Exact Solutions to the Generalized Hirota – Satsuma KdV Equations Using the Extended Trial Equation Method

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***Abstract*-In this paper, we study the traveling wave solutions of the generalized Hirota – Satsuma KdV equations by using the modified extended trial equation method. We construct the exact solutions for the nonlinear partial differential equations when the balance number is a positive integer via the generalized Hirota–Satsuma KdV equations using different types of functions such as: hyperbolic functions, trigonometric functions, Jacobi elliptic functions, and rational functional. The performance of this method is reliable, effective, and powerful for solving more complicated nonlinear partial differential equations in mathematical physics. The balance amount in this method is not constant and changes whenever the derivative definition of the trial equation changes. This method allowed us to construct many new types of exact solutions. We show by using the Maple software package that all obtained solutions satisfy the original partial differential equations.**

***Keywords*- Extended trial equation method; Exact solutions; Traveling wave solutions, Balance number, Soliton solutions, Jacobi elliptic functions.**

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I. INTRODUCTION

The exact solutions of nonlinear differential equations play an important role in understanding most of the nonlinear physical phenomena. In recent years, exact solutions of nonlinear PDEs have been investigated by many authors (see for example [1-28]) who are interested in nonlinear physical phenomena. Many powerful methods have been presented by those authors such as the inverse scattering transform [1], the Backland transform [2], the Darboux transform [3], the generalized Riccati equation [4,5], the Jacobi elliptic function expansion method [6,7], the Painlev´e expansion method [8], the extended tanh- function method [9], the modification of Fan sub- Equation method [10], the F- expansion method [11,12], the expansion function method [13,14], the sub-ODE method [15,16], the extended sinh- cosh and sine-cosine methods [17,18], and the (G′/G) -expansion method [19,20]. There are also many methods for finding the analytic approximate solutions for nonlinear partial differential equations such as the homotopy perturbation method [21,22], the Adomain decomposition method [23,24], the Variation iteration, and the homotopy analysis method [25,26].

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Other methods for solving the nonlinear partial differential equations have also been developed (see for example [27-35]).

Recently, Gurefe et al [36] have presented a direct method, namely the extended trial equation method for solving the nonlinear partial differential equations. The main objective of this paper is to modify the extended trial equation method to construct a series of some new analytic exact solutions for the following generalized Hirota – Satsuma KdV system of equations which was introduced by Wu et al. [37]:

$$\begin{aligned} & u_t - \frac{1}{2} u_{xxx} + 3uu_x - 3(vw)_x = 0 , \\ & v_t + v_{xxx} - 3uv_x = 0 , \\ & w_t + w_{xxx} - 3uw_x = 0, \end{aligned} \qquad (1.1)$$

This system of equations describes the interaction of two long waves with different dispersion relations. Eq. (1.1) is reduced to a new complex coupled KdV equation [37] and the Hirota– Satsuma equation [38], with $w = v^*$ and $w = v$ respectively. In this paper, we construct the exact solutions for different types of roots of the trial equation. We obtain many different kinds of exact solutions in hyperbolic function, trigonometric function, Jacobi elliptic functions, and rational functions. In these solutions, the balance number is not constant and changes when the trial equation derivative of the nonlinear partial differential equations also changes.

II. DESCRIPTION OF THE EXTENDED TRIAL EQUATION METHOD

Suppose that we have a nonlinear partial differential equation in the following form:

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \qquad (2.1)$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u = u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. Let us now give the main steps for solving Eq. (2.1) using the extended trial equation method as in [36,39]:

<u>**Step 1**</u>.
Let the traveling wave variable be defined by

$$u(x,t) = u(\xi) , \ \xi = x + \beta t, \qquad (2.2)$$

where β is a nonzero constant.
The transformation (2.2) permits us to reduce (2.1) to the following ODE

$$P(u, \beta u', u', \beta^2 u'', \beta u'', u'', ...) = 0, \qquad (2.3)$$

where P is a polynomial of $u(\xi)$ and its total derivatives.

Step 2.
Suppose the solution takes the form:

$$u(\xi)=\sum_{i=0}^{\delta}\tau_i Y^i, \tag{2.4}$$

where Y satisfies the following nonlinear trial differential equation:

$$\begin{aligned}(Y')^2&=\Lambda(Y)=\frac{\Phi(Y)}{\Psi(Y)}\\ &=\frac{\xi_\theta Y^\theta+\xi_{\theta-1}Y^{\theta-1}+\ldots+\xi_1 Y+\xi_0}{\zeta_\varepsilon Y^\varepsilon+\zeta_{\varepsilon-1}Y^{\varepsilon-1}+\ldots+\zeta_1 Y+\zeta_0},\end{aligned} \tag{2.5}$$

where ξ_i,ζ_j are constants to be determined later. Using (2.4) and (2.5), we have

$$\begin{aligned}u''(\xi)&=\frac{\Phi'(Y)\Psi(Y)-\Phi(Y)\Psi'(Y)}{2\Psi^2(Y)}\left(\sum_{i=0}^{\delta} i\,\tau_i Y^{i-1}\right)\\ &+\frac{\Phi(Y)}{\Psi(Y)}\left(\sum_{i=0}^{\delta} i\,(i-1)\tau_i Y^{i-2}\right),\end{aligned} \tag{2.6}$$

where $\Phi(Y),\Psi(Y)$ are polynomials in Y.

Step 3.
Balancing the highest derivative term with the nonlinear terms, we can find the relations between δ,θ and ε. We can calculate some values of δ,θ and ε.

Step 4.
Substituting (2.4) - (2.6) into (2.3) yields a polynomial $\Omega(y)$ of Y as follows:

$$\Omega(y)=\rho_s Y^s+\ldots+\rho_1 Y+\rho_0=0. \tag{2.7}$$

Step 5.
Setting the coefficients of the polynomial $\Omega(y)$ to zero yields a set of algebraic equations:

$$\rho_i=0,\qquad i=0,\ldots,s. \tag{2.8}$$

Solving this system of algebraic equations to determine the values of $\xi_\theta,\xi_{\theta-1},\ldots,\xi_1,\xi_0,\zeta_\varepsilon,\zeta_{\varepsilon-1},\ldots,\zeta_1,\zeta_0$ and $\tau_\delta,\tau_{\delta-1},\ldots,\tau_1,\tau_0$.

