Permanence of a Delayed Biological System with Stage Structure and Density-dependent Juvenile Birth Rate

Qin Yue

Abstract—**A delayed biological system with stage structure and density-dependent juvenile birth rate is revisited in this paper. By establishing a new lemma and using the comparison theorem of the differential equation, a new set of sufficient condition which ensure the permanence of the system is obtained. Our result supplements and complements some known results.**

Index Terms—**Stage structure, Predator-prey, Permanence**

I. Introduction

THE dynamic behaviors of the stage structured ecosystem has recently studied by many scholars, see [1]-[22] and the references cited therein. The most simple singe species stage structure model takes the form:

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1,\\ \frac{dx_2}{dt} &= \beta x_1 - \gamma x_2^2,\end{aligned} \tag{1.1}$$

where $\alpha, \beta, \delta_1, \delta_2$ and γ are all positive constants, $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature species at time t. The per capita birth rate of the immature population is $\alpha > 0$; The per capita death rate of the immature population is $\delta_1 > 0$; $\beta > 0$ denotes the surviving rate of immaturity to reach maturity; The mature species is density dependent with the parameter $\gamma > 0$. Cui, Chen and Wang[31] had showed that above system admits a unique positive equilibrium, which is globally asymptotically stable, which means that the dynamic behaviors of the system (1.1) is similarly to the traditional Logistic model.

Xiao and Lei[21] argued that a suitable model should considered the influence of the harvesting, and they proposed the following single species stage structure system incorporating partial closure for the populations and non-selective harvesting:

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1 - q_1 E m x_1,\\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - q_2 E m x_2,\end{aligned} \tag{1.2}$$

where all the other coefficients has the same meaning as the system, and δ_2 represents the per capita death rate of the mature population, E is the combined fishing effort used to harvest and $m (0 < m < 1)$ is the fraction of the stock available for harvesting. They showed that the birth rate of the immature species and the fraction of the stocks for the harvesting plays crucial role on the dynamic behaviors of the system.

In some stage-structured populations, the intraspecific and interspecific competitions occur within each stage structure. In two-stage single-species population, Abrams and Quince[33] have demonstrated that adult population competition makes a low birth rate of juvenile population. They proposed the following single species stage structured model:

$$\begin{aligned}\frac{dN_1}{dt} &= B_2(1-\alpha_2 N_2)N_2 - d_1 N_1 - G(1-\alpha_1 N_1)N_1,\\ \frac{dN_2}{dt} &= G(1-\alpha_1 N_1)N_1 - d_2 N_2.\end{aligned} \tag{1.3}$$

For a range of parameter values, the authors declared that the model (1.3) possesses two locally stable equilibria. Hence, compared with system (1.1), by introducing the nonlinear birth rate term, the system admits the quite different dynamics behaviors. Obviously, in this case, it is impossible for the system to admits a unique globally asymptotically stable positive equilibrium.

Based on model (1.1), many scholars invested the dynamic behaviors of the stage structured predator prey model. For example, Yang, Li and Bai[20] proposed the following model:

$$\begin{aligned}\frac{dx_1}{dt} &= a(t)x_2(t) - b(t)x_1(t)\\ &\quad -d_1(t)(x_1(t))^2 - \frac{c_1(t)x_1(t)y(t)}{m(t)+x_1^2(t)},\\ \frac{dx_2}{dt} &= b(t)x_1(t) - d_2(t)x_2^2(t),\\ \frac{dy}{dt} &= y(t)\Big(-d_3(t) + \frac{c_2(t)x_1(t-\tau)}{m(t)+x_1^2(t-\tau)}\\ &\quad -q(t)y(t)\Big).\end{aligned} \tag{1.4}$$

Sufficient and necessary conditions are obtained for the permanence of the system.

We mention here that in the study of biomathematics, such topics as the extinction, persistent and stability of the ecosystem are the most important study area, and they were extensively studied by many scholars, see [1]-[40] and the references cited therein.

