Numerical Solutions of Fractional Partial Differential Equations by Using Legendre Wavelets

Hao Song, Mingxu Yi*, Jun Huang, and Yalin Pan

***Abstract*— A numerical method based on Legendre wavelets is proposed for fractional partial differential equations. Legendre wavelets operational matrices of fractional order integration and fractional order differentiation are derived. By using these matrices, each term of the problem was converted into matrix form. Lastly, the equation was transformed into a Sylvester equation. The error estimation of the Legendre wavelets method is given in Theorem 5.1. Three numerical examples are shown to demonstrate the validity and applicability of the method.**

***Index Terms*— Fractional partial differential equation, Legendre wavelets, Operational matrix, Sylvester equation, Error analysis**

I. Introduction

In science and engineering, many dynamical systems can be described by fractional-order equations [1-3]. These dynamical systems generally originates in the fields of electrode-electrolyte [4], dielectric polarization [5], electromagnetic waves [6], viscoelastic systems [7] etc. Various materials and processes have been found to be described using fractional calculus. Anomalous diffusion has been discussed in various physical fields [8-10]. The features of anomalous diffusion include history dependence, long-range correlation and heavy tail characteristics. These features can be accommodated well by using fractional calculus. In order to model these phenomena, fractional derivatives and fractional partial differential equations were proposed. Nowadays, fractional partial differential equations have been employed as a powerful tool in complex anomalous diffusion modelling.

Apart from modelling aspects of these fractional partial differential equations, the numerical solution techniques are rather more significant aspects. Various numerical methods and approaches are available to solve linear and nonlinear fractional partial differential equations. Some analytic methods are proposed. However, numerical methods are in demand since it is difficult to obtain analytic solutions for each and every fractional partial differential equation originating from real life problems. Until now, to the best of the author's knowledge, the main approach for solving fractional partial equations were the finite difference method [11, 12], Laplace transform method [13], and generalized differential transform method [14]. These approximations are valuable for researchers and scientists.

This research considered a class of fractional partial equations:

$$\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^\beta u}{\partial y^\beta}=f(x,y)\,. \tag{1}$$

such that

$$u(0,y)=u(x,0)=0\,. \tag{2}$$

where $\partial^\alpha u(x,y)/\partial x^\alpha$ and $\partial^\beta u(x,y)/\partial y^\beta$ are fractional derivatives, $f(x,y)$ is the known continuous function, $u(x,y)$ is the unknown function, and $0<\alpha,\beta\le 1$.

II. Legendre Wavelets

Legendre wavelets $\psi_{nm}(x)$ are expressed as follows [15]

$$\psi_{nm}(x)=\begin{cases}\left(\dfrac{2m+1}{2}\right)^{1/2}2^{\frac{k}{2}}P_m(2^k x-\hat{n}), & \dfrac{\hat{n}-1}{2^k}\le x<\dfrac{\hat{n}+1}{2^k};\\ 0, & otherwise.\end{cases} \tag{3}$$

where $k=1,2,\ldots$, $\hat{n}=2n-1$, $n=1,2,\ldots,2^{k-1}$, $m=0,1,\ldots,M-1$ is the degree of the Legendre polynomials and M is a fixed positive integer; $P_m(x)$ are Legendre polynomials of degree m.

For any function, $f(x)\in L^2[0,1)$ may be given by the Legendre wavelets as

$$f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(x)\,. \tag{4}$$

where $c_{nm}=\langle f(x),\psi_{nm}(x)\rangle$, and $\langle,\rangle$ is the inner product of $f(x)$ and $\psi_{nm}(x)$.

If the infinite series in Equation (4) is truncated, then

$$f(x)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x)=C^T\Psi(x). \tag{5}$$

where C and $\Psi(x)$ are $\hat{m}=2^{k-1}M$ column vectors

$$\begin{aligned}C=[&c_{10},c_{11},\ldots,c_{1M-1},c_{20},c_{21},\ldots,\\ &c_{2M-1},\ldots,c_{2^{k-1}0},c_{2^{k-1}1},\ldots,c_{2^{k-1}M-1}]^T.\end{aligned} \tag{6}$$

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$$\Psi(x)=[\psi_{10},\psi_{11},\ldots,\psi_{1M-1},\psi_{20},\psi_{21},\ldots,\psi_{2M-1},\ldots,\psi_{2^{k-1}0},\psi_{2^{k-1}1},\ldots,\psi_{2^{k-1}M-1}]^T. \tag{7}$$

For simplicity, Equation (5) is rewritten as

$$f(x)\approx\sum_{i=1}^{\hat{m}}c_i\psi_i(x)=C^T\Psi(x). \tag{8}$$

where $c_i=c_{nm}$, $\psi_i=\psi_{nm}$. The index i is determined by the relation $i=M(n-1)+m+1$.