Step 6.
Reduce (2.5) to the elementary integral form:

$$\pm(\xi-\eta_0)=\int\frac{dY}{\sqrt{\Lambda(y)}}=\int\sqrt{\frac{\Psi(Y)}{\Phi(Y)}}dY. \tag{2.9}$$

where η_0 is an arbitrary constant.

Using a discriminant for the polynomial to classify the roots of $\Phi(Y)$, we solve (2.9) to determine Y. In addition, we can write the corresponding exact traveling wave solutions to (2.1).

III. EXTENDED TRIAL EQUATION METHOD FOR THE GENERALIZED HIROTA–SATSUMA KDV EQUATIONS

In this section, we consider a generalized Hirota–Satsuma Korteweg – de Vries (KdV) equation which was introduced by Wu et al.. One of the typical equations in the hierarchy is a new generalized Hirota–Satsuma KdV equations, which we reproduce below:

$$\begin{aligned}&u_t-\frac{1}{2}u_{xxx}+3uu_x-3(vw)_x=0,\\ &v_t+v_{xxx}-3uv_x=0,\\ &w_t+w_{xxx}-3uw_x=0.\end{aligned} \tag{3.1}$$

The traveling wave variables

$$u(x,t)=u(\xi),\ v(x,t)=v(\xi), w(x,t)=w(\xi), \xi=x+\beta t, \tag{3.2}$$

where $u(\xi),v(\xi)$ and $w(\xi)$ are arbitrary functions of ξ, and β is an arbitrary constant. The traveling wave transformation (3.2) permit us to convert (3.1) into the following system of ODE's:

$$\begin{aligned}&\beta u'-\frac{1}{2}u'''+3uu'-3vw'-3wv'=0,\\ &\beta v'+v'''-3uv'=0,\\ &\beta w'+w'''-3uw'=0.\end{aligned} \tag{3.3}$$

From (2.4)-(2.9), we can write the exact solution of (3.3) into the following form:

$$u(\xi)=\sum_{i=0}^{\delta_1}\tau_i Y^i,\ v(\xi)=\sum_{i=0}^{\delta_2}T_i Y^i, w(\xi)=\sum_{i=0}^{\delta_3}a_i Y^i, \tag{3.4}$$

where Y satisfies (2.5) and $\delta_1,\delta_2,\delta_3$ are arbitrary positive integers. From balancing the highest derivative terms with the nonlinear terms in (3.3), we obtain:

$$\delta_1=\delta_2=\delta_3=\theta-\varepsilon-2 \tag{3.5}$$

Equations (3.5) have infinitely many solutions. We suppose some of these solutions as follows:

Case 1.
In the special case $\varepsilon=0$, $\theta=3$, we get $\delta_1=\delta_2=\delta_3=1$. Equations (2.4)-(2.9) lead to:

$$\begin{aligned}&u(\xi)=\tau_0+\tau_1 Y,\\ &(u')^2=\frac{\tau_1^2(\xi_3 Y^3+\xi_2 Y^2+\xi_1 Y+\xi_0)}{\zeta_0},\\ &u''=\frac{\tau_1(3\xi_3 Y^2+2\xi_2 Y+\xi_1)}{2\zeta_0},\end{aligned} \tag{3.6}$$

The higher order derivatives can be found in the same manner. Similarly, we find:

$$\begin{aligned}&v(\xi)=T_0+T_1 Y,\\ &(v')^2=\frac{T_1^2(\xi_3 Y^3+\xi_2 Y^2+\xi_1 Y+\xi_0)}{\zeta_0},\\ &v''=\frac{T_1(3\xi_3 Y^2+2\xi_2 Y+\xi_1)}{2\zeta_0},\end{aligned} \tag{3.7}$$

and

$$w(\xi) = a_0 + a_1 Y,$$
$$(w')^2 = \frac{a_1^2(\xi_3 Y^3 + \xi_2 Y^2 + \xi_1 Y + \xi_0)}{\zeta_0},$$
$$w'' = \frac{a_1(3\xi_3 Y^2 + 2\xi_2 Y + \xi_1)}{2\zeta_0}, \tag{3.8}$$

Substituting (3.6), (3.7) and (3.8) into (3.3), we get a system of algebraic equations which can be solved to obtain the following results:

$$a_0 = -\frac{\tau_1(-8\beta T_1\zeta_0 + 3\tau_1 T_0\zeta_0 - 2T_1\xi_2)}{12T_1^2\zeta_0},$$
$$a_1 = \frac{\tau_1^2}{4T_1}, \quad \xi_3 = \tau_1\zeta_0, \ \tau_0 = \frac{\beta\zeta_0 + \xi_2}{3\zeta_0}, \tag{3.9}$$

where $\zeta_0, \xi_0, \xi_1, \xi_2, T_0, T_1$ and τ_1 are arbitrary constants. Substituting these results (3.9) into (2.5) and (2.9), we have:

$$\pm(\xi - \eta_0) = L\int \frac{dY}{\sqrt{Y^3 + \frac{\xi_2}{\xi_3}Y^2 + \frac{\xi_1}{\xi_3}Y + \frac{\xi_o}{\xi_3}}}, \tag{3.10}$$

where $L = \sqrt{\frac{\zeta_0}{\xi_3}}$. Now, we will discuss the roots of the following equation:

$$Y^3 + \frac{\xi_2}{\tau_1\zeta_0}Y^2 + \frac{\xi_1}{\tau_1\zeta_0}Y + \frac{\xi_0}{\tau_1\zeta_0} = 0 \tag{3.11}$$

to integrate equations (3.10) as the following families:

Family 1.