II. Model

Recently, stimulated by the work of Yang, Li and Bai[20] and Abrams and Quince[33], Zhang and Zhang[22] proposed

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the following delayed biological system with stage structure and density-dependent juvenile birth rate

$$\begin{aligned}\frac{dx_1}{dt} &= a(t)\big(1-\beta(t)x_2(t)\big)x_2(t)-b(t)x_1(t)\\ &\quad -d_1(t)(x_1(t))^2-\frac{c_1(t)x_1(t)y(t)}{m(t)+x_1^2(t)},\\ \frac{dx_2}{dt} &= b(t)x_1(t)-d_2(t)x_2^2(t),\\ \frac{dy}{dt} &= y(t)\Big(-d_3(t)+\frac{c_2(t)x_1(t-\tau)}{m(t)+x_1^2(t-\tau)}\\ &\quad -q(t)y(t)\Big),\end{aligned} \tag{2.1}$$

where $x_1(t), x_2(t)$ and $y(t)$ represent the density of immature prey, mature prey and predator species, respectively. The coefficients in system (2.1) are all continuous positive T periodic functions. The parameter $\beta(t)$ is the proportional rate of decrease in per capita births with increased mature prey density and takes a value between 0 and 1. For more background of system (2.1), one could refer to [22].

Concerned with the persistent property of the system (2.1), the authors obtained the following result.

Theorem A.*System (2.1) is uniformly persistent and has at least one T periodic solution provided that*

$$A_T\Big(-d_3(t)+\frac{c_2(t)x_1^*(t-\tau)}{m(t)+(x_1^*(t-\tau))^2}\Big)>0, \tag{2.2}$$

where $(x_1^*(t), x_2^*(t))$ is the unique positive periodic solution of the following system

$$\begin{aligned}\frac{dx_1}{dt} &= a(t)x_2(t)-b(t)x_1(t)-d_1(t)(x_1(t))^2,\\ \frac{dx_2}{dt} &= b(t)x_1(t)-d_2(t)x_2^2(t).\end{aligned} \tag{2.3}$$

Now let's consider the following example.

Example 2.1. Consider the following system

$$\begin{aligned}\frac{dx_1}{dt} &= (1-0.6x_2(t))x_2(t)-x_1(t)\\ &\quad -(x_1(t))^2-\frac{2x_1(t)y(t)}{1+x_1^2(t)},\\ \frac{dx_2}{dt} &= x_1(t)-\frac{1}{4}x_2^2(t),\\ \frac{dy}{dt} &= y(t)\Big(-\frac{1}{3}+\frac{1}{10}\cos(t)\\ &\quad +\frac{x_1(t)}{1+x_1^2(t)}-y(t)\Big).\end{aligned} \tag{2.4}$$

Here, we assume that $a(t)=b(t)=d_1(t)=m(t)=c_2(t)=1, c_1(t)=2, d_3(t)=\frac{1}{3}, \beta(t)=0.6, d_2(t)=\frac{1}{4}$. Then

$$\begin{aligned}\frac{dx_1}{dt} &= x_2(t)-x_1(t)-(x_1(t))^2,\\ \frac{dx_2}{dt} &= x_1(t)-\frac{1}{4}x_2^2(t)\end{aligned} \tag{2.5}$$

has a unique positive equilibrium $E(1,2)$, which is globally asymptotically stable, and

$$\begin{aligned}&A_{2\pi}\Big(-d_3(t)+\frac{c_2(t)x_1^*(t-\tau)}{m(t)+(x_1^*(t-\tau))^2}\Big)\\ &= -\frac{1}{3}+\frac{1}{1+1}=\frac{1}{6}>0.\end{aligned} \tag{2.6}$$

That is, the condition of Theorem A holds, however, numeric simulation (Fig. 1) shows that in this case, the predator species will be driven to extinction.