Therefore, the vectors can also be written as

$$C=[c_1,c_2,\ldots,c_M,c_{M+1},\ldots,c_{2M},\ldots,c_{M(2^{k-1}-1)+1},\ldots,c_{\hat{m}}]^T. \tag{9}$$

$$\Psi(x)=[\psi_1,\psi_2,\ldots,\psi_M,\psi_{M+1},\ldots,\psi_{2M},\ldots,\psi_{M(2^{k-1}-1)+1},\ldots,\psi_{\hat{m}}]^T. \tag{10}$$

Similarly, the function $u(x,y)$ over $[0,1)\times[0,1)$ can be expressed as follows

$$u(x,y)\approx\sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}}u_{ij}\psi_i(x)\psi_j(y)=\Psi^T(x)U\Psi(y). \tag{11}$$

where $U=[u_{ij}]$ and $u_{ij}=\langle\psi_i(x),\langle u(x,y),\psi_j(y)\rangle\rangle$.

Theorem 2.1[16] *Any function $f(x)$, defined over $[0,1]$, is with bounded second derivative, say $|f''(x)|\le\tilde{M}$, can be expressed as the sum of Legendre wavelets, and the series converges uniformly to the function $f(x)$. That is*

$$f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(x).$$

where $c_{nm}=\langle f(x),\psi_{nm}(x)\rangle$, and $\langle,\rangle$ is the inner product of $f(x)$ and $\psi_{nm}(x)$.

Theorem 2.2[17] *If a continuous function $u(x,y)$ defined over $[0,1)\times[0,1)$ has bounded mixed fourth partial derivative $\left|\frac{\partial^4u(x,y)}{\partial x^2\partial y^2}\right|\le M'$, then the Legendre wavelets expansion of $u(x,y)$ converges uniformly to it.*

III. Operational Matrices of Integration and Differentiation for Legendre Wavelets

3.1 Fractional Calculus

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha\ge0$ of a function is defined as [13]

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}f(\tau)d\tau,\quad \alpha>0. \tag{12}$$

$$J^0f(x)=f(x). \tag{13}$$

Definition 2. The fractional differential operator in Caputo sense is defined as

$$D^\alpha f(x)=\begin{cases}\dfrac{d^rf(x)}{dx^r}, & \alpha=r\in N;\\ \dfrac{1}{\Gamma(r-\alpha)}\displaystyle\int_0^x\frac{f^{(r)}(\tau)}{(x-\tau)^{\alpha-r+1}}d\tau, & 0\le r-1<\alpha<r.\end{cases} \tag{14}$$

The Caputo fractional derivative of order α is also given by $D^\alpha f(x)=J^{r-\alpha}D^rf(x)$, where D^r is the usual integer differential operator of order r. The relation between the Caputo operator and Riemann-Liouville operator is given as follows:

$$D^\alpha J^\alpha f(x)=f(x). \tag{15}$$

$$J^\alpha D^\alpha f(x)=f(x)-\sum_{k=0}^{r-1}f^{(k)}(0^+)\frac{x^k}{k!},\quad x>0. \tag{16}$$

3.2 Fractional Order Operational Matrix of Integration and Differentiation for Legendre Wavelets.

This section simply presents the operational matrix of fractional integration of Legendre wavelets [15].

Firstly, the basis set of block pulse functions is considered. These functions, defined over $[0,1)$, are given as follows [18]

$$b_i(x)=\begin{cases}1, & ih\le x<(i+1)h;\\ 0, & otherwise.\end{cases} \tag{17}$$

Note: $i=0,1,2,\ldots,\hat{m}-1$ and is a positive integer value for $\hat{m}$ and $h=\frac{1}{\hat{m}}$.

Let $B(x)=[b_0(x),b_1(x),\ldots,b_{\hat{m}-1}(x)]^T$. Accordingly, suppose

$$J^\alpha\left(B(x)\right)\approx F^\alpha B(x). \tag{18}$$

where F^α is the fractional integration block pulse operational matrix [18], where

$$F^\alpha=h^\alpha\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix}1 & \xi_1 & \xi_2 & \cdots & \xi_{\hat{m}-1}\\ 0 & 1 & \xi_1 & \cdots & \xi_{\hat{m}-2}\\ 0 & 0 & 1 & \cdots & \xi_{\hat{m}-3}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}.$$

Here, $\xi_k=(k+1)^{\alpha+1}-2k^{\alpha+1}+(k-1)^{\alpha+1}$, $k=1,2,\ldots,\hat{m}-1$.