If equation (3.11) has three equal repeated roots α_1, consequently we can write (3.11) in the following form:

$$Y^3 + \frac{\xi_2}{\tau_1\zeta_0}Y^2 + \frac{\xi_1}{\tau_1\zeta_0}Y + \frac{\xi_0}{\tau_1\zeta_0} - (Y - \alpha_1)^3 = 0 \tag{3.12}$$

By equating the coefficients of Y in both sides of (3.12), we get a system of algebraic equations in $\zeta_0, \xi_0, \xi_1, \xi_2$ and τ_1 which can be solved by using the Maple software package to get the following results :

$$\xi_0 = -\alpha_1^3\zeta_0, \ \xi_1 = 3\alpha_1^2\zeta_0, \xi_2 = -3\alpha_1\zeta_0, \ \tau_1 = 1. \tag{3.13}$$

Equations (3.13) , (3.9) and (3.10) lead to:

$$a_0 = -\frac{-8\beta T_1\zeta_0 + 3T_0\zeta_0 + 6\alpha_1 T_1\zeta_0}{12T_1^2\zeta_0},$$
$$a_1 = \frac{1}{4T_1}, \qquad \xi_3 = \zeta_0, \qquad \tau_0 = \frac{\beta}{3} - \alpha_1, \tag{3.14}$$

where ζ_0, T_0 and T_1 are arbitrary constants, and

$$\pm(\xi - \eta_0) = \int \frac{dY}{(Y - \alpha_1)^{3/2}} = \frac{-2}{\sqrt{Y - \alpha_1}}, \tag{3.15}$$

or

$$Y = \alpha_1 + \frac{4}{(x + \beta t - \eta_0)^2}. \tag{3.16}$$

Substituting (3.16), (3.14) and (3.13) into (3.6), (3.7), and (3.8), we get the exact solutions of the generalized Hirota–Satsuma equations (3.1) in the form:

$$u_1(\xi) = \frac{\beta}{3} + \frac{4}{(x + \beta t - \eta_0)^2}, \tag{3.17}$$

$$v_1(\xi) = T_0 + T_1\alpha_1 + \frac{4T_1}{(x + \beta t - \eta_0)^2}, \tag{3.18}$$

and

$$w_1(\xi) = -\frac{-8\beta T_1 + 3T_0 + 6\alpha_1 T_1}{12T_1^2} + \frac{\alpha_1}{4T_1} + \frac{1}{T_1(x + \beta t - \eta_0)^2}. \tag{3.19}$$

Family 2.

If the equation (3.11) has two distinct roots α_1 a double root, and α_2 a simple root, such that $\alpha_1 \neq \alpha_2$, we can write (3.11) in the following form:

$$Y^3 + \frac{\xi_2}{\tau_1\zeta_0}Y^2 + \frac{\xi_1}{\tau_1\zeta_0}Y + \frac{\xi_0}{\tau_1\zeta_0} - (Y - \alpha_1)^2(Y - \alpha_2) = 0. \tag{3.20}$$

Equating the coefficients of Y from both sides of (3.20), we get a system of algebraic equations in $\zeta_0, \xi_0, \xi_1, \xi_2$ and τ_1 which can be solved by using the Maple software package to get the following results:

$$\xi_0 = -\alpha_1^2\alpha_2\zeta_0, \quad \xi_1 = \alpha_1(\alpha_1 + 2\alpha_2)\zeta_0,$$
$$\xi_2 = -(2\alpha_1 + \alpha_2)\zeta_0, \quad \tau_1 = 1. \tag{3.21}$$

Equations (3.21) , (3.9) and (3.10) lead to:

$$a_0 = -\frac{-8\beta T_1 + 3T_0 + 2T_1(2\alpha_1 + \alpha_2)}{12T_1^2}, \quad a_1 = \frac{1}{4T_1},$$
$$\tau_0 = \frac{\beta - (2\alpha_1 + \alpha_2)}{3}, \qquad \xi_3 = \zeta_0. \tag{3.22}$$

where ζ_0, T_0 and T_1 are arbitrary constants. If $\alpha_2 > \alpha_1$ in this family, the solution of Eq.(3.10) has the form :

$$\pm(\xi-\eta_0)=\int\frac{dY}{(Y-\alpha_1)\sqrt{Y-\alpha_2}}=\frac{2}{\sqrt{\alpha_2-\alpha_1}}\tan^{-1}\left[\frac{\sqrt{Y-\alpha_2}}{\sqrt{\alpha_2-\alpha_1}}\right], \tag{3.23}$$

or

$$Y=\alpha_2+(\alpha_2-\alpha_1)\tan^2\left[\frac{\sqrt{\alpha_2-\alpha_1}}{2}(x+\beta t-\eta_0)\right], \tag{3.24}$$

Substituting equations (3.24), (3.22) and (3.21) into (3.6) , (3.7) and (3.8), we get the exact solutions of the generalized Hirota–Satsuma equations (3.1) in the form:

$$u_2(\xi)=\frac{\beta-(2\alpha_1+\alpha_2)}{3}+\alpha_2+(\alpha_2-\alpha_1)\tan^2\left[\frac{\sqrt{\alpha_2-\alpha_1}}{2}(x+\beta t-\eta_0)\right], \tag{3.25}$$

$$v_2(\xi)=T_0+T_1\{\alpha_2+(\alpha_2-\alpha_1)\tan^2\left[\frac{\sqrt{\alpha_2-\alpha_1}}{2}(x+\beta t-\eta_0)\right]\}, \tag{3.26}$$

and

$$w_2(\xi)=-\frac{-8\beta T_1+3T_0+2T_1(2\alpha_1+\alpha_2)}{12T_1^2}+\frac{1}{4T_1}\{\alpha_2+(\alpha_2-\alpha_1)\tan^2\left[\frac{\sqrt{\alpha_2-\alpha_1}}{2}(x+\beta t-\eta_0)\right]\}. \tag{3.27}$$

If $\alpha_1>\alpha_2$ in this family, the solution of (3.10) has the form :

$$Y=\alpha_1+(\alpha_1-\alpha_2)\operatorname{csch}^2\left[\frac{\sqrt{\alpha_1-\alpha_2}}{2}(x+\beta t-\eta_0)\right], \tag{3.28}$$