Above example shows that although the conditions of

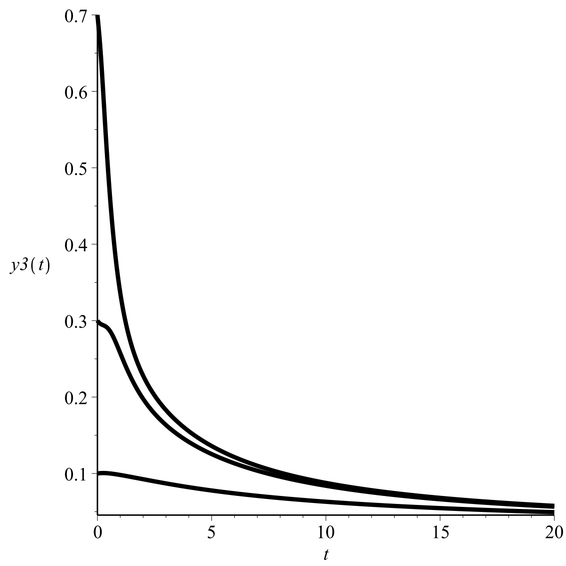

Fig. 1. Dynamics behaviors of the third component of system (2.4), the initial conditions $(x_1(0), x_2(0), y(0))=$ $(1,2,0.7)$,$(2,1,0.3)$ and $(0.5,0.2,0.1)$, $t\in[0,20]$, respectively.

Theorem A holds, the result of Theorem A may still not hold, Hence, the conclusion of Theorem A may not be hold, indeed, by carefully checking the proof of Theorem A in [22], we found that the authors directly applying Lemma 2.2 to the system (2.9) and (2.14) in [22], however, this is incorrect. That is to say, the persistent property of the system (2.1) need to be revisited.

The aim of this paper is to revisit the persistent property of system (2.1).

III. Main result

We adopt the following notations through this paper:

$$\begin{aligned}A_T(g) &= \frac{1}{T}\int_0^T g\,dt,\\ g^M &= \sup_{t\in[0,T]} g(t),\\ g^L &= \inf_{t\in[0,T]} g(t),\end{aligned} \tag{3.1}$$

where $g(t)$ is a continuous T-periodic function.

We first introduce several Lemmas.

Lemma 3.1. (see[30]) *If $a(t)$ and $b(t)$ are all continuous T*

periodic functions for all $t \in R$, and $A_T(a(t)) > 0, b(t) > 0$, then the system

$$\dot{x}(t) = x(t)(a(t) - b(t)x(t)) \tag{3.2}$$

has a unique T periodic solution which is globally asymptotically stable.

Lemma 3.2. (see [31])*If $a(t), b(t), d_1(t)$ and $d_2(t)$ are all positive and continuous T periodic functions for all $t \in R$, then the system*

$$\begin{aligned} \frac{dx_1}{dt} &= a(t)x_2(t) - b(t)x_1(t) - d_1(t)(x_1(t))^2, \\ \frac{dx_2}{dt} &= b(t)x_1(t) - d_2(t)x_2^2(t). \end{aligned} \tag{3.3}$$

has a T periodic solution $(x_1^*(t), x_2^*(t))$, which is globally asymptotically stable with respect to $R_+^2 = \{(x, y)|x > 0, y > 0\}$.

Lemma 3.3. *The system*

$$\begin{aligned} \frac{dx_1}{dt} &= ax_2(t) - bx_2^2(t) - cx_1(t) \\ &\quad -d(x_1(t))^2 \overset{\text{def}}{=} P_1(x_1, x_2), \end{aligned} \tag{3.4}$$

$$\frac{dx_2}{dt} = ex_1(t) - fx_2^2(t) \overset{\text{def}}{=} P_2(x_1, x_2)$$

admits a unique positive equilibrium $E(x_1^*, x_2^*)$, which is globally asymptotically stable, where a, b, c, d, e, f are all positive constants.

Proof. Since

$$\begin{aligned} \frac{dx_1}{dt} &\le ax_2(t) - cx_1(t) - d(x_1(t))^2, \\ \frac{dx_2}{dt} &= ex_1(t) - fx_2^2(t), \end{aligned} \tag{3.5}$$

while from Lemma 2.2 the system

$$\begin{aligned} \frac{du_1}{dt} &= au_2(t) - cu_1(t) - d(u_1(t))^2, \\ \frac{du_2}{dt} &= eu_1(t) - fu_2^2(t) \end{aligned} \tag{3.6}$$

has an positive equilibrium $E(x_1^{**}, x_2^{**})$, which is globally asymptotically stable, it then follows from the comparison theorem of the differential equation that the solution of (3.4) are all uniformly bounded.