There is a relationship between the Legendre wavelets and block pulse functions [19]

$$\Psi(x)=\Phi B(x). \tag{19}$$

where $\Phi=[\Psi(x_0),\Psi(x_1),\ldots,\Psi_{\hat{m}-1}]$, $x_i=\frac{i}{\hat{m}}$, $i=0,1,\ldots,\hat{m}-1$.

Legendre wavelets fractional integration operator J^α satisfies

$$J^\alpha\Psi(x)\approx P^\alpha\Psi(x). \tag{20}$$

where P^α is the Legendre wavelets fractional integration operational matrix . Equation (18) and Equation (19) result in

$$J^\alpha\Psi(x)\approx J^\alpha\Phi B(x)=\Phi J^\alpha B(x)\approx\Phi F^\alpha B(x). \tag{21}$$

Using Equation (20) and Equation (21),

$$P^\alpha\Psi(x)=P^\alpha\Phi B(x)=\Phi F^\alpha B(x). \tag{22}$$

Then, matrix P^α is as follows

$$P^\alpha=\Phi F^\alpha\Phi^{-1}. \tag{23}$$

The fractional derivative of order α in the Caputo sense of the vector $\Psi(x)$ can be expressed as

$$D^\alpha\left(\Psi(x)\right)\approx Q^\alpha\Psi(x). \tag{24}$$

where Q^α is the $\hat{m}\times\hat{m}$ Legendre wavelets operational matrix of fractional differentiation. Due to the relationship in fractional calculus, $Q^\alpha P^\alpha=I$, matrix Q^α can easily be acquired by inverting matrix P^α.

The fractional order integration and differentiation of the function t are selected to verify the effectiveness of matrix P^α and Q^α. The fractional order integration and differentiation of $u(t)=t$ are obtained as follows: $J^\alpha u(t)=\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}$ and $D^\alpha u(t)=\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}$.

When $\alpha=0.5,\hat{m}=32$, comparative results for the fractional integration and differentiation are shown in **Fig. 1** and **Fig. 2**, respectively.

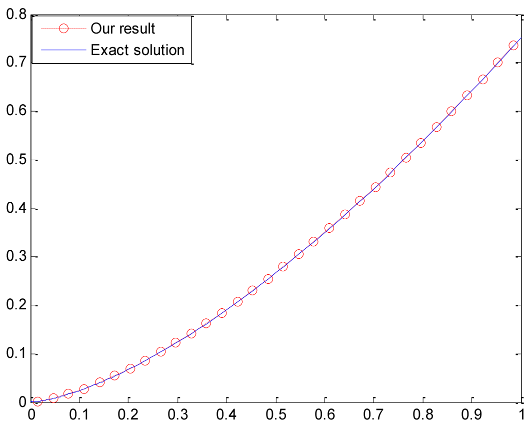

Fig. 1. 1/2 order integration of $u(t)=t$.

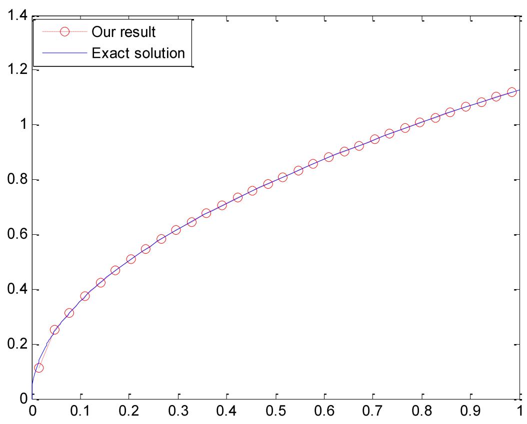

Fig. 2. 1/2 order differentiation of $u(t)=t$.

IV. Solution of the Fractional Partial Differential Equation

Consider the fractional partial differential equation Equation (1) in section 1. If it is assumed the function $u(x,y)$ in terms of Legendre series, it can be written as Equation (11).

Then the following can be obtained:

$$\frac{\partial^\alpha u}{\partial x^\alpha} \cong \frac{\partial^\alpha(\Psi^T(x)U\Psi(y))}{\partial x^\alpha} = \left[\frac{\partial^\alpha \Psi(x)}{\partial x^\alpha}\right]^T U\Psi(y) = [Q^\alpha \Psi(x)]^T U\Psi(y) = \Psi^T(x)[Q^\alpha]^T U\Psi(y). \tag{25}$$

$$\frac{\partial^\beta u}{\partial y^\beta} \cong \frac{\partial^\beta(\Psi^T(x)U\Psi(y))}{\partial y^\beta} = \Psi^T(x)U\frac{\partial^\beta(\Psi(y))}{\partial y^\beta} = \Psi^T(x)UQ^\beta\Psi(y). \tag{26}$$

The function $f(x,y)$ of Equation (1) may also be written as

$$f(x,y) \cong \Psi^T(x)\cdot F\cdot\Psi(y). \tag{27}$$

where $F=[f_{i,j}]_{\hat{m}\times\hat{m}}$.

