Adaptive Complex Modified Projective Synchronization of Two Fractional-order Complex-variable Chaotic Systems with Unknown Parameters

Xiaomin Tian, Zhong Yang

Abstract—**In this paper, the complex modified projective synchronization (CMPS) of two different fractional-order chaotic complex systems with unknown parameters is firstly investigated. We assume that the slave system is perturbed by external disturbances. The master and slave systems achieved CMPS can be synchronized up to a complex scaling matrix. On the basis of a novel stability theory, a robust adaptive control law is designed to realize the CMPS for two different fractional-order chaotic complex systems. Meanwhile, to deal with these unknown parameters, some fractional-order type update laws are provided. The CMPS can be regarded as the generalization of several types of synchronization reported in existing literatures. Simulation results are given to verify the effectiveness and feasibility of the proposed synchronization scheme.**

Index Terms—**Complex modified projective synchronization, Fractional-order chaotic complex system, Adaptive control, Fractional-order type update law**

I. Introduction

Although fractional calculus is a mathematical topic with more than 300 years history, its applications in the fields of physics and engineering have attracted lots of attentions only in the recent years. It was found that, with the help of fractional calculus, many systems in interdisciplinary fields can be described more accurately, such as viscoelastic system [1], dielectric polarization [2], electrode-electrolyte polarization [3], finance systems and electromagnetic waves [4]. That is to say, fractional calculus provides a superb instrument for the description of memory and hereditary properties of various materials and processes. Many literatures have proven that some fractional-order differential systems can behave chaotically, e.g., the fractional-order Duffing system [5], fractional-order Chen-Lee system [6], fractional-order Lorenz system [7], fractional-order hyperchaotic Chen system [8], fractional-order Qi system [9], and so on.

The research on chaotic systems has grown significantly over past decades and has become a popular topic. For example, Gyorgyi [10] calculated the entropy in the chaotic systems. Steeb et al. [11] applied the maximum entropy formalism into the study of chaotic systems. Aghababa [12] used the finite-time theory to realize the finite-time synchronization of chaotic systems. Lu [13] developed a nonlinear observer to synchronize the chaotic systems. Chen et al. [14,15] researched the chaos synchronization of fractional-order chaotic neural network.

However, all of aforementioned researches only focus on the systems with real variables, while chaotic complex systems are not involved. In practical, chaotic complex systems can be widely used to describe a variety of physical phenomenon such as population inversion [16], detuned laser systems [17], thermal convections of liquid flows [18], etc. At present, some control schemes [19-24] have been proposed to synchronize two chaotic systems with complex variables. It is should be noted that all of chaotic complex nonlinear systems in above mentioned literatures are integer-order systems. So, it is a challenging and meaningful problem for researchers to realize the synchronization of fractional-order chaotic complex nonlinear systems. Recently, Luo et al. [25,26] firstly studied the dynamic properties of fractional-order complex Lorenz and fractional-order complex Chen systems. Liu et al. [27] researched the control and synchronization of fractional-order complex T system. But, in [25-27], the system parameters are assumed to be known for the synchronization of two identical fractional-order chaotic complex systems. As a matter of fact, many systems' parameters cannot be exactly known in advance. The synchronization will be not achieved under the effect of unknown uncertainties. Therefore, it is urgent to consider the influence of unknown parameters in synchronizing two chaotic complex systems.

On the other hand, in many practical systems, the master (drive) system and slave (response) system may evolve in different directions with a constant intersection angle. Thus, the modified projective synchronization with respect to a complex scaling matrix, namely, the complex modified projective synchronization (CMPS) should be taken into consideration. This kind of CMPS is deemed to be the generalization of several types of synchronization, such as complete synchronization (CS) [28], anti-synchronization (AS) [29], modified projective synchronization (MPS) [30], and projective synchronization (PS) [31], etc. For this reason, CMPS of two different chaotic complex systems is more

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essential. To the best of our knowledge, until now, there is no information available about the CMPS between two different fractional-order chaotic complex systems with unknown parameters.

Motivated by the above discussions, in this paper, the CMPS of two different fractional-order chaotic complex systems with unknown parameters and external disturbances is firstly investigated. Inspired by [31], we extend the theoretical results of [31] to the fractional-order chaotic complex systems. As we all known that the stability theory of integer-order nonlinear systems cannot be directly applied to fractional-order systems, so, we use a novel fractional-order system's stability theory to demonstrate the applicability of the proposed synchronization scheme. It is worth noting that in [31] the effect of external disturbances is not taken into account, and the feedback control gains are chosen as fixed values wherever the initial points start, which result in the theoretical results in [31] are much conservative. Fortunately, on the basis of the novel stability theory and fractional-order adaptive control method, we propose a new adaptive control law and some parametric update rules to address the above problems. Therefore, our approach is more meaningful and practical than that of [31]. We applied this proposed control scheme, as an example, to research the CMPS between fractional-order complex Lorenz system and fractional-order complex Chen system.

The rest structure of this paper is as follows. In Section 2, the relevant definitions, lemma and description of n-dimensional fractional-order chaotic complex system are introduced. In Section 3, the adaptive controller and unknown parameters update laws are designed in detail. Simulation results about the CMPS between fractional-order complex Lorenz system and fractional-order complex Chen system are provided in Section 4. Finally, conclusions are included in Section 5.

II. Preliminaries

A. Definitions and Lemma

The most frequently used definitions for the general fractional calculus are Riemann-Liouville definition and Caputo definition.

Definition 1 The α th-order Riemann-Liouville fractional integration is given by

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 For $n-1<\alpha\le n$, $n\in R$, the Riemann-Liouville fractional derivative definition of order α is defined as

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau = \frac{d^n}{dt^n} I^{n-\alpha} f(t) \tag{2}$$

Definition 3 The Caputo fractional derivative definition of order α is written as

$$ {}_{t_0}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1<\alpha<m \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \tag{3}$$

Lemma 1 (see [32, 33]) Consider the autonomous system

$$D^{\alpha} x = Ax \text{ or } D^{\alpha} x = f(x) \tag{4}$$

where $\alpha\in(0,1]$ is the fractional order and $x=(x_1, x_2, ..., x_n)^T$ is the state variable. $A\in R^{n\times n}$ is a constant matrix. If there is a real symmetric positive definite matrix P such that the inequation $J = x^T P D^{\alpha} x \le 0$ always holds for any states, then system (4) is asymptotically stable.

For the detailed application of the above lemma in fractional-order chaotic systems, the reader can refer to Refs. [32-36].