The equilibrium of system (3.4) is determined by

$$\begin{aligned} x_2 - bx_2^2 - cx_1 - dx_1^2 &= 0, \\ ex_1 - fx_2^2 &= 0. \end{aligned} \tag{3.7}$$

From the second equation of (3.7), one has

$$x_1 = \frac{fx_2^2}{e}, \tag{3.8}$$

Substituting (3.8) into the first equation of (3.7) leads to

$$H(x_2) = df^2x_2^3 + be^2x_2 + cefx_2 - ae^2 = 0. \tag{3.9}$$

Since $H(0) = -ae^2 < 0$, and $H^{'}(x_2) = 3df^2x_2^2 + be^2 + cef > 0$, it follows that $H(x_2)$ is a strictly increasing function for all $x_2 > 0$ and so, there exists a unique positive solution x_2^*, consequently, from (3.8), we can obtain the unique x_1^*. That is, the system (3.4) admits a unique positive equilibrium $E(x_1^*, x_2^*)$.

The Jacobian matrix at $E(x_1^*, x_2^*)$ is

$$J(x_1^*, x_2^*) = \begin{pmatrix} -2dx_1^* - c & -2bx_2^* + a \\ e & -2fx_2^* \end{pmatrix}. \tag{3.10}$$

From the fact x_2^* satisfies (2.9), it immediately follows that

$$\begin{aligned} & tr(J(x_1^*, x_2^*)) \\ &= -2dx_1^* - c - 2fx_2^* < 0. \end{aligned} \tag{3.11}$$

$$\begin{aligned} & det(J(x_1^*, x_2^*)) \\ &= \frac{1}{e}\Big(4df^2(x_2^*)^3 + 2cefx_2^* + 2be^2x_2^* - ea^2\Big) \\ &= \frac{1}{e}\Big(3df^2(x_2^*)^3 + cefx_2^* + be^2x_2^*\Big) > 0. \end{aligned} \tag{3.12}$$

Hence, $J(x_1^*, x_2^*)$ has two negative characteristic root, and $E(x_1^*, x_2^*)$ is locally asymptotically stable.

Now, to ensure $E(x_1^*, x_2^*)$ is globally asymptotically stable, it is enough to show that system (3.4) has no limit cycle. We consider the Dulac function $u(x_1, x_2) = 1$, then

$$\frac{\partial(uP_1)}{\partial x_1} + \frac{\partial(uP_2)}{\partial x_2} = -2dx_1 - 2fx_2 - c < 0.$$

By Bendixson-Dulac principle, there is no closed orbit in area R_2^+. So $E(x_1^*, x_2^*)$ is globally asymptotically stable. This completes the proof of Lemma 3.3.

Lemma 3.4. *There exists positive constants M_{ix} and M_y such that*

$$\limsup_{t\to+\infty} x_i(t) < M_{ix}, i = 1, 2;$$

Also, if

$$A_T\Big(-d_3(t) + \frac{c_2(t)M_{1x}}{m(t)}\Big) > 0, \tag{3.13}$$

where M_{1x} is defined in (3.16). Then

$$\limsup_{t\to+\infty} y(t) < M_y. \tag{3.14}$$

Proof. In Proposition 2.1 of Zhang and Zhang[22], the authors had proved that

$$x_i(t) \le x_i^*(t) + \varepsilon, i = 1, 2, \ \ t \ge T_1, \tag{3.15}$$

where $(x_1^*(t), x_2^*(t))$ is the unique T-periodic solution of the system (3.3). Let $M_{ix} = \max\limits_{t\in[0,T]}\Big\{x_i^*(t) + \varepsilon\Big\}, i = 1, 2,$ then

$$\limsup_{t\to+\infty} x_i(t) \le M_{ix}, i = 1, 2. \tag{3.16}$$

From the third equation of system (2.1) and (3.16), for all $t \ge T_1$, we have

$$\frac{dy}{dt} \le y(t)\Big(-d_3(t) + \frac{c_2(t)M_{1x}}{m(t)} - q(t)y(t)\Big), \tag{3.17}$$