Substituting Equation (25), Equation (26) and Equation (27) into Equation (1), then

$$\Psi^T(x)[Q^\alpha]^T U\Psi(y)+\Psi^T(x)UQ^\beta\Psi(y) = \Psi^T(x)F\Psi(y). \tag{28}$$

Dispersing Equation (28) by points (x_i,y_j) , $i=1,2,\cdots,\hat{m}$ and $j=1,2,\cdots,\hat{m}$, then

$$[Q^\alpha]^T U+UQ^\beta = F. \tag{29}$$

Equation (28) is a Sylvester equation. The Sylvester equation can be solved easily using Matlab2011a.

V. Error Analysis

In this part, error analysis of the method is employed. Let $\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}$ be the following approximation of $\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}$, $\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}=\sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}}u_{ij}\psi_i(x)\psi_j(y)$.

Then $\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}=\sum_{i=\hat{m}+1}^{\infty}\sum_{j=\hat{m}+1}^{\infty}u_{ij}\psi_i(x)\psi_j(y)$.

Theorem 5.1 *Let the function* $\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}$ *obtained using Legendre wavelets be the approximation of* $\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}$, *and* $u(x,y)$ *has bounded mixed fractional partial derivative* $\left|\frac{\partial^{4+\alpha+\beta}u(x,y)}{\partial x^{2+\alpha}\partial y^{2+\beta}}\right|\le \hat{M}$, *then*

$$\left\|\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}\right\|_E \le \frac{\hat{M}\cdot\hat{N}^{1/2}}{2^{4k}},$$

wher $\|u(x,y)\|_E=\left(\int_0^1\int_0^1 u^2(x,y)dxdy\right)^{1/2}$,

$u_{ij}=\left\langle \psi_i(x),\left\langle\frac{\partial^\alpha u(x,y)}{\partial x^\alpha},\psi_j(y)\right\rangle\right\rangle$, *and* $\hat{N}$ *is a constant.*

Proof.

The orthonormality of the series $\{\psi_i(x)\}$, defined on $[0,1)$, implies $\int_0^1\Psi(x)[\Psi(x)]^T dx = I$, where I is the identify matrix, then

$$\begin{aligned}
&\left\|\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}\right\|_E^2 \\
&=\int_0^1\int_0^1\left[\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}\right]^2 dxdy \\
&=\int_0^1\int_0^1\left[\sum_{i=\hat{m}+1}^{\infty}\sum_{j=\hat{m}+1}^{\infty}u_{ij}\psi_i(x)\psi_j(y)\right]^2 dxdy \\
&=\sum_{i=\hat{m}+1}^{\infty}\sum_{j=\hat{m}+1}^{\infty}\sum_{i'=\hat{m}+1}^{\infty}\sum_{j'=\hat{m}+1}^{\infty}u_{ij}u_{i'j'}\int_0^1\int_0^1\psi_i(x)\psi_j(y)dxdy\int_0^1\int_0^1\psi_{i'}(x)\psi_{j'}(y)dxdy \\
&=\sum_{i=\hat{m}+1}^{\infty}\sum_{j=\hat{m}+1}^{\infty}u_{ij}^2\int_0^1\psi_i^2(x)dx\int_0^1\psi_j^2(y)dy \\
&=\sum_{i=\hat{m}+1}^{\infty}\sum_{j=\hat{m}+1}^{\infty}u_{ij}^2.
\end{aligned}$$

The Legendre wavelets coefficients of function $u(x,y)$ are defined by

$$\begin{aligned}
u_{ij}&=\int_0^1\int_0^1\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}\psi_i(x)\psi_j(y)dxdy \\
&=\int_0^1\int_{I_{nk}}\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}\left(\frac{2m+1}{2}\right)^{1/2}2^{k/2}P_m(2^k x-\hat{n})\psi_j(y)dxdy.
\end{aligned}$$