Substituting equations. (3.28), (3.21) and (3.22) into (3.6)-(3.8), we get the exact solutions of the generalized Hirota–Satsuma equations (3.1) in the form:

$$u_3(\xi)=\frac{\beta-(2\alpha_1+\alpha_2)}{3}+\alpha_1+(\alpha_1-\alpha_2)\operatorname{csch}^2\left[\frac{\sqrt{\alpha_1-\alpha_2}}{2}(x+\beta t-\eta_0)\right], \tag{3.29}$$

$$v_3(\xi)=T_0+T_1\{\alpha_1+(\alpha_1-\alpha_2)\operatorname{csch}^2\left[\frac{\sqrt{\alpha_1-\alpha_2}}{2}(x+\beta t-\eta_0)\right]\}, \tag{3.30}$$

and

$$w_3(\xi)=-\frac{-8\beta T_1+3T_0+2T_1(2\alpha_1+\alpha_2)}{12T_1^2}+\frac{1}{4T_1}\{\alpha_1+(\alpha_1-\alpha_2)\operatorname{csch}^2\left[\frac{\sqrt{\alpha_1-\alpha_2}}{2}(x+\beta t-\eta_0)\right]\}. \tag{3.31}$$

Family 3.

If the equation (3.11) has three distinct roots α_1,α_2 and α_3, we can write equation (3.11) in the following form:

$$Y^3+\frac{\xi_2}{\tau_1\zeta_0}Y^2+\frac{\xi_1}{\tau_1\zeta_0}Y+\frac{\xi_0}{\tau_1\zeta_0}-(Y-\alpha_1)(Y-\alpha_2)(Y-\alpha_3)=0. \tag{3.32}$$

By equating the coefficients of Y in both sides of (3.32), we get a system of algebraic equations in $\zeta_0,\xi_0,\xi_1,\xi_2$ and τ_1 which can be solved by using the Maple software package to get the following results:

$$\xi_0=-\alpha_1\alpha_2\alpha_3\zeta_0,\quad \xi_1=(\alpha_1\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3)\zeta_0,\quad \xi_2=-(\alpha_1+\alpha_2+\alpha_3)\zeta_0,\quad \tau_1=1. \tag{3.33}$$

Equations (3.33) , (3.9) and (3.10) lead to:

$$a_0=-\frac{-8\beta T_1+3T_0+2T_1(\alpha_1+\alpha_2+\alpha_3)}{12T_1^2},\quad \tau_0=\frac{\beta-(\alpha_1+\alpha_2+\alpha_3)}{3},\ a_1=\frac{1}{4T_1},\ \xi_3=\zeta_0, \tag{3.34}$$

where ζ_0,T_0 and T_1 are arbitrary constants . In this family the solution of (3.10) has the following form:

$$\pm(\xi-\eta_0)=\int\frac{dY}{\sqrt{(Y-\alpha_1)(Y-\alpha_2)(Y-\alpha_3)}}=\frac{2}{\sqrt{\alpha_3-\alpha_1}}\mathit{EllipticF}\left[\frac{\sqrt{Y-\alpha_1}}{\sqrt{\alpha_2-\alpha_1}},\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}}\right], \tag{3.35}$$

or

$$Y=\alpha_1+(\alpha_2-\alpha_1)sn^2\left[\frac{\sqrt{\alpha_3-\alpha_1}}{2}(x+\beta t-\eta_0),\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}}\right], \tag{3.36}$$

Substituting (3.36), (3.34) and (3.33) into (3.6)-(3.8), we get the exact solutions of the generalized Hirota–Satsuma equations (3.1) in the form:

$$u_4(\xi)=\frac{\beta-(\alpha_1+\alpha_2+\alpha_3)}{3}+\alpha_1+(\alpha_2-\alpha_1)\Phi_1^2(x,t), \tag{3.37}$$

$$v_4(\xi) = T_0 + T_1\{\alpha_1 + (\alpha_2 - \alpha_1)\Phi_1^2(x,t)\}, \tag{3.38}$$

and

$$w_4(\xi) = -\frac{-8\beta T_1 + 3T_0 + 2T_1(\alpha_1+\alpha_2+\alpha_3)}{12T_1^2} + \frac{1}{4T_1}\{\alpha_1 + (\alpha_2-\alpha_1)\Phi_1^2(x,t)\}. \tag{3.39}$$

where

$$\Phi_1(x,t) = sn\left[\frac{\sqrt{\alpha_3-\alpha_1}}{2}(x+\beta t-\eta_0), \sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}}\right].$$

Family 4.

If the equation (3.11) has one real root α_1 and two imaginary roots $\alpha_2 = N_1 + i\,N_2$, $\alpha_3 = N_1 - i\,N_2$, where N_1 and N_2 are real numbers, we can then write the equation (3.11) in the following form:

$$Y^3 + \frac{\xi_2}{\tau_1\zeta_0}Y^2 + \frac{\xi_1}{\tau_1\zeta_0}Y + \frac{\xi_0}{\tau_1\zeta_0} - (Y-\alpha_1)(Y^2 - 2N_1Y + N_1^2 + N_2^2) = 0. \tag{3.40}$$

From equating the coefficients of Y to both sides of Eq. (3.40) , we get a system of algebraic equations in $\zeta_0, \xi_0, \xi_1, \xi_2$ and τ_1 which can be solved by using the Maple software package to get the following results:

$$\begin{aligned} \xi_0 &= -\alpha_1\zeta_0(N_1^2+N_2^2), \\ \xi_1 &= \zeta_0(2\alpha_1 N_1 + N_1^2 + N_2^2), \\ \xi_2 &= -(\alpha_1+2N_1)\zeta_0, \quad \tau_1 = 1 \end{aligned} \tag{3.41}$$

