-GARCH(1, 1) with the skewed student's t innovations, are shown in figure 1. The results of the worst 5% return loss are not surprising and are shown in figure 2. As seen in figure 2 and figure 3 below, we realize that the financial sector caused more risk distribution during the subprime crisis from 2008 to 2009, while the consumer staples sector is the major risk receiver. The results present that this measure is a simplified and efficient methodology to analyze systemic risk.

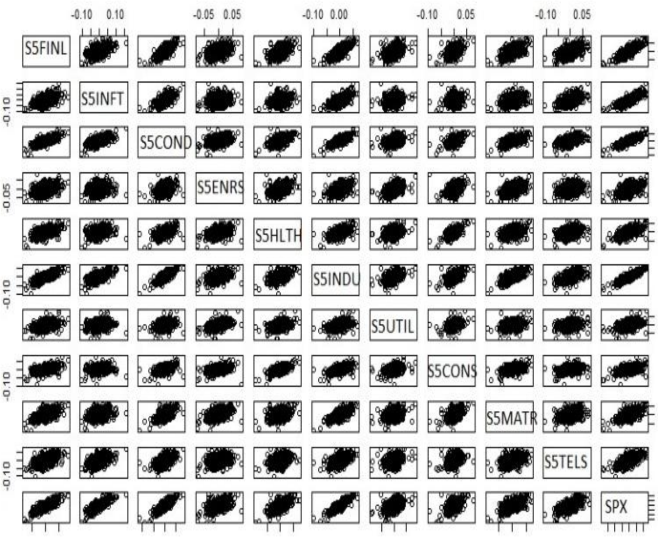

Fig. 1. The scatter plot of residuals

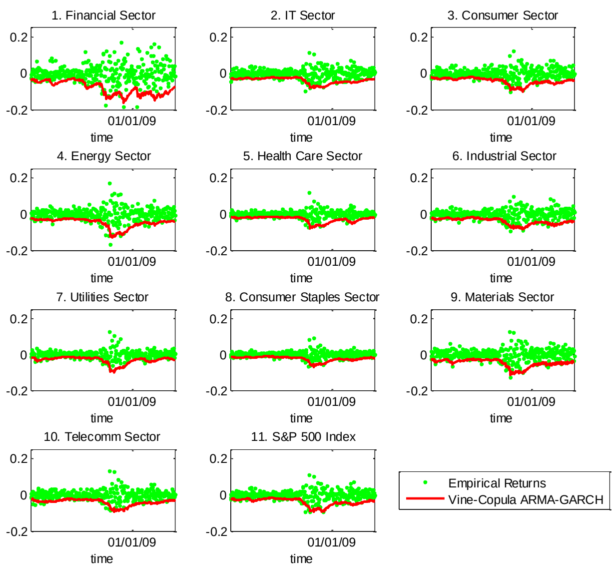

Fig. 2. The one-day ahead worst 5% return loss for each sector index

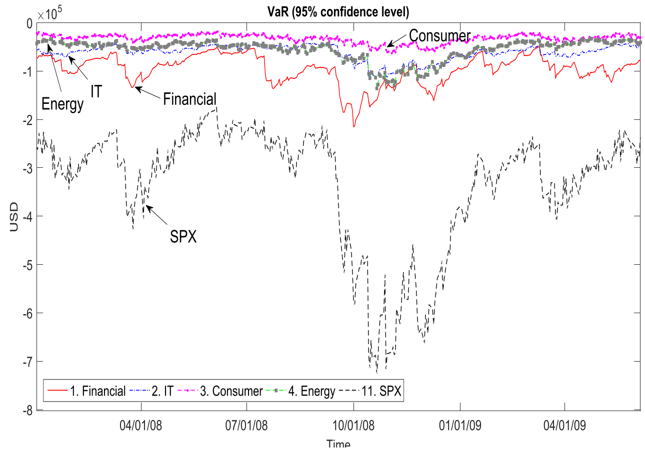

Fig. 3. The one-day ahead VaR at the 95% confidence level

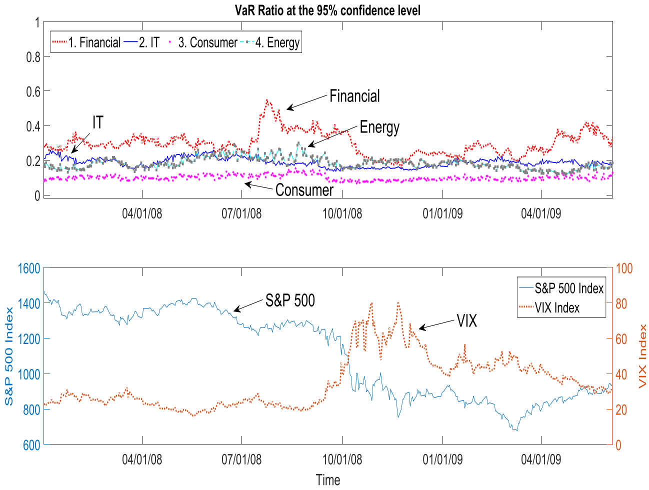

Fig. 4. The one-day ahead VaR ratio at the 95% confidence level

IV. Conclusion

The evidence in our paper shows that not only that vine Copula-based ARMA-GARCH (1, 1) has a better performance than the Gaussian and the Student's *t* copula-based ARMA-GARCH (1, 1) based on AIC values, but also that using the skewed student's t innovations is much more appropriate than using the student's t innovations.

In addition, using vine Copula-based ARMA-GARCH model to forecast Copula VaR and Copula VaR ratio, we develop a real-time and useful way with sector indices data. Moreover, the VaR/ES ratio provides the information of the risk contribution from each sectors. This approach is very general and can be tailored to any underlying country and financial market easily.

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