Let $2^k x-\hat{n}=t$ .By changing $2^k x-\hat{n}=t$ and $dx=\frac{1}{2^k}dt$, then

$$
\begin{aligned}
& u_{ij} \\
& =\left(\frac{2m+1}{2}\right)^{1/2} 2^{-k/2}\cdot\int_0^1 \psi_j(y)\int_{-1}^1 \frac{\partial^\alpha}{\partial t^\alpha} u\left(\frac{\hat{n}+t}{2^k}, y\right) P_m(t)\,dtdy \\
& =\left(\frac{1}{2^{k+1}(2m+1)}\right)^{1/2}\cdot \\
& \quad \int_0^1 \psi_j(y)\int_{-1}^1 \frac{\partial^\alpha}{\partial t^\alpha} u\left(\frac{\hat{n}+t}{2^k}, y\right) d(P_{m+1}(t)-P_{m-1}(t))\,dy \\
& =-\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{1/2}\cdot \\
& \quad \int_0^1 \psi_j(y)\int_{-1}^1 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} u\left(\frac{\hat{n}+t}{2^k}, y\right)(P_{m+1}(t)-P_{m-1}(t))\,dtdy \\
& =-\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{1/2}\cdot \\
& \quad \int_0^1 \psi_j(y)\int_{-1}^1 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} u\left(\frac{\hat{n}+t}{2^k}, y\right) d\left(\frac{P_{m+2}(t)-P_m(t)}{2m+3}-\frac{P_m(t)-P_{m-2}(t)}{2m-1}\right) dy \\
& =\left(\frac{1}{2^{5k+1}(2m+1)}\right)^{1/2}\cdot \\
& \quad \int_0^1 \psi_j(y)\int_{-1}^1 \frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}} u\left(\frac{\hat{n}+t}{2^k}, y\right)\left(\frac{P_{m+2}(t)-P_m(t)}{2m+3}-\frac{P_m(t)-P_{m-2}(t)}{2m-1}\right) dtdy.
\end{aligned}
$$

Now,
let $\tau_m(t)=(2m-1)P_{m+2}(t)-2(2m+1)P_m(t)+(2m+3)P_{m-2}(t)$,
then

$$
u_{ij}=\left(\frac{1}{2^{5k+1}(2m+1)}\right)^{1/2}\frac{1}{(2m-1)(2m+3)}\cdot
\int_0^1 \psi_j(y)\int_{-1}^1 \frac{\partial^{2+\alpha}}{\partial t^{2+\alpha}} u\left(\frac{\hat{n}+t}{2^k}, y\right)\tau_m(t)\,dtdy.
$$

By solving this equation,

$$
u_{ij}=A(k,m)\int_{-1}^1\int_{-1}^1 \frac{\partial^{4+\alpha+\beta}}{\partial t^{2+\alpha}\partial s^{2+\beta}} u\left(\frac{\hat{n}+t}{2^k}, \frac{\hat{n}+s}{2^k}\right)\tau_m(t)\tau_m(s)\,dtds.
$$

where $A(k,m)=\dfrac{1}{2^{5k+1}(2m+1)}\dfrac{1}{(2m-1)^2(2m+3)^2}$.

Therefore

$$
|u_{ij}|\le A(k,m)\int_{-1}^1\int_{-1}^1 \left|\frac{\partial^{4+\alpha+\beta}}{\partial t^{2+\alpha}\partial s^{2+\beta}} u\left(\frac{\hat{n}+t}{2^k}, \frac{\hat{n}+s}{2^k}\right)\right| |\tau_m(t)||\tau_m(s)|\,dtds.
$$

Furthermore, the above equation reveals

$$
\int_{-1}^1 |\tau_m(t)|dt \le \sqrt{24}\,\frac{2m+3}{\sqrt{2m-3}}.
$$

Thus,

$$
\begin{aligned}
|u_{ij}| &\le A(k,m)\frac{24\hat{M}(2m+3)^2}{2m-3} \\
&\le \frac{1}{2^{5k}(2m+1)}\frac{1}{(2m-3)(2m-1)^2} \le \frac{12\hat{M}}{(2n)^5(2m-3)^4}.
\end{aligned}
$$