Equations (3.41) , (3.9) and (3.10) lead to:

$$a_0 = -\frac{-8\beta T_1 + 3T_0 + 2T_1(\alpha_1+2N_1)}{12T_1^2},$$

$$a_1 = \frac{1}{4T_1}, \; \xi_3 = \zeta_0, \; \tau_0 = \frac{\beta-(\alpha_1+2N_1)}{3}. \tag{3.42}$$

where ζ_0, T_0 and T_1, are arbitrary constants. With the help of Maple software package, the integration of equation.(3.10) in this family take the following form:

$$\pm(\xi-\eta_0) = \int\frac{dY}{\sqrt{(Y-\alpha_1)(Y^2-2N_1Y+N_1^2+N_2^2)}} = \frac{2}{\sqrt{N_1+iN_2-\alpha_1}}EllipticF\left[\frac{\sqrt{Y-\alpha_1}}{\sqrt{N_1-iN_2-\alpha_1}}, M\right], \tag{3.43}$$

or

$$Y = \alpha_1 + (N_1 - iN_2 - \alpha_1)\Phi_2^2(x,t), \tag{3.44}$$

where $M = \sqrt{\frac{N_1-iN_2-\alpha_1}{N_1+iN_2-\alpha_1}}$ and

$$\Phi_2(x,t) = sn\left[\frac{\sqrt{N_1+iN_2-\alpha_1}}{2}(x+\beta t-\eta_0), M\right].$$

Substituting (3.44), (3.42) and (3.41) into (3.6)- (3.8), we get the exact solutions of the generalized Hirota–Satsuma equations (3.1) in the form:

$$u_5(\xi) = \frac{\beta-(\alpha_1+2N_1)}{3} + \alpha_1 + (N_1 - iN_2 - \alpha_1)\Phi_2^2(x,t), \tag{3.45}$$

$$v_5(\xi) = T_0 + T_1\{\alpha_1 + (N_1 - iN_2 - \alpha_1)\Phi_2^2(x,t), \tag{3.46}$$

and

$$w_5(\xi) = -\frac{-8\beta T_1 + 3T_0 + 2T_1(\alpha_1+2N_1)}{12T_1^2} + \frac{1}{4T_1}\{\alpha_1 + (N_1 - iN_2 - \alpha_1)\Phi_2^2(x,t)\}. \tag{3.47}$$

Case 2. In the special case when $\varepsilon = 0$ and $\theta = 4$, we get $\delta_1 = \delta_2 = 2$. Equations (2.4)- (2.9) lead to:

$$u(\xi) = \tau_0 + \tau_1 Y + \tau_2 Y^2,$$

$$(u')^2 = \frac{(\tau_1 + 2\tau_2 Y)^2}{\zeta_0}(\xi_4Y^4 + \xi_3Y^3 + \xi_2Y^2 + \xi_1Y + \xi_0),$$

$$u'' = \frac{\tau_1(4\xi_4Y^3 + 3\xi_3Y^2 + 2\xi_2Y + \xi_1)}{2\zeta_0} + \frac{\tau_2}{\zeta_0}(6\xi_4Y^4 + 5\xi_3Y^3 + 4\xi_2Y^2 + 3\xi_1Y + 2\xi_0), \tag{3.48}$$

The higher order derivatives can be computed in the same manner. Similarly, we can deduce the following:

$$v(\xi) = T_0 + T_1Y + T_2Y^2,$$

$$v'^2 = \frac{(T_1 + 2T_2Y)^2}{\zeta_0}(\xi_4Y^4 + \xi_3Y^3 + \xi_2Y^2 + \xi_1Y + \xi_0),$$

$$v''(\xi) = \frac{T_1(4\xi_4Y^3 + 3\xi_3Y^2 + 2\xi_2Y + \xi_1)}{2\zeta_0} + \frac{T_2}{\zeta_0}(6\xi_4Y^4 + 5\xi_3Y^3 + 4\xi_2Y^2 + 3\xi_1Y + 2\xi_0), \tag{3.49}$$

and

$$w(\xi) = a_0 + a_1Y + a_2Y^2,$$

$$w'^2(\xi) = \frac{(a_1 + 2a_2Y)^2}{\zeta_0}(\xi_4Y^4 + \xi_3Y^3 + \xi_2Y^2 + \xi_1Y + \xi_0),$$

$$w''(\xi) = \frac{a_1(4\xi_4Y^3 + 3\xi_3Y^2 + 2\xi_2Y + \xi_1)}{2\zeta_0}$$

$$+\frac{a_2}{\zeta_0}(6\xi_4Y^4+5\xi_3Y^3+4\xi_2Y^2+3\xi_1Y+2\xi_0). \quad (3.50)$$

Substituting (3.48) ,(3.49) and (3.50) into equation.(3.3), we get a system of algebraic equations which can be solved to obtain the following results:

$$T_1=\frac{\tau_1T_2}{\tau_2},\quad a_1=\frac{\tau_2\tau_1}{4T_2},\quad a_2=\frac{\tau_2^2}{4T_2},$$

$$\xi_1=\frac{\tau_1(-\tau_1^2\zeta_0+4\tau_2\xi_2)}{4\tau_2^2},\quad \xi_3=\frac{\tau_1\zeta_0}{2},$$

$$\tau_0=\frac{-3\tau_1^2\zeta_0+4\beta\tau_2\zeta_0+16\tau_2\xi_2}{12\tau_2\zeta_0},\xi_4=\frac{\tau_2\zeta_0}{4}, \quad (3.51)$$

$$a_0=-\frac{1}{24T_2^2\zeta_0}(3\tau_1^2\zeta_0T_2-16\beta\tau_2\zeta_0T_2$$
$$-16\tau_2\xi_2T_2+6T_0\tau_2^2\zeta_0),$$

where $\zeta_0,\xi_0,\xi_2,\tau_2,\tau_1,T_0$ and T_2 are arbitrary constants. Substituting these results (3.51) into (2.5) and (2.9), we have:

$$(\xi-\eta_0)=\int\frac{LdY}{\sqrt{\xi_4Y^4+\xi_3Y^3+\xi_2Y^2+\xi_1Y+\xi_0}}, \quad (3.52)$$

where $L=\sqrt{\zeta_0}$. Now we will discuss the roots of the following equation:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3+\frac{4\xi_2}{\tau_2\zeta_0}Y^2+\frac{\tau_1(-\tau_1^2\zeta_0+4\tau_2\xi_2)}{\tau_2^3\zeta_0}Y+\frac{4\xi_0}{\tau_2\zeta_0}=0, \quad (3.53)$$

to integrate equations (3.52). We discuss the roots of Eq.(3.53) as following families:

Family 5.