Consider the following auxiliary equation:

$$\frac{dv}{dt} = v(t)\Big(-d_3(t) + \frac{c_2(t)M_{1x}}{m(t)} - q(t)v(t)\Big). \tag{3.18}$$

If

$$A_T\Big(-d_3(t) + \frac{c_2(t)M_{1x}}{m(t)}\Big) > 0, \tag{3.19}$$

then by Lemma 3.1, we obtain that system (3.17) has a unique positive T periodic solution $y^*(t) > 0$, which is globally asymptotically stable. Similarly to the above analysis, for the above ε, there exists a $T_2 > T_1$, such that

$$y(t) < y^*(t) + \varepsilon \;\; t \geq T_2. \tag{3.20}$$

Set $M_y = \max_{t\in[0,T]} \Big\{y^*(t) + \varepsilon\Big\}$, then

$$\limsup_{t\to+\infty} y(t) \leq M_y. \tag{3.21}$$

Lemma 3.5. *There exists positive constants m_{ix} such that*

$$\liminf_{t\to+\infty} x_i(t) > x_{i*}, i = 1, 2. \tag{3.22}$$

where (x_{1*}, x_{2*}) is the unique positive equilibrium of the system (3.24).

Proof. For $t \geq T_2$, from Lemma 3.4 and the first two equation of system (2.1), one has

$$\begin{aligned} \frac{dx_1}{dt} &\geq a^L x_2(t) - a^U\beta^U(x_2(t))^2 \\ &\quad -\Big(b^U + \frac{c_1^U M_y}{m^L}\Big)x_1(t) - d_1^U(x_1(t))^2, \quad (3.23) \\ \frac{dx_2}{dt} &= b^L x_1(t) - d_2^U x_2^2(t). \end{aligned}$$

Consider the following auxiliary equation:

$$\begin{aligned} \frac{dv_1}{dt} &= a^L v_2(t) - a^U\beta^U(v_2(t))^2 \\ &\quad -\Big(b^U + \frac{c_1^U M_y}{m^L}\Big)v_1(t) - d_1^U(v_1(t))^2, \quad (3.24) \\ \frac{dv_2}{dt} &= b^L v_1(t) - d_2^U v_2^2(t). \end{aligned}$$

It follows from Lemma 3.3 that system (3.24) admits a unique positive equilibrium $E(x_{1*}, x_{2*})$, which is globally asymptotically stable. By applying the comparison theorem of differential equation, it immediately follows that

$$\liminf_{t\to+\infty} x_i(t) \geq x_{i*}. \tag{3.25}$$

This ends the proof of Lemma 3.5.

Lemma 3.6. *Assume that*

$$A_T\Big(-d_3(t) + \frac{c_2(t)x_{1*}}{m(t) + (M_{1x})^2}\Big) > 0, \tag{3.26}$$

then there exists positive constants m_y, which is independent of the solution of system (2.1), such that

$$\liminf_{t\to+\infty} y(t) > m_y. \tag{3.27}$$

Proof. Condition (3.26) implies that for enough small positive constant ε, the inequality

$$A_T\Big(-d_3(t) + \frac{c_2(t)(x_{1*} - \varepsilon)}{m(t) + (M_{1x} + \varepsilon)^2}\Big) > 0. \tag{3.28}$$

It follows from Lemma 3.4 and 3.5 that there exists a $T_3 > T_2$ such that

$$x_i(t) < M_{ix} + \varepsilon, \;\; x_i(t) > x_{i*} - \varepsilon, \;\; i = 1, 2. \tag{3.29}$$

From the third equation of system (2.1) and (3.29), for all $t \geq T_3$, we have

$$\frac{dy}{dt} \geq y(t)\Big(-d_3(t) + \frac{c_2(t)(x_{1*} - \varepsilon)}{m(t) + (M_{1x} + \varepsilon)^2} - q(t)y(t)\Big), \tag{3.30}$$