Namely,

$$
|u_{ij}|^2 \le \frac{144\hat{M}^2}{(2n)^{10}(2m-3)^8}.
$$

Therefore,

$$
\begin{aligned}
& \left\|\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}\right\|_E^2 \\
& \le \sum_{i=\hat{m}}^{\infty}\sum_{j=\hat{m}}^{\infty} u_{ij}^2 \le \sum_{i=\hat{m}}^{\infty}\sum_{j=\hat{m}}^{\infty}\frac{144\hat{M}^2}{(2n)^{10}(2m-3)^8} \\
& \le \sum_{i=\hat{m}}^{\infty}\sum_{j=\hat{m}}^{\infty}\frac{144\hat{M}^2}{(2^k)^{10}(2M-5)^8} = \sum_{i=M2^{k-1}}^{\infty}\sum_{j=M2^{k-1}}^{\infty}\frac{144\hat{M}^2}{(2^k)^{10}(2M-5)^8} \\
& \le \sum_{M=1}^{\infty}\sum_{p=k-1}^{\infty}\left(\sum_{i=M2^p}^{M2^{p+1}-1}\sum_{j=M2^p}^{M2^{p+1}-1}\frac{144\hat{M}^2}{(2^k)^{10}(2M-5)^8}\right) \\
& \le \sum_{M=1}^{\infty}\sum_{p=k-1}^{\infty}\left(\frac{144\hat{M}^2}{(2^{p+1})^{10}(2M-5)^8}M^2 2^{2p}\right) \le \frac{\hat{M}^2\hat{N}}{2^{1+8k}}.
\end{aligned}
$$

where $\hat{N}$ is a constant.
Next,

$$
\left\|\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}\right\|_E^2 \le \frac{\hat{M}^2\hat{N}}{2^{8k}}.
$$

Thus

$$
\left\|\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}\right\|_E \le \frac{\hat{M}\cdot\hat{N}^{1/2}}{2^{4k}}.\ \square
$$

From this theorem, it is evident

$$
\left\|\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}-\frac{\partial^\alpha u_{\hat{m}}(x,y)}{\partial x^\alpha}\right\|_E \to 0 \text{ when } k\to+\infty.
$$

VI. Numerical Examples

Example 1. Consider the nonhomogeneous partial differential equation

$$
\frac{\partial^{1/5}u}{\partial x^{1/5}}+\frac{\partial^{1/5}u}{\partial y^{1/5}}=f(x,y), \qquad x,y\ge 0.
$$

Such that $u(0,t)=u(x,0)=0$ and $f(x,y)=\dfrac{5(x^{4/5}y+xy^{4/5})}{4\Gamma(4/5)}$. The numerical results for $\hat{m}=8$, $\hat{m}$=16, $\hat{m}$=32 are shown in **Fig. 3**, **Fig. 4**, **Fig. 5**. The exact solution is xy, shown in **Fig. 6**. **Fig. 3-6** illustrate the numerical solutions are in very good coincidence with the exact solution.

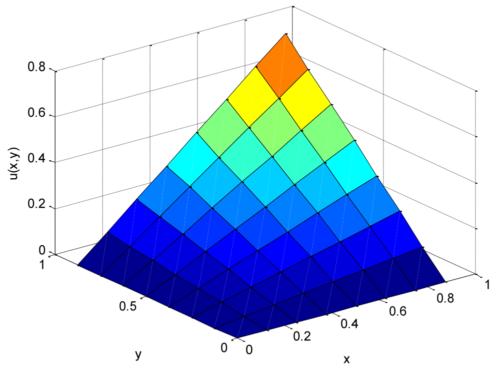

Fig. 3. Numerical solution of $\hat{m}=8$.

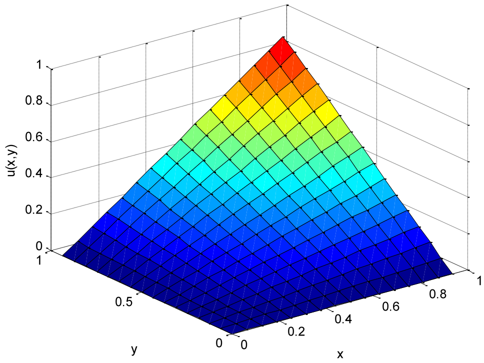

Fig. 4. Numerical solution of $\hat{m}=16$.

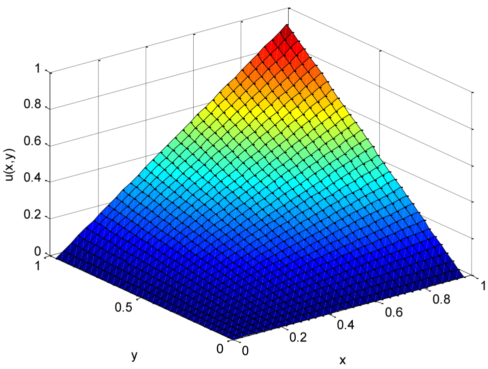

Fig. 5. Numerical solution of $\hat{m}=32$.

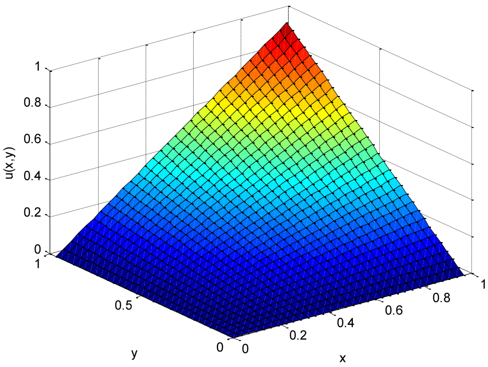

Fig. 6. Exact solution for Example 1.