If equation (3.53) has one single repeated real root α_1 (the root being repeated four times), we can write equation (3.53) in the following form:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3+\frac{4\xi_2}{\tau_2\zeta_0}Y^2+\frac{\tau_1(-\tau_1^2\zeta_0+4\tau_2\xi_2)}{\tau_2^3\zeta_0}Y$$
$$+\frac{4\xi_0}{\tau_2\zeta_0}-(Y-\alpha_1)^4=0. \quad (3.54)$$

From equating the coefficients of Y to both sides of Eq.(3.54) , we get a system of algebraic equations in $\xi_0,\xi_2,\zeta_0,\tau_1$ and τ_2, which can be solved by using the Maple software package to get the following results:

$$\xi_0=\alpha_1^4\zeta_0,\quad \xi_2=6\alpha_1^2\zeta_0,\quad \tau_1=-8\alpha_1,\quad \tau_2=4. \quad (3.55)$$

Equations (3.55) , (3.51) and (3.52) lead to:

$$T_1=-2\alpha_1T_2,\quad a_1=\frac{-8\alpha_1}{T_2},\ a_2=\frac{4}{T_2},$$

$$\xi_1=-4\zeta_0\alpha_1^3,\xi_3=-4\zeta_0\alpha_1,\quad \xi_4=\zeta_0,$$

$$\tau_0=\frac{192\alpha_1^2+16\beta}{48},$$

$$a_0=-\frac{1}{24T_2^2}(192\alpha_1^2T_2-64\beta T_2-16\tau_2\xi_2T_2 \quad (3.56)$$
$$+96T_0),$$

where ζ_0,T_0 and T_2 are arbitrary constants and

$$\pm(\xi-\eta_0)=\int\frac{dY}{(Y-\alpha_1)^2}=\frac{-1}{Y-\alpha_1}. \quad (3.57)$$

or

$$Y=\alpha_1\mp\frac{1}{(x+\beta t-\eta_0)}. \quad (3.58)$$

Substituting (3.58), (3.56) and (3.55) into (3.48)- (3.50), we get the exact solutions of the generalized Hirota– Satsuma KdV equations (3.1) in the following form :

$$u_6(\xi)=-8\alpha_1\left[\alpha_1\mp\frac{1}{(x+\beta t-\eta_0)}\right]$$
$$+4\left[\alpha_1\mp\frac{1}{(x+\beta t-\eta_0)}\right]^2-\frac{192\alpha_1^2+16\beta}{48}, \quad (3.59)$$

and

$$v_6(\xi)=T_0-2\alpha_1T_2\left[\alpha_1\mp\frac{1}{(x+\beta t-\eta_0)}\right]$$
$$+T_2\left[\alpha_1\mp\frac{1}{(x+\beta t-\eta_0)}\right]^2, \quad (3.60)$$

$$w_6(\xi)=-\frac{1}{24T_2^2}\Big(192\alpha_1^2T_2-64\beta T_2$$
$$-16\tau_2\xi_2T_2+96T_0\Big)-\frac{8\alpha_1}{T_2}\left[\alpha_1\mp\frac{1}{(x+\beta t-\eta_0)}\right]$$
$$+\frac{4}{T_2}\left[\alpha_1\mp\frac{1}{(x+\beta t-\eta_0)}\right]^2. \quad (3.61)$$

Family 6.

If the equation (3.53) has two twice-repeated roots α_1 and α_2, $\alpha_1\neq\alpha_2$, we can write equation (3.53) in the following form:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3+\frac{4\xi_2}{\tau_2\zeta_0}Y^2+\frac{\tau_1(-\tau_1^2\zeta_0+4\tau_2\xi_2)}{\tau_2^3\zeta_0}Y$$
$$+\frac{4\xi_0}{\tau_2\zeta_0}-(Y-\alpha_1)^2(Y-\alpha_2)^2=0. \quad (3.62)$$

By equating the coefficients of Y in both sides of (3.62), we get a system of algebraic equations in $\xi_0,\xi_2,\zeta_0,\tau_1$ and τ_2 which can be solved by using the Maple software package to get the following results:

$$\xi_0=\zeta_0\alpha_1^2\alpha_2^2,\quad \xi_2=\zeta_0(\alpha_1^2+4\alpha_1\alpha_2+\alpha_2^2),$$
$$\tau_1=-4(\alpha_1+\alpha_2),\quad \tau_2=4. \quad (3.63)$$

Equations (3.63) , (3.51) and (3.52) lead to:

$$T_1=-(\alpha_1+\alpha_2)T_2,\quad a_0=-\frac{2}{3T_2^2}(-T_2\alpha_1^2-4\beta T_2$$
$$-10\alpha_1\alpha_2T_2-\alpha_2^2T_2+6T_0),\quad a_2=\frac{4}{T_2},\ \xi_4=\zeta_0,$$
$$a_1=\frac{-4(\alpha_1+\alpha_2)}{T_2},\ \xi_1=-2(\alpha_1+\alpha_2)\alpha_1\alpha_2\zeta_0, \quad (3.64)$$
$$\xi_3=\frac{-4(\alpha_1+\alpha_2)\zeta_0}{2},\ \tau_0=\frac{\beta+10\alpha_1\alpha_2+\alpha_2^2+\alpha_1^2}{3}.$$

where ζ_0,T_0 and T_2 are arbitrary constants and

$$\pm(\xi-\eta_0)=\int\frac{dY}{(Y-\alpha_1)(Y-\alpha_2)}$$
$$=\frac{1}{\alpha_1-\alpha_2}\ln\left|\frac{Y-\alpha_1}{Y-\alpha_2}\right| \quad (3.65)$$

or

$$Y=\frac{-\alpha_1+\alpha_2e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}{-1+e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}. \quad (3.66)$$