B. Description of N-dimensional Fractional-order Chaotic Complex System

Consider a fractional-order chaotic complex system, described by

$$D^{\alpha} x = F(x)\psi + f(x) \tag{5}$$

where $\alpha\in(0,1)$ is the fractional order of the system. $x=(x_1, x_2, ..., x_n)^T$ is the state complex vector, and $x = x^r + jx^i$, $x^r = (u_1, u_3, ..., u_{2n-1})^T$, $x^i = (u_2, u_4, ..., u_{2n})^T$, $j=\sqrt{-1}$. $F(x)\in C^{n\times m}$, and the elements of this matrix are the functions of state complex variables. $\psi\in R^m$ (or $\in C^m$) is the vector of unknown system parameters. $f=(f_1, f_2, ..., f_n)^T$ is the vector of nonlinear complex functions. The superscripts r and i represent the real and imaginary parts of the state complex vector, respectively.

Remark 1 System (5) is the generalization of fractional-order chaotic complex systems. In this paper, as an example, the fractional-order complex Lorenz and fractional-order complex Chen systems are considered, and both systems can be described by (5).

Take the fractional-order complex Lorenz system as master system, described by

$$\begin{aligned} D^{\alpha} x_1 &= a_1(x_2 - x_1) \\ D^{\alpha} x_2 &= a_2 x_1 - x_2 - x_1 x_3 \\ D^{\alpha} x_3 &= \frac{1}{2}(\bar{x}_1 x_2 + x_1 \bar{x}_2) - a_3 x_3 \end{aligned} \tag{6}$$

and the fractional-order complex Chen system as slave system, determined by

$$\begin{aligned} D^{\alpha} y_1 &= b_1(y_2 - y_1) \\ D^{\alpha} y_2 &= (b_2 - b_1) y_1 - y_1 y_3 + b_2 y_2 \\ D^{\alpha} y_3 &= \frac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - b_3 y_3 \end{aligned} \tag{7}$$

where $x_1 = u_{1m} + ju_{2m}$, $x_2 = u_{3m} + ju_{4m}$ are complex state variables, and $x_3 = u_{5m}$ is the real state variable of system (6). $y_1 = u_{1s} + ju_{2s}$, $y_2 = u_{3s} + ju_{4s}$ are complex state variables, and $y_3 = u_{5s}$ is the real state variable of system (7). The subscripts m and s denote the master and slave systems, respectively.

Through separating real and imaginary parts of (6) and (7), two five-dimensional continuous real systems can be obtained [25, 26], given by

$$
\begin{aligned}
D^{\alpha}u_{1m} &= a_1(u_{3m}-u_{1m})\\
D^{\alpha}u_{2m} &= a_1(u_{4m}-u_{2m})\\
D^{\alpha}u_{3m} &= a_2u_{1m}-u_{3m}-u_{1m}u_{5m}\\
D^{\alpha}u_{4m} &= a_2u_{2m}-u_{4m}-u_{2m}u_{5m}\\
D^{\alpha}u_{5m} &= u_{1m}u_{3m}+u_{2m}u_{4m}-a_3u_{5m}
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
D^{\alpha}u_{1s} &= b_1(u_{3s}-u_{1s})\\
D^{\alpha}u_{2s} &= b_1(u_{4s}-u_{2s})\\
D^{\alpha}u_{3s} &= (b_2-b_1)u_{1s}-u_{1s}u_{5s}+b_2u_{3s}\\
D^{\alpha}u_{4s} &= (b_2-b_1)u_{2s}-u_{2s}u_{5s}+b_2u_{4s}\\
D^{\alpha}u_{5s} &= u_{1s}u_{3s}+u_{2s}u_{4s}-b_3u_{5s}
\end{aligned} \tag{9}
$$

According to the analysis results in [25] and [26], when $a_1=10$, $a_2=28$, $a_3=8/3$, $b_1=35$, $b_2=28$, $b_3=3$, $\alpha=0.998$, system (8) and (9) behave chaotically, the attractors are shown in Fig. 1.

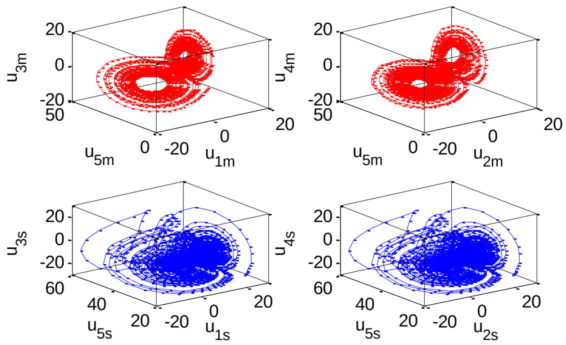


Fig. 1. The 3D projections of chaotic attractors of system (8) and (9)

III. MAIN RESULTS

In this section, a robust adaptive control law and the parameter adaptation rules are designed to achieve the CMPS between two different fractional-order chaotic complex systems with unknown parameters.

According to system (5), the master and slave systems with unknown parameters can be represented as

Master system:

$$D^{\alpha}x = D^{\alpha}x^{r} + jD^{\alpha}x^{i} = F(x)\psi + f(x) \tag{10}$$

Slave system with external disturbances:

$$D^{\alpha}y = D^{\alpha}y^{r} + jD^{\alpha}y^{i} = G(y)\delta + g(y) + d(t) + W(t) \tag{11}$$

where $\psi \in R^{l}$, $\delta \in R^{9}$ are the vectors of unknown parameters in master system and slave system, respectively, $d(t)=(d_1(t), d_2(t), ..., d_n(t))^{T} \in R^{n}$ is the vector of external disturbances, $W(t)=(w_1(t), w_2(t), ..., w_n(t))^{T}$, and $w_k(t)=w_k^{r}+jw_k^{i}$ $(k=1,2,...,n)$ are the controller to be designed later.

Before introducing our approach, we firstly give the definition of CMPS.

Definition 4 Consider the master (10) and slave (11) systems, the error of CMPS is defined as

$$
\begin{aligned}
e &= e^{r}+je^{i} = y-Hx\\
&= y^{r}-H^{r}x^{r}+H^{i}x^{i}+j(y^{i}-H^{r}x^{i}-H^{i}x^{r})
\end{aligned} \tag{12}
$$

where $e^{r}=(e_{u_1}, e_{u_3}, ..., e_{u_{2n-1}})^{T} \in R^{n}$, $e^{i}=(e_{u_2}, e_{u_4}, ..., e_{u_{2n}})^{T} \in R^{n}$, $H=diag(h_1, h_2, ..., h_n) \in C^{n\times n}$ is a complex constant matrix, and $h_k=h_k^{r}+jh_k^{i}$, $(k=1,2,...,n)$. If $e \to 0$ as $t \to \infty$, that is $\lim_{t\to\infty}\|e^{r}\| = \lim_{t\to\infty}\|y^{r}-H^{r}x^{r}+H^{i}x^{i}\|=0$ and $\lim_{t\to\infty}\|e^{i}\| = \lim_{t\to\infty}\|y^{i}-H^{r}x^{i}-H^{i}x^{r}\|=0$, then the CMPS between system (10) and (11) is achieved.

Remark 2 It is obvious that when $H \in R^{n\times n}$, CMPS reduced to MPS, when $h_1=h_2=...=h_n=1$, CS is achieved, when $h_1=h_2=...=h_n=-1$, we can get AS.

Our goal in this paper is to design an adaptive controller and appropriate parametric update laws to achieve CMPS between master system (10) and slave system (11). To the best of the authors' knowledge, so far, there is no information available in this field.

In order to make the proposed method more reasonable and effective, an assumption is necessary.

Assumption 1 In general, it is assumed that the external disturbances $d_k(t) \in R$ are bounded by

$$|d_k(t)| \le \varphi_k,\ k=1,\ 2,\ ...,\ n$$

where φ_k are positive constants.

Next, we will design an adaptive controller to achieve CMPS for two fractional-order chaotic complex systems.

Theorem 1 *Consider the master system (10) and slave system (11), if the controller is designed as*

$$
\begin{aligned}
w_k(t) &= w_k^{r}(t)+jw_k^{i}(t)\\
w_k^{r}(t) &= -G_k^{r}(y)\hat{\delta}+\left(h_k^{r}F_k^{r}(x)-h_k^{i}F_k^{i}(x)\right)\hat{\psi}-g_k^{r}(y)\\
&\quad +h_k^{r}f_k^{r}(x)-h_k^{i}f_k^{i}(x)-\xi_k\,\mathrm{sgn}(e_{u_{2k-1}})\\
w_k^{i}(t) &= -G_k^{i}(y)\hat{\delta}+\left(h_k^{r}F_k^{i}(x)+h_k^{i}F_k^{r}(x)\right)\hat{\psi}-g_k^{i}(y)\\
&\quad +h_k^{r}f_k^{i}(x)+h_k^{i}f_k^{r}(x)-\eta_k\,\mathrm{sgn}(e_{u_{2k}})\\
&\qquad k=1,\ 2,\ ...,\ n
\end{aligned} \tag{13}
$$

where sgn is the sign function, G_k^{r}, G_k^{i}, F_k^{r}, F_k^{i}, g_k^{r}, g_k^{i}, f_k^{r}, f_k^{i} are the ith row vector of G^{r}, G^{i}, F^{r}, F^{i}, g^{r}, g^{i}, f^{r}, f^{i} respectively, h_k^{r}, h_k^{i} are given constants. ξ_k and η_k are control gains, which are updated by

$$
\begin{aligned}
D^{\alpha}\xi_k &= \beta_k|e_{u_{2k-1}}|\\
D^{\alpha}\eta_k &= \sigma_k|e_{u_{2k}}|
\end{aligned} \tag{14}
$$

in which, β_k and σ_k are positive constants.

The parametric update laws are selected as

$$
\begin{aligned}
D^{\alpha}\hat{\psi} &= -\left(H^{r}F^{r}(x)-H^{i}F^{i}(x)\right)^{T}e^{r}-\left(H^{r}F^{i}(x)+H^{i}F^{r}(x)\right)^{T}e^{i}\\
D^{\alpha}\hat{\delta} &= \left(G^{r}(y)\right)^{T}e^{r}+\left(G^{i}(y)\right)^{T}e^{i}
\end{aligned} \tag{15}
$$

Then the CMPS between master system (10) and slave system (11) can be achieved.

Proof According to the definition of CMPS, we obtain

$$
\begin{aligned}
D^\alpha e &= D^\alpha e^r + jD^\alpha e^i = D^\alpha (y - Hx) = D^\alpha y - HD^\alpha x \\
&= G(y)\delta + g(y) + d(t) + W(t) - H(F(x)\psi + f(x)) \\
&= G^r(y)\delta + g^r(y) + d(t) + W^r(t) + j(G^i(y)\delta + g^i(y) + W^i(t)) \\
&\quad - (H^r + jH^i)[F^r(x)\psi + f^r(x) + j(F^i(x)\psi + f^i(x))] \\
&= G^r(y)\delta + g^r(y) + d(t) + W^r(t) - H^r(F^r(x)\psi + f^r(x)) \\
&\quad + H^i(F^i(x)\psi + f^i(x)) + j[G^i(y)\delta + g^i(y) + W^i(t) \\
&\quad - H^r(F^i(x)\psi + f^i(x)) - H^i(F^r(x)\psi + f^r(x))]
\end{aligned} \tag{16}
$$

Namely, the real and imaginary parts of complex synchronization error system (16) can be equivalently converted to the following real error systems

$$
\begin{aligned}
D^\alpha e_{u_{2k-1}} &= G_k^r(y)\delta + g_k^r(y) + d_k(t) + w_k^r(t) - h_k^r(F_k^r(x)\psi + f_k^r(x)) \\
&\quad + h_k^i(F_k^i(x)\psi + f_k^i(x)) \\
D^\alpha e_{u_{2k}} &= G_k^i(y)\delta + g_k^i(y) + w_k^i(t) - h_k^r(F_k^i(x)\psi + f_k^i(x)) \\
&\quad - h_k^i(F_k^r(x)\psi + f_k^r(x))
\end{aligned} \tag{17}
$$

where $k = 1, 2, ..., n$, $H = diag(h_1, h_2, ..., h_n)$, $h_k = h_k^r + jh_k^i$. Denote $\tilde{\psi} = \hat{\psi} - \psi$, $\tilde{\delta} = \hat{\delta} - \delta$ ($\hat{\psi}$ and $\hat{\delta}$ are the estimation of ψ and δ, respectively). Further denote $X^T = (E^T, \tilde{\psi}^T, \tilde{\delta}^T, \tilde{\xi}^T, \tilde{\eta}^T) \in R^{1\times(4n+l+\vartheta)}$, where

$$E^T = ((e^r)^T, (e^i)^T) \in R^{1\times 2n}$$

$$(e^r)^T = (e_{u_1}, e_{u_3}, ..., e_{u_{2n-1}}) \in R^{1\times n}$$

$$(e^i)^T = (e_{u_2}, e_{u_4}, ..., e_{u_{2n}}) \in R^{1\times n}$$

$$\tilde{\psi}^T \in R^{1\times l},\ \tilde{\delta}^T \in R^{1\times\vartheta}$$

$$\tilde{\xi}^T = (\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in R^{1\times n},\ \tilde{\xi}_k = \xi_k - \xi_k^*$$

$$\tilde{\eta}^T = (\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n) \in R^{1\times n},\ \tilde{\eta}_k = \eta_k - \eta_k^*$$

$$\xi_k^* \ge \varphi_k + 1$$

$$\eta_k^* \ge 1$$

Choose a real symmetric positive finite matrix P in the form of

$$P = diag\left(I_{2n+l+\vartheta}, \frac{1}{\beta_1}, \frac{1}{\beta_2}, ..., \frac{1}{\beta_n}, \frac{1}{\sigma_1}, \frac{1}{\sigma_2}, ..., \frac{1}{\sigma_n}\right)$$

Then we can construct a function to prove the stability of the closed-loop system, written as

$$J = X^T P D^\alpha X$$

That is

$$
\begin{aligned}
J = X^T P D^\alpha X &= (e^r)^T D^\alpha e^r + (e^i)^T D^\alpha e^i + \tilde{\psi}^T D^\alpha \tilde{\psi} + \tilde{\delta}^T D^\alpha \tilde{\delta} \\
&\quad + \sum_{k=1}^n \frac{1}{\beta_k} \tilde{\xi}_k D^\alpha \tilde{\xi}_k + \sum_{k=1}^n \frac{1}{\sigma_k} \tilde{\eta}_k D^\alpha \tilde{\eta}_k \\
&= \sum_{k=1}^n e_{u_{2k-1}} D^\alpha e_{u_{2k-1}} + \sum_{k=1}^n e_{u_{2k}} D^\alpha e_{u_{2k}} + \tilde{\psi}^T D^\alpha \hat{\psi} + \tilde{\delta}^T D^\alpha \hat{\delta} \\
&\quad + \sum_{k=1}^n \frac{1}{\beta_k} \tilde{\xi}_k D^\alpha \xi_k + \sum_{k=1}^n \frac{1}{\sigma_k} \tilde{\eta}_k D^\alpha \eta_k
\end{aligned} \tag{18}
$$

Inserting (17) into (18), it yields

$$
\begin{aligned}
J &= \sum_{k=1}^n e_{u_{2k-1}} \left[G_k^r(y)\delta + g_k^r(y) + d_k(t) + w_k^r(t) - h_k^r(F_k^r(x)\psi + f_k^r(x)) + h_k^i(F_k^i(x)\psi + f_k^i(x))\right] \\
&\quad + \sum_{k=1}^n e_{u_{2k}} \left[G_k^i(y)\delta + g_k^i(y) + w_k^i(t) - h_k^r(F_k^i(x)\psi + f_k^i(x)) - h_k^i(F_k^r(x)\psi + f_k^r(x))\right] \\
&\quad + \tilde{\psi}^T D^\alpha \hat{\psi} + \tilde{\delta}^T D^\alpha \hat{\delta} + \sum_{k=1}^n \frac{1}{\beta_k} \tilde{\xi}_k D^\alpha \xi_k + \sum_{k=1}^n \frac{1}{\sigma_k} \tilde{\eta}_k D^\alpha \eta_k
\end{aligned} \tag{19}
$$

Substituting the control law (13) into (19), we obtain

$$
\begin{aligned}
J &= \sum_{k=1}^n e_{u_{2k-1}} \left[G_k^r(y)(\delta - \hat{\delta}) + d_k(t) - (h_k^r F_k^r(x) - h_k^i F_k^i(x))(\psi - \hat{\psi}) - \xi_k \operatorname{sgn}(e_{u_{2k-1}})\right] \\
&\quad + \sum_{k=1}^n e_{u_{2k}} \left[G_k^i(y)(\delta - \hat{\delta}) - (h_k^r F_k^i(x) + h_k^i F_k^r(x))(\psi - \hat{\psi}) - \eta_k \operatorname{sgn}(e_{u_{2k}})\right] \\
&\quad + \tilde{\psi}^T D^\alpha \hat{\psi} + \tilde{\delta}^T D^\alpha \hat{\delta} + \sum_{k=1}^n \frac{1}{\beta_k}(\xi_k - \xi_k^*) D^\alpha \xi_k + \sum_{k=1}^n \frac{1}{\sigma_k}(\eta_k - \eta_k^*) D^\alpha \eta_k
\end{aligned} \tag{20}
$$

On the basis of the parametric update laws (15), we can derivate that

$$
\begin{aligned}
&\sum_{k=1}^n e_{u_{2k-1}} G_k^r(y)(\delta - \hat{\delta}) + \sum_{k=1}^n e_{u_{2k}} G_k^i(y)(\delta - \hat{\delta}) \\
&= (e^r)^T \left[G^r(y)(\delta - \hat{\delta})\right] + (e^i)^T \left[G^i(y)(\delta - \hat{\delta})\right] \\
&= (\delta - \hat{\delta})^T (G^r(y))^T e^r + (\delta - \hat{\delta})^T (G^i(y))^T e^i \\
&= -\tilde{\delta}^T D^\alpha \hat{\delta}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^n e_{u_{2k-1}} \left[-(h_k^r F_k^r(x) - h_k^i F_k^i(x))(\psi - \hat{\psi})\right] + \sum_{k=1}^n e_{u_{2k}} \left[-(h_k^r F_k^i(x) + h_k^i F_k^r(x))(\psi - \hat{\psi})\right] \\
&= (e^r)^T \left[-(H^r F^r(x) - H^i F^i(x))(\psi - \hat{\psi})\right] + (e^i)^T \left[-(H^r F^i(x) + H^i F^r(x))(\psi - \hat{\psi})\right] \\
&= (\psi - \hat{\psi})^T \left[-(H^r F^r(x) - H^i F^i(x))\right]^T e^r + (\psi - \hat{\psi})^T \left[-(H^r F^i(x) + H^i F^r(x))\right]^T e^i \\
&= (\psi - \hat{\psi})^T \left[-(H^r F^r(x) - H^i F^i(x))^T e^r - (H^r F^i(x) + H^i F^r(x))^T e^i\right] \\
&= -\tilde{\psi}^T D^\alpha \hat{\psi}
\end{aligned}
$$

So, substituting the adaptive laws (15) into (20), we obtain

$$
\begin{aligned}
J &= \sum_{k=1}^n e_{u_{2k-1}} (d_k(t) - \xi_k \operatorname{sgn}(e_{u_{2k-1}})) + \sum_{k=1}^n e_{u_{2k}} (-\eta_k \operatorname{sgn}(e_{u_{2k}})) + \sum_{k=1}^n \frac{1}{\beta_k}(\xi_k - \xi_k^*) D^\alpha \xi_k + \sum_{k=1}^n \frac{1}{\sigma_k}(\eta_k - \eta_k^*) D^\alpha \eta_k \\
&\le \sum_{k=1}^n |d_k(t)| |e_{u_{2k-1}}| - \sum_{k=1}^n \xi_k |e_{u_{2k-1}}| - \sum_{k=1}^n \eta_k |e_{u_{2k}}| + \sum_{k=1}^n \frac{1}{\beta_k}(\xi_k - \xi_k^*) D^\alpha \xi_k + \sum_{k=1}^n \frac{1}{\sigma_k}(\eta_k - \eta_k^*) D^\alpha \eta_k
\end{aligned} \tag{21}
$$

Inserting the update rules (14) of ξ_k, η_k into (21), then according to Assumption 1, we have

$$
\begin{aligned}
J &\le \sum_{k=1}^n \varphi_k |e_{u_{2k-1}}| - \sum_{k=1}^n \xi_k |e_{u_{2k-1}}| - \sum_{k=1}^n \eta_k |e_{u_{2k}}| + \sum_{k=1}^n (\xi_k - \xi_k^*) |e_{u_{2k-1}}| + \sum_{k=1}^n (\eta_k - \eta_k^*) |e_{u_{2k}}| \\
&= \sum_{k=1}^n (\varphi_k - \xi_k^*) |e_{u_{2k-1}}| - \sum_{k=1}^n \eta_k^* |e_{u_{2k}}|
\end{aligned} \tag{22}
$$

Since $\xi_k^* \ge \phi_k + 1$, $\eta_k^* \ge 1$, so (22) can be further derived as

$$J \le \sum_{k=1}^n (\varphi_k - \varphi_k - 1) |e_{u_{2k-1}}| - \sum_{k=1}^n |e_{u_{2k}}| = -\sum_{k=1}^n |e_{u_{2k-1}}| - \sum_{k=1}^n |e_{u_{2k}}| = -(\| e^r \| + \| e^i \|) < 0$$

Owing to $J < 0$, according to Lemma 1, the error system (17) is asymptotically stable. Consequently, the master system (10) and slave system (11) can be synchronized up to a complex matrix. Therefore, the proof is completed.

Remark 3 Apparently, the theoretical results in Refs. [19, 31] are the special cases of our scheme. It is should be noted that in practical applications, the feedback control gain is desired as small as possible, however, the theoretical feedback control gains in [19, 31] are fixed values no matter where the initial values start, thus the gains must be larger than the values needed, which means a kind of waste in practice. In our method, we use an adaptive controller to overcome the above drawbacks. The control gains ξ_k, η_k can be automatically adapted to the suitable values, which can simplify the task of design and reduce the cost of control.

If ψ and δ are two unknown complex parameter vectors, the ψ and δ can be rewritten as $\psi = \psi^r + j\psi^i$, $\delta = \delta^r + j\delta^i$. Then master system (10) and slave system (11) become

$$D^\alpha x = D^\alpha x^r + jD^\alpha x^i = F(x)(\psi^r + j\psi^i) + f(x) \tag{23}$$

and

$$D^{\alpha}y = D^{\alpha}y^{r} + jD^{\alpha}y^{i} = G(y)(\delta^{r} + j\delta^{i}) + g(y) + d(t) + W(t) \quad (24)$$

According to the definition of CMPS, we can obtain the synchronization error system

$$\begin{aligned} D^{\alpha}e &= D^{\alpha}e^{r} + jD^{\alpha}e^{i} = D^{\alpha}y^{r} - H^{r}D^{\alpha}x^{r} + H^{i}D^{\alpha}x^{i} + j(D^{\alpha}y^{i} - H^{r}D^{\alpha}x^{i} - H^{i}D^{\alpha}x^{r}) \\ &= G^{r}(y)\delta^{r} - G^{i}(y)\delta^{i} + g^{r}(y) + d(t) + W^{r}(t) - H^{r}(F^{r}(x)\psi^{r} - F^{i}(x)\psi^{i} + f^{r}(x)) \\ &\quad + H^{i}(F^{r}(x)\psi^{i} + F^{i}(x)\psi^{r} + f^{i}(x)) + j[G^{r}(y)\delta^{i} + G^{i}(y)\delta^{r} + g^{i}(y) + W^{i}(t) \\ &\quad - H^{r}(F^{r}(x)\psi^{i} + F^{i}(x)\psi^{r} + f^{i}(x)) - H^{i}(F^{r}(x)\psi^{r} - F^{i}(x)\psi^{i} + f^{r}(x))] \end{aligned} \quad (25)$$

That is, the real and imaginary parts of (25) can be written as

$$\begin{aligned} D^{\alpha}e_{u_{2k-1}} &= G_{k}^{r}(y)\delta^{r} - G_{k}^{i}(y)\delta^{i} + g_{k}^{r}(y) + d_{k}(t) + w_{k}^{r}(t) - h_{k}^{r}\left(F_{k}^{r}(x)\psi^{r} - F_{k}^{i}(x)\psi^{i} + f_{k}^{r}(x)\right) \\ &\quad + h_{k}^{i}\left(F_{k}^{r}(x)\psi^{i} + F_{k}^{i}(x)\psi^{r} + f_{k}^{i}(x)\right) \\ D^{\alpha}e_{u_{2k}} &= G_{k}^{r}(y)\delta^{i} + G_{k}^{i}(y)\delta^{r} + g_{k}^{i}(y) + w_{k}^{i}(t) - h_{k}^{r}\left(F_{k}^{r}(x)\psi^{i} + F_{k}^{i}(x)\psi^{r} + f_{k}^{i}(x)\right) \\ &\quad - h_{k}^{i}\left(F_{k}^{r}(x)\psi^{r} - F_{k}^{i}(x)\psi^{i} + f_{k}^{r}(x)\right) \\ & k = 1,\, 2,\, \ldots,\, n \end{aligned} \quad (26)$$

Similarly, for stabilizing the error system (26), we give the following theorem.

Theorem 2 *Consider the master system (23) and slave system (24) with unknown complex parameters, if the controller is designed as*

$$\begin{aligned} w_{k}(t) &= w_{k}^{r}(t) + jw_{k}^{i}(t) \\ w_{k}^{r}(t) &= -G_{k}^{r}(y)\hat{\delta}^{r} + G_{k}^{i}(y)\hat{\delta}^{i} + \left(h_{k}^{r}F_{k}^{r}(x) - h_{k}^{i}F_{k}^{i}(x)\right)\hat{\psi}^{r} - \left(h_{k}^{r}F_{k}^{i}(x) + h_{k}^{i}F_{k}^{r}(x)\right)\hat{\psi}^{i} \\ &\quad - g_{k}^{r}(y) + h_{k}^{r}f_{k}^{r}(x) - h_{k}^{i}f_{k}^{i}(x) - \xi_{k}\,\mathrm{sgn}(e_{u_{2k-1}}) \\ w_{k}^{i}(t) &= -G_{k}^{r}(y)\hat{\delta}^{i} - G_{k}^{i}(y)\hat{\delta}^{r} + \left(h_{k}^{r}F_{k}^{i}(x) + h_{k}^{i}F_{k}^{r}(x)\right)\hat{\psi}^{r} + \left(h_{k}^{r}F_{k}^{r}(x) - h_{k}^{i}F_{k}^{i}(x)\right)\hat{\psi}^{i} \\ &\quad - g_{k}^{i}(y) + h_{k}^{r}f_{k}^{i}(x) + h_{k}^{i}f_{k}^{r}(x) - \eta_{k}\,\mathrm{sgn}(e_{u_{2k}}) \end{aligned} \quad (27)$$

where $k = 1, 2, \ldots, n$. ξ_k *and* η_k *are control gains, which are updated by*

$$D^{\alpha}\xi_{k} = \beta_{k}\,|e_{u_{2k-1}}|$$
$$D^{\alpha}\eta_{k} = \sigma_{k}\,|e_{u_{2k}}|$$

where β_k *and* σ_k *are two positive constants.*

The parametric update laws are selected as

$$\begin{aligned} D^{\alpha}\hat{\psi}^{r} &= -\left(H^{r}F^{r}(x) - H^{i}F^{i}(x)\right)^{T}e^{r} - \left(H^{r}F^{i}(x) + H^{i}F^{r}(x)\right)^{T}e^{i} \\ D^{\alpha}\hat{\psi}^{i} &= \left(H^{r}F^{i}(x) + H^{i}F^{r}(x)\right)^{T}e^{r} - \left(H^{r}F^{r}(x) - H^{i}F^{i}(x)\right)^{T}e^{i} \\ D^{\alpha}\hat{\delta}^{r} &= \left(G^{r}(y)\right)^{T}e^{r} + \left(G^{i}(y)\right)^{T}e^{i} \\ D^{\alpha}\hat{\delta}^{i} &= \left(-G^{i}(y)\right)^{T}e^{r} + \left(G^{r}(y)\right)^{T}e^{i} \end{aligned} \quad (28)$$

Then the CMPS between master system (23) and slave system (24) can be achieved.

Proof It is similar to that of Theorem 1. Limited by the length of this paper, the proof is omitted here.

Remark 4 It is not hard to see that the synchronization schemes in [37-39] are also special cases of the proposed scheme. Therefore the proposed method is applicable to achieve the CMPS of two integer-order chaotic complex systems with unknown complex parameters.

In this paper, for simplicity, we take the fractional-order chaotic complex systems with unknown real parameters as example to verify the effectiveness of the proposed synchronization scheme.

IV. SIMULATION RESULTS

In this section, we investigate numerical simulation for the CMPS of two different fractional-order chaotic complex systems. Let us take the fractional-order complex Lorenz system as master system, and the fractional-order complex Chen system as slave system.

From the generalized form of fractional-order chaotic complex systems (10) and (11), we get

$$F(x) = \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{pmatrix} = \begin{pmatrix} u_{3m} - u_{1m} & 0 & 0 \\ 0 & u_{1m} & 0 \\ 0 & 0 & -u_{5m} \end{pmatrix} + j\begin{pmatrix} u_{4m} - u_{2m} & 0 & 0 \\ 0 & u_{2m} & 0 \\ 0 & 0 & 0 \end{pmatrix} = F^{r}(x) + jF^{i}(x)$$

$$G(y) = \begin{pmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & y_1 + y_2 & 0 \\ 0 & 0 & -y_3 \end{pmatrix} = \begin{pmatrix} u_{3s} - u_{1s} & 0 & 0 \\ -u_{1s} & u_{1s} + u_{3s} & 0 \\ 0 & 0 & -u_{5s} \end{pmatrix} + j\begin{pmatrix} u_{4s} - u_{2s} & 0 & 0 \\ -u_{2s} & u_{2s} + u_{4s} & 0 \\ 0 & 0 & 0 \end{pmatrix} = G^{r}(y) + jG^{i}(y)$$

$$f(x) = \begin{pmatrix} 0 \\ -x_2 - x_1x_3 \\ \frac{1}{2}(\bar{x}_1x_2 + x_1\bar{x}_2) \end{pmatrix} = \begin{pmatrix} 0 \\ -u_{3m} - u_{1m}u_{5m} \\ u_{1m}u_{3m} + u_{2m}u_{4m} \end{pmatrix} + j\begin{pmatrix} 0 \\ -u_{4m} - u_{2m}u_{5m} \\ 0 \end{pmatrix} = f^{r}(x) + jf^{i}(x)$$

$$g(y) = \begin{pmatrix} 0 \\ -y_1y_3 \\ \frac{1}{2}(\bar{y}_1y_2 + y_1\bar{y}_2) \end{pmatrix} = \begin{pmatrix} 0 \\ -u_{1s}u_{5s} \\ u_{1s}u_{3s} + u_{2s}u_{4s} \end{pmatrix} + j\begin{pmatrix} 0 \\ -u_{2s}u_{5s} \\ 0 \end{pmatrix} = g^{r}(y) + jg^{i}(y)$$

$$\psi = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \delta = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

In this study, disturbances $d_k(t) = 0.5\cos(t)$, $k = 1, 2, 3$ are attached to the slave system. The complex matrix $H = diag(h_1, h_2, h_3)$, and $h_1 = h_1^r + jh_1^i$, $h_2 = h_2^r + jh_2^i$ are two complex constants, h_3 is a real constant.

The adaptive controller designed by Theorem 1 as

$$\begin{aligned} w_1^r &= -\hat{b}_1(u_{3s} - u_{1s}) + \hat{a}_1[h_1^r(u_{3m} - u_{1m}) - h_1^i(u_{4m} - u_{2m})] - \xi_1\,\mathrm{sgn}(e_{u_1}) \\ w_1^i &= -\hat{b}_1(u_{4s} - u_{2s}) + \hat{a}_1[h_1^r(u_{4m} - u_{2m}) + h_1^i(u_{3m} - u_{1m})] - \eta_1\,\mathrm{sgn}(e_{u_2}) \\ w_2^r &= \hat{b}_1u_{1s} - \hat{b}_2(u_{1s} + u_{3s}) + \hat{a}_2(h_2^ru_{1m} - h_2^iu_{2m}) + u_{1s}u_{5s} \\ &\quad + h_2^r(-u_{3m} - u_{1m}u_{5m}) - h_2^i(-u_{4m} - u_{2m}u_{5m}) - \xi_2\,\mathrm{sgn}(e_{u_3}) \\ w_2^i &= \hat{b}_1u_{2s} - \hat{b}_2(u_{2s} + u_{4s}) + \hat{a}_2(h_2^ru_{2m} + h_2^iu_{1m}) + u_{2s}u_{5s} \\ &\quad + h_2^r(-u_{4m} - u_{2m}u_{5m}) + h_2^i(-u_{3m} - u_{1m}u_{5m}) - \eta_2\,\mathrm{sgn}(e_{u_4}) \\ w_3^r &= \hat{b}_3u_{5s} - \hat{a}_3h_3u_{5m} - (u_{1s}u_{3s} + u_{2s}u_{4s}) + h_3(u_{1m}u_{3m} + u_{2m}u_{4m}) - \xi_3\,\mathrm{sgn}(e_{u_5}) \\ w_3^i &= 0 \end{aligned} \quad (29)$$

where

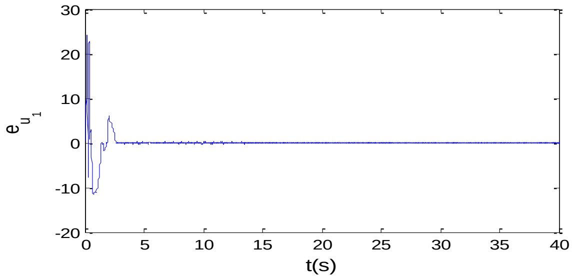
$$\begin{aligned} D^{\alpha}\xi_1 &= \beta_1\,|e_{u_1}| \\ D^{\alpha}\xi_2 &= \beta_2\,|e_{u_3}| \\ D^{\alpha}\xi_3 &= \beta_3\,|e_{u_5}| \\ D^{\alpha}\eta_1 &= \sigma_1\,|e_{u_2}| \\ D^{\alpha}\eta_2 &= \sigma_2\,|e_{u_4}| \\ D^{\alpha}\eta_3 &= 0 \end{aligned}$$

On the basis of (15), we can also obtain the parametric update laws

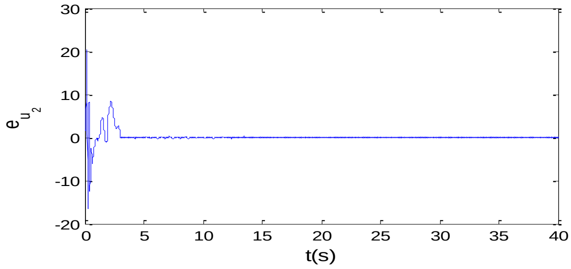
$$\begin{aligned} D^{\alpha}\hat{a}_1 &= -\left(h_1^r(u_{3m} - u_{1m}) - h_1^i(u_{4m} - u_{2m})\right)e_{u_1} - \left(h_1^r(u_{4m} - u_{2m}) + h_1^i(u_{3m} - u_{1m})\right)e_{u_2} \\ D^{\alpha}\hat{a}_2 &= -(h_2^ru_{1m} - h_2^iu_{2m})e_{u_3} - (h_2^ru_{2m} + h_2^iu_{1m})e_{u_4} \\ D^{\alpha}\hat{a}_3 &= h_3u_{5m}e_{u_5} \\ D^{\alpha}\hat{b}_1 &= (u_{3s} - u_{1s})e_{u_1} - u_{1s}e_{u_3} + (u_{4s} - u_{2s})e_{u_2} - u_{2s}e_{u_4} \\ D^{\alpha}\hat{b}_2 &= (u_{1s} + u_{3s})e_{u_3} + (u_{2s} + u_{4s})e_{u_4} \\ D^{\alpha}\hat{b}_3 &= -u_{5s}e_{u_5} \end{aligned} \quad (30)$$

In this simulation, let $\alpha = 0.998$, $H = diag(1, j, -1)$, the unknown parameters are $(a_1, a_2, a_3)^T = (10, 28, 8/3)^T$,

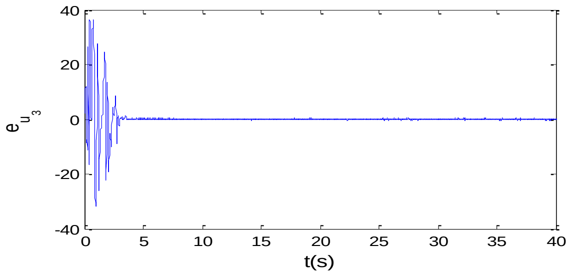
$(b_1, b_2, b_3)^T = (35, 28, 3)^T$, the initial values are randomly chosen as $x(0) = (1-j, -2+2j, 3)^T$, $y(0) = (1+j, 1+j, 1)^T$, $\hat{\psi}(0) = (0, 0, 0)^T$, $\hat{\delta}(0) = (0, 0, 0)^T$, $\xi(0) = (0, 0, 0)^T$, $\eta(0) = (0, 0, 0)^T$. The positive constants are chosen as $\beta_1 = \beta_2 = \beta_3 = \sigma_1 = \sigma_2 = 0.5$. The simulation results are depicted in Figs. 2-4. Fig. 2 shows the time evolution of synchronization errors between fractional-order complex Lorenz system and fractional-order complex Chen system.



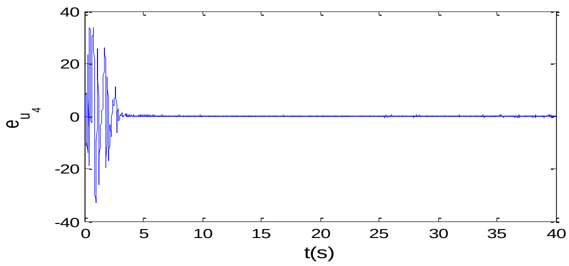
(a) e_{u_1}



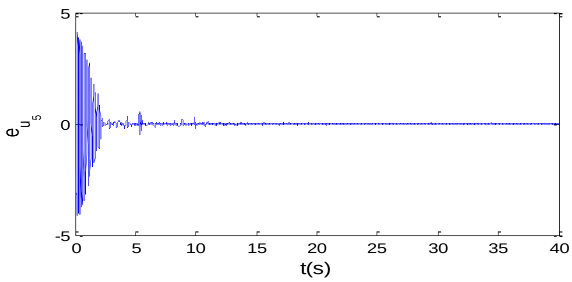
(b) e_{u_2}



(c) e_{u_3}



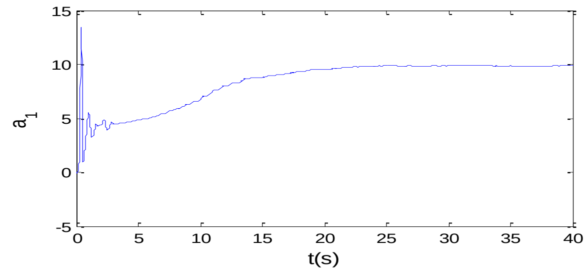
(d) e_{u_4}



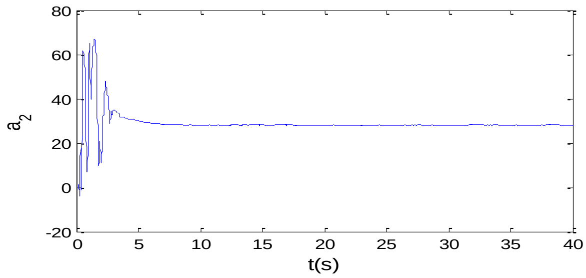
(e) e_{u5}

Fig. 2. Time response of synchronization error system (17)

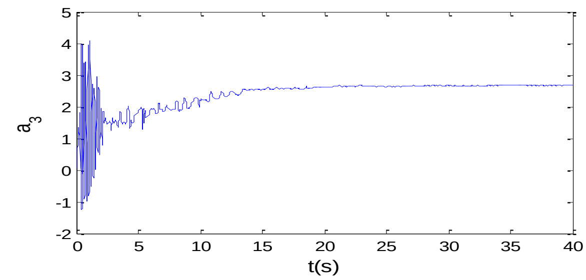
Obviously, the synchronization errors e_{u_i} ($i = 1, 2, 3, 4, 5$) converge to zero asymptotically, which illustrates that the CMPS between fractional-order complex Lorenz system and fractional-order complex Chen system is achieved. Fig. 3 displays the time evolution of estimate parameters. It is not hard to see all unknown parameters gradually converge to their actual values, which implies that the proposed parametric update laws are applicable.



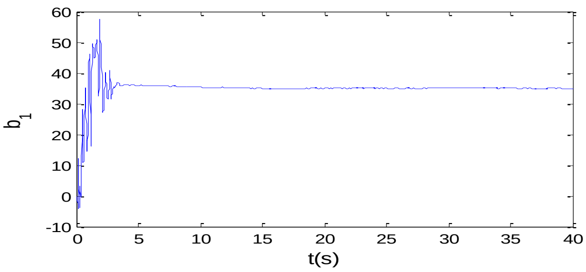
(a) a_1



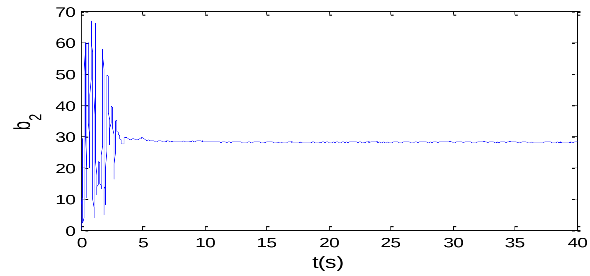
(b) a_2



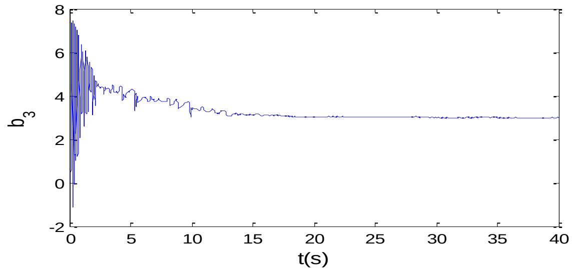
(c) a_3



(d) b_1



(e) b_2



(f) b_3

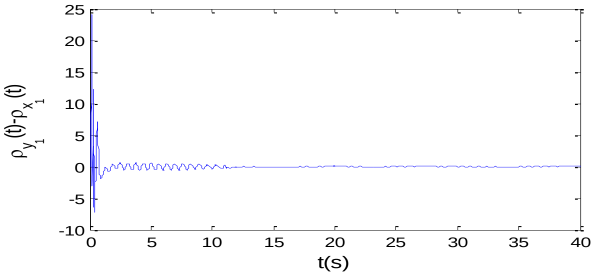
Fig. 3. Time evolution of estimate parameters for master and slave systems

Further, we can calculate the module errors and phases of master system and slave system, respectively. For any complex number $x = x^r + jx^i \in C$, the module ρ_x and phase θ_x are calculated as follows [40]

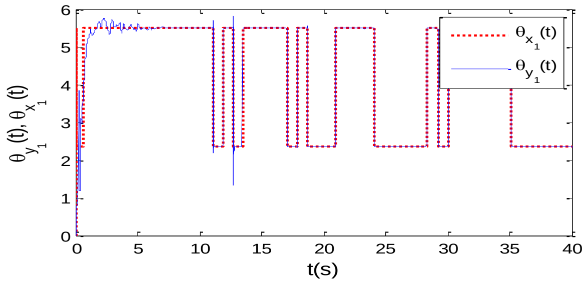
$$\rho_x = \sqrt{(x^r)^2 + (x^i)^2}$$

$$\theta_x = \begin{cases} \arctan(x^i / x^r), & x^r > 0, \ x^i \geq 0, \\ 2\pi + \arctan(x^i / x^r), & x^r > 0, \ x^i < 0, \\ \pi + \arctan(x^i / x^r), & x^r < 0, \\ \pi - \arctan(x^i / x^r), & x^r = 0. \end{cases}$$

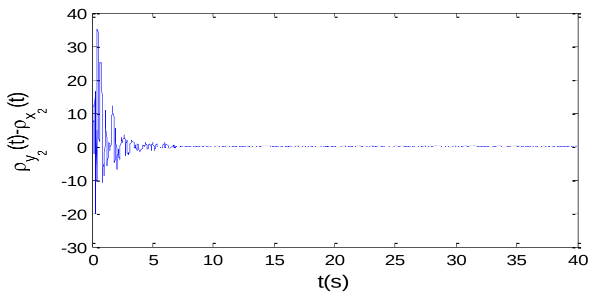
Since we are merely interested in the complex variables, thus the results of module errors and phases for state complex variables in master and slave systems are illustrated in Fig. 4.



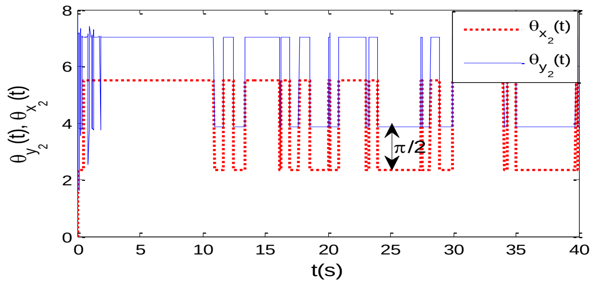
(a) $\rho_{y_1}(t) - \rho_{x_1}(t)$



(b) $\theta_{y_1}(t), \theta_{x_1}(t)$



(c) $\rho_{y_2}(t) - \rho_{x_2}(t)$



(d) $\theta_{y_2}(t), \theta_{x_2}(t)$

Fig. 4. The module errors and phases of master and slave systems

All simulation results demonstrate that the adaptive CMPS between two nonidentical fractional-order chaotic complex systems has been achieved, and all unknown parameters of master and slave systems are fully identified.

Remark 5 In the simulation, the random choice of complex matrix H will not affect the theoretic results.

V. Conclusions

In this paper, a novel synchronization scheme called complex modified projective synchronization (CMPS) is introduced, and the CMPS between two nonidentical fractional-order chaotic complex systems with unknown parameters is firstly investigated. Moreover, the slave system is assumed to be perturbed by bounded external disturbances. It is worth noting that CMPS is the generalization of several types of synchronization, such as CS, PS, MPS, AS, etc. Thus, the research about CMPS is meaningful and essential. A new stability theory is applied to prove the asymptotic stability of fractional-order error system. Simulation results are provided to verify the effectiveness and correctness of the proposed method. Because our results contain and extend most existing works, we believe that there are high potential in the proposed method.

Acknowledgement

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