Example 2. Consider the following fractional partial differential equation [20]

$$\frac{\partial^{1/3}u}{\partial x^{1/3}}+\frac{\partial^{1/2}u}{\partial y^{1/2}}=f(x,y), \qquad x,y\geq 0.$$

subject to $u(0,t)=u(x,0)=0$, $f(x,y)=\dfrac{9x^2y^{5/3}}{5\Gamma(2/3)}+\dfrac{8x^{3/2}y^2}{3\Gamma(1/2)}$.

Fig. 7-10 show the numerical solutions for various m and the exact solution x^2y^2 . The absolute errors obtained by Block Pulse Method (BPM) and Legendre Wavelets Method (LWM) for different $\hat{m}$ are shown in **Table 1**, respectively. From **Fig. 7-10** and **Table 1**, the absolute errors between numerical solutions and the exact solution are clearly decreasingly smaller when $\hat{m}$ increases. Compared with the approximations obtained by BPM, LWM can achieve a higher degree of accuracy.

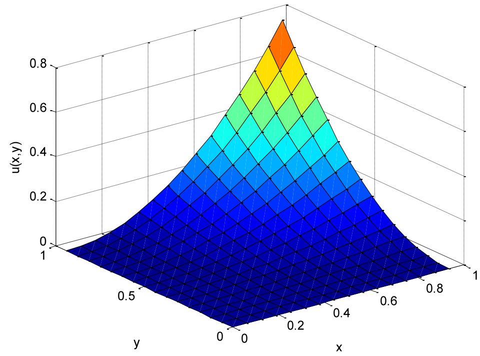

Fig. 7. Numerical solution of $\hat{m}=16$.

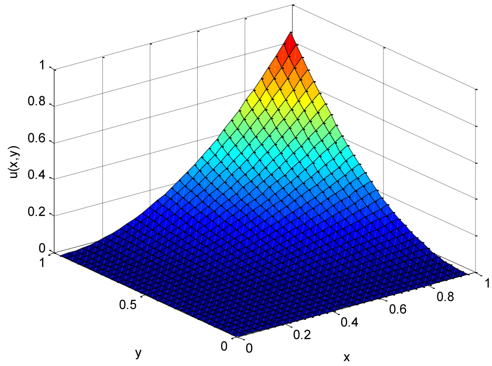

Fig. 8. Numerical solution of $\hat{m}=32$.

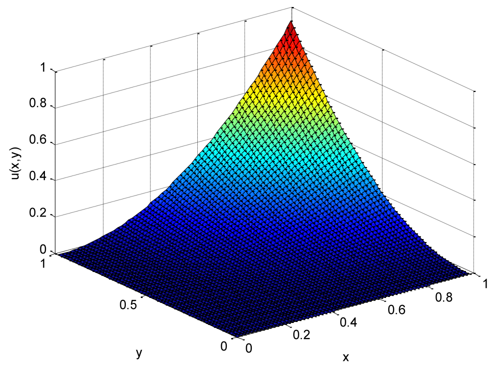

Fig. 9. Numerical solution of $\hat{m}=64$.

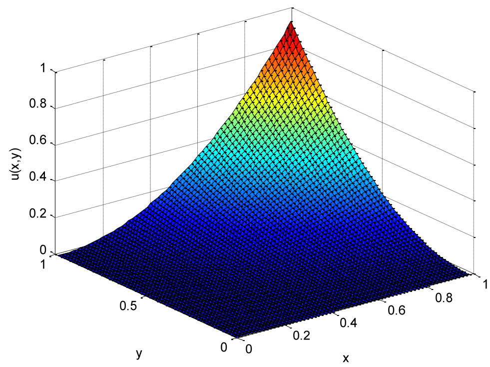

Fig. 10. Exact solution for Example 2.