Substituting (3.66) (3.64) and (3.63) into (3.48)- (3.50), we get the exact solutions of generalized Hirota–Satsuma KdV equations (3.1) in the following form:

$$u_7(\xi)=\frac{\beta+10\alpha_1\alpha_2+\alpha_2^2+\alpha_1^2}{3}$$
$$-4(\alpha_1+\alpha_2)\left[\frac{-\alpha_1+\alpha_2e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}{-1+e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}\right]$$
$$+4\left[\frac{-\alpha_1+\alpha_2e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}{-1+e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}\right]^2, \quad (3.67)$$

$$v_7(\xi)=-(\alpha_1+\alpha_2)T_2\left[\frac{-\alpha_1+\alpha_2e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}{-1+e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}\right]$$
$$+T_2\left[\frac{-\alpha_1+\alpha_2e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}{-1+e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}\right]^2+T_0 \quad (3.68)$$

and

$$w_7(\xi)=\frac{2}{3T_2^2}(-T_2\alpha_1^2-4\beta T_2-10\alpha_1\alpha_2T_2-\alpha_2^2T_2$$
$$+6T_0)+\frac{4}{T_2}\left[\frac{-\alpha_1+\alpha_2e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}{-1+e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}\right]^2$$
$$-\frac{4(\alpha_1+\alpha_2)}{T_2}\left[\frac{-\alpha_1+\alpha_2e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}{-1+e^{\pm(\alpha_1-\alpha_2)(x+\beta t-\eta_0)}}\right] \quad (3.69)$$

Family 7.

If equation (3.53) has four different real roots α_1, α_2,α_3 and α_4, we can write the equation (3.53) in the following form:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3+\frac{4\xi_2}{\tau_2\zeta_0}Y^2+\frac{\tau_1(-\tau_1^2\zeta_0+4\tau_2\xi_2)}{\tau_2^3\zeta_0}Y$$
$$+\frac{4\xi_0}{\tau_2\zeta_0}=(Y-\alpha_1)(Y-\alpha_2)(Y-\alpha_3)(Y-\alpha_4). \quad (3.70)$$

By equating the coefficients of Y in both sides of equation. (3.70), we get a system of algebraic equations in $\xi_0,\xi_2,\zeta_0,\tau_1$ and τ_2 which can be solved by using the Maple software package to get the following results:

$$\xi_0=\zeta_0\alpha_2\alpha_3\alpha_4(\alpha_2+\alpha_3-\alpha_4),\quad \tau_2=4,$$
$$\alpha_1=-\alpha_2+\alpha_3+\alpha_4,\quad \tau_1=-4(\alpha_3+\alpha_4),$$
$$\xi_2=\zeta_0(3\alpha_4\alpha_3+\alpha_2\alpha_3+\alpha_3^2+\alpha_2\alpha_4+\alpha_4^2-\alpha_2^2), \quad (3.71)$$

Equations (3.71) , (3.51) and (3.52) lead to:

$$a_0=-\frac{2}{3T_2^2}(-T_2\alpha_3^2-4\beta T_2-6\alpha_3\alpha_4T_2-\alpha_4^2T_2$$
$$+4T_2\alpha_2^2-4\alpha_2\alpha_4T_2-4\alpha_2\alpha_3T_2+6T_0),$$
$$T_1=-(\alpha_3+\alpha_4)T_2,\qquad a_2=\frac{4}{T_2},$$
$$\xi_1=(\alpha_3+\alpha_4)(-\alpha_3\alpha_4+\alpha_2^2-\alpha_2\alpha_4-\alpha_2\alpha_3)\zeta_0,$$
$$\xi_3=-2(\alpha_3+\alpha_4)\zeta_0,\ a_1=\frac{-4(\alpha_3+\alpha_4)}{T_2},\ \xi_4=\zeta_0,$$
$$\tau_0=2\alpha_3\alpha_4+\frac{1}{3}\left(\beta+4\alpha_2\alpha_3+4\alpha_2\alpha_4+\alpha_3^2+\alpha_4^2-4\alpha_2^2\right) \quad (3.72)$$

where ζ_0, T_0 and T_2 are arbitrary constants and

$$\pm(\xi-\eta_0)=\int\frac{dY}{\sqrt{(Y-(-\alpha_2+\alpha_3+\alpha_4))(Y-\alpha_2)(Y-\alpha_3)(Y-\alpha_4)}}=\frac{2i}{(\alpha_2-\alpha_4)}EllipticF\left[\sqrt{\frac{(\alpha_4-\alpha_2)(Y-\alpha_4)}{(\alpha_3-\alpha_2)(Y-\alpha_3)}},\frac{(\alpha_2-\alpha_3)}{(\alpha_2-\alpha_4)}\right], \tag{3.73}$$

or

$$Y=\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)\Phi_3^2(x,t)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)\Phi_3^2(x,t)}. \tag{3.74}$$

where

$$\Phi_3(x,t)=sn\left(\frac{i}{2}(\alpha_2-\alpha_4)(x+\beta t-\eta_0),\frac{(\alpha_2-\alpha_3)}{(\alpha_2-\alpha_4)}\right)$$

Substituting (3.74), (3.72) and (3.71) into (3.48)- (3.50), we get the exact solutions of the generalized Hirota–Satsuma KdV equations (3.1) in the form:

$$u_8(\xi)=\frac{\beta+4\alpha_2\alpha_3+4\alpha_2\alpha_4+\alpha_3^2+\alpha_4^2-4\alpha_2^2+6\alpha_4\alpha_3}{3}-4(\alpha_3+\alpha_4)\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)\Phi_3^2(x,t)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)\Phi_3^2(x,t)}\right]+4\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)\Phi_3^2(x,t)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)\Phi_3^2(x,t)}\right]^2, \tag{3.75}$$

$$v_8(\xi)=T_0+T_2\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)\Phi_3^2(x,t)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)\Phi_3^2(x,t)}\right]^2,-(\alpha_3+\alpha_4)T_2\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)\Phi_3^2(x,t)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)\Phi_3^2(x,t)}\right], \tag{2.76}$$

and

$$w_8(\xi)=+\frac{4}{T_2}\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)\Phi_3^2(x,t)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)\Phi_3^2(x,t)}\right]^2.-\frac{4(\alpha_3+\alpha_4)}{T_2}\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)\Phi_3^2(x,t)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)\Phi_3^2(x,t)}\right]+\frac{2}{3T_2^2}(-T_2\alpha_3^2-4\beta T_2-6\alpha_3\alpha_4T_2-\alpha_4^2T_2+4T_2\alpha_2^2-4\alpha_2\alpha_4T_2-4\alpha_2\alpha_3T_2+6T_0) \tag{2.77}$$

Family 8.