Consider the following auxiliary equation:

$$\frac{dv}{dt} = v(t)\Big(-d_3(t) + \frac{c_2(t)(x_{1*} - \varepsilon)}{m(t) + (M_{1x} + \varepsilon)^2} - q(t)v(t)\Big). \tag{3.31}$$

From (3.26) and Lemma 3.1, we obtain that system (3.31) has a unique positive T periodic solution $v^*(t) > 0$, which is globally asymptotically stable. Hence, for the above ε, there exists a $T_4 > T_3$, such that

$$y(t) > v^*(t) - \varepsilon, \;\; t \geq T_4. \tag{3.32}$$

Set $m_y = \min_{t\in[0,T]} \Big\{v^*(t) - \varepsilon\Big\}$, then

$$\liminf_{t\to+\infty} y(t) \geq m_y. \tag{3.33}$$

This ends the proof of Lemma 3.6.

Noting that under the assumption (3.26) holds, then (3.13) always holds. As a direct corollary of Lemma 3.4-3.6, we have

Theorem 3.1.*Assume that (3.26) holds, then system (2.1) is permanent.*

IV. Numeric simulations

Example 4.1. Consider the following stage structure predator prey system

$$\begin{aligned} \frac{dx_1}{dt} &= (1 - 0.6x_2(t))x_2(t) - x_1(t) - (x_1(t))^2 \\ &\quad - \frac{(\frac{3}{2} + \frac{1}{2}\cos(t))x_1(t)y(t)}{1 + x_1^2(t)}, \\ \frac{dx_2}{dt} &= x_1(t) - \frac{1}{4}x_2^2(t), \\ \frac{dy}{dt} &= y(t)\Big(-\frac{1}{100} + \frac{x_1(t)}{1 + x_1^2(t)} - y(t)\Big). \end{aligned} \tag{4.1}$$

Here, we assume that $a(t) = b(t) = d_1(t) = m(t) = c_2(t) = 1, c_1(t) = \frac{3}{2} + \frac{1}{2}\cos(t), d_3(t) = \frac{1}{100}, \beta(t) = 0.6, d_2(t) = \frac{1}{4}$. Then

$$\begin{aligned} \frac{dx_1}{dt} &= x_2(t) - x_1(t) - (x_1(t))^2, \\ \frac{dx_2}{dt} &= x_1(t) - \frac{1}{4}x_2^2(t) \end{aligned} \tag{4.2}$$

has a unique positive equilibrium $E(1, 2)$, which is globally asymptotically stable. From the third equation of (4.1), we have

$$\frac{dy}{dt} \leq y(t)\Big(1 - y(t)\Big).$$

and so,

$$\limsup_{t\to+\infty} y(t) \leq 1. \tag{4.3}$$

From (4.3) and the first and second equation we have

$$\begin{aligned} \frac{dx_1}{dt} &\geq (1-0.6x_2(t))x_2(t)-x_1(t) \\ &\quad -(x_1(t))^2-2x_1(t), \qquad (4.4) \\ \frac{dx_2}{dt} &= x_1(t)-\frac{1}{4}x_2^2(t). \end{aligned}$$

Noting that the system

$$\begin{aligned} \frac{dv_1}{dt} &= (1-0.6v_2(t))v_2(t)-v_1(t) \\ &\quad -(v_1(t))^2-2v_1(t), \qquad (4.5) \\ \frac{dv_2}{dt} &= v_1(t)-\frac{1}{4}v_2^2(t). \end{aligned}$$

admits a unique positive equilibrium $E_1(0.065, 0.509)$, which is globally asymptotically stable. also,

$$A_{2\pi}\Big(-d_3(t)+\frac{c_2(t)x_{1*}}{m(t)+(M_{1x})^2}\Big) = -0.01+\frac{0.065}{2} \geq 0.02 > 0. \qquad (4.6)$$

That is, the condition of Theorem 2.1 holds, consequently, system (4.1) is permanent. Fig. 2-4 also supports this assertion.