TABLE 1
THE ABSOLUTE ERROR OF DIFFERENT $\hat{m}$ FOR EXAMPLE 2

(x, y)	$\hat{m}=8$		$\hat{m}=16$		$\hat{m}=32$	
	LWM	BPM	LWM	BPM	LWM	BPM
(0,0)	0	0	0	0	0	0
(1/8,1/8)	7.4931e-006	3.0719e-005	5.2376e-008	6.3969e-006	3.2276e-010	1.6065e-006
(2/8,2/8)	3.3456e-006	1.0113e-004	6.4260e-007	2.5382e-005	5.3431e-009	6.4515e-006
(3/8,3/8)	2.1453e-005	2.2617e-004	3.6551e-007	5.7263e-005	7.0156e-009	1.4572e-005
(4/8,4/8)	5.4356e-005	4.0185e-004	4.0276e-006	1.0214e-004	5.8820e-008	2.5983e-005
(5/8,5/8)	7.2573e-005	6.2905e-004	6.1355e-006	1.6009e-004	6.3285e-008	4.0692e-005
(6/8,6/8)	9.3462e-005	9.0799e-004	8.8762e-006	2.3113e-004	4.1455e-007	5.8709e-005
(7/8,7/8)	1.2653e-004	1.2388e-003	1.1992e-005	3.1532e-004	2.4376e-006	8.0040e-005

Example 3. Consider the fractional partial differential equation as follows

$$\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^\beta u}{\partial y^\beta}=\sin(x+y), \qquad x, y \geq 0.$$

Such that $u(0,t)=u(x,0)=0$. The exact solution of this equation is $\sin x \sin y$ when $\alpha=\beta=1$. The numerical solution is shown in **Fig. 11**, and the exact solution is displayed in **Fig. 12**, **Fig. 13** and **Fig. 14** show the approximations for various values of α, β . They demonstrate the simplicity and power of the proposed method. Compared with the generalized differential transform method in Ref. [14], using the aforementioned method can greatly reduce computation.

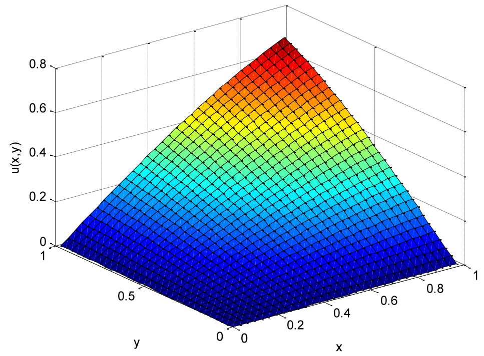

Fig. 11. Numerical solution of $\alpha=\beta=1$.

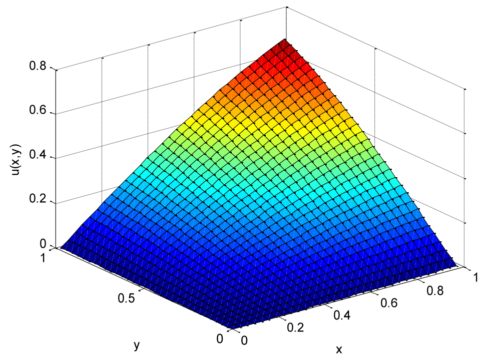

Fig. 12. Exact solution of $\alpha=\beta=1$.

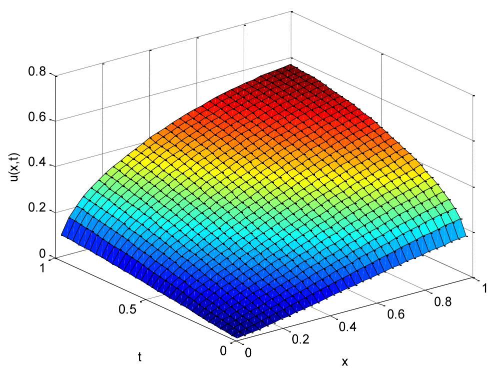

Fig. 13. Numerical solution of $\alpha=1/2, \beta=1/3$.

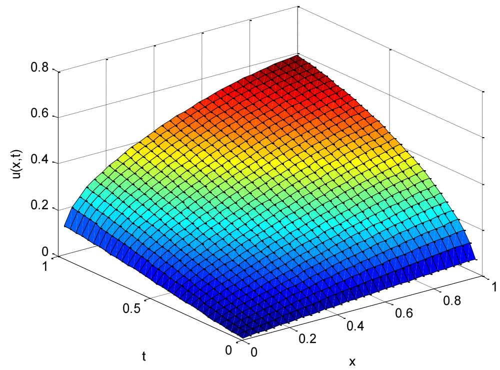

Fig. 14. Numerical solution of $\alpha=3/7, \beta=3/5$.

VII. CONCLUSION

This article introduced Legendre wavelets and wavelets operational matrices of fractional integration and fractional differentiation. The fractional partial differential equations improved numerically via the operational matrices. By solving the Sylvester system, numerical solutions were obtained. In addition, the error analysis of Legendre wavelets was proposed. The solution obtained using the suggested method showed numerical solutions were in very good agreement with the exact solution.

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