If equation (3.53) has four complex roots $\alpha_1=N_1+iN_2$

$\alpha_2=N_1-iN_2$

$\alpha_3=N_3+iN_4$

$\alpha_4=N_3-iN_4$

where N_j, $j=1...,4$ are real numbers, we can write the equation (3.53) in the following form:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3+\frac{4\xi_2}{\tau_2\zeta_0}Y^2+\frac{\tau_1(-\tau_1^2\zeta_0+4\tau_2\xi_2)}{\tau_2^3\zeta_0}Y+\frac{4\xi_0}{\tau_2\zeta_0}-(Y-(N_1+iN_2))(Y-(N_1-iN_2))(Y-(N_3+iN_4))(Y-(N_3-iN_4))=0. \tag{3.78}$$

By equating the coefficients of Y in both sides of Eq.(3.78), we get a system of algebraic equations in $\xi_0,\xi_2,\zeta_0,\tau_1$ and τ_2 which can be solved by using the Maple software package to get the following results:

$$N_1=N_3,\quad \tau_1=-8N_3,\quad \tau_2=4,$$
$$\xi_0=\zeta_0(N_3^2N_4^2+N_2^2N_3^2+N_2^2N_4^2+N_3^4),$$
$$\xi_2=(6N_3^2+N_4^2+N_2^2)\zeta_0. \tag{3.79}$$

Equations (3.79) , (3.51) and (3.52) lead to:

$$T_1=-2N_3T_2,\quad a_1=\frac{-8N_3}{T_2},\quad a_2=\frac{4}{T_2},$$
$$\xi_1=-2N_3(2N_3^2+N_2^2+N_4^2)\zeta_0,\ \xi_4=\zeta_0,$$
$$\tau_0=\frac{\beta+4N_4^2+4N_2^2}{3}+4N_3^2,\ \xi_3=-4N_3\zeta_0.$$
$$a_0=\frac{4}{3T_2^2}(6N_3^2T_2+2\beta T_2+2N_4^2T_2+2T_2N_2^2-3T_0), \tag{3.80}$$

where ζ_0, T_0 and T_2 are arbitrary constants and

$$\pm(\xi-\eta_0)=\int\frac{dY}{\sqrt{(Y^2-2N_3Y+N_3^2+N_2^2)(Y^2-2N_3Y+N_3^2+N_4^2)}}=\frac{2}{(N_2-N_4)}EllipticF\left[\sqrt{\frac{(N_2-N_4)(Y-N_3-iN_4)}{(N_2+N_4)(Y-N_3+iN_4)}},\frac{(N_2+N_4)}{(N_2-N_4)}\right]. \tag{3.81}$$

or

$$Y=\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)\Phi_4^2}{(N_4-N_2)+(N_4+N_2)\Phi_4^2}, \tag{3.82}$$

where

$$\Phi_4(x,t)=sn\left(\frac{1}{2}(N_2-N_4)(x+\beta t-\eta_0),\frac{(N_2+N_4)}{(N_2-N_4)}\right).$$

Substituting (3.82), (3.80) and (3.79) into (3.48)- (3.50), we get the exact solutions of generalized Hirota–Satsuma KdV equations(3.1) in the form:

$$u_9(\xi)=\frac{\beta+4N_4^2+4N_2^2}{3}+4N_3^2$$
$$-8N_3\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)\Phi_4^2(x,t)}{(N_4-N_2)+(N_4+N_2)\Phi_4^2(x,t)}\right]$$
$$+4\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)\Phi_4^2(x,t)}{(N_4-N_2)+(N_4+N_2)\Phi_4^2(x,t)}\right]^2, \tag{3.83}$$

and

$$v_9(\xi)=T_0$$
$$-2N_3T_2\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)\Phi_4^2(x,t)}{(N_4-N_2)+(N_4+N_2)\Phi_4^2(x,t)}\right]$$
$$+T_2\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)\Phi_4^2(x,t)}{(N_4-N_2)+(N_4+N_2)\Phi_4^2(x,t)}\right]^2, \tag{3.84}$$

$$w_9(\xi)=\frac{4}{3T_2^2}(6N_3^2T_2+2\beta T_2+2N_4^2T_2+2T_2N_2^2-3T_0)$$
$$-\frac{8N_3}{T_2}\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)\Phi_4^2(x,t)}{(N_4-N_2)+(N_4+N_2)\Phi_4^2(x,t)}\right] \tag{3.85}$$
$$+\frac{4}{T_2}\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)\Phi_4^2(x,t)}{(N_4-N_2)+(N_4+N_2)\Phi_4^2(x,t)}\right]^2.$$

Remarks:

1- This method allowed us to construct many types of the traveling wave solutions in the hyperbolic functions, trigonometric functions, and Jacobian elliptic functions.

2- The balance number of this method is not constant as in other methods but changes when the trial equation changes.

3- This method has generalized the tanh-function method, Jacobian elliptic functions methods, and Exp function method.

IV.CONCLUSION

In this paper, we used the extended trial equation method to construct a series of new analytic solutions for some nonlinear partial differential equations in mathematical physics when the balance number is a positive integer. We constructed the exact solutions in many different functions such as hyperbolic functions, trigonometric functions, Jacobian elliptic functions, and rational solutions for the nonlinear Hirota –Satsuma KdV equations. This method is more powerful than other methods for solving the nonlinear partial differential equations, and can be used to solve many other nonlinear partial differential equations in mathematical physics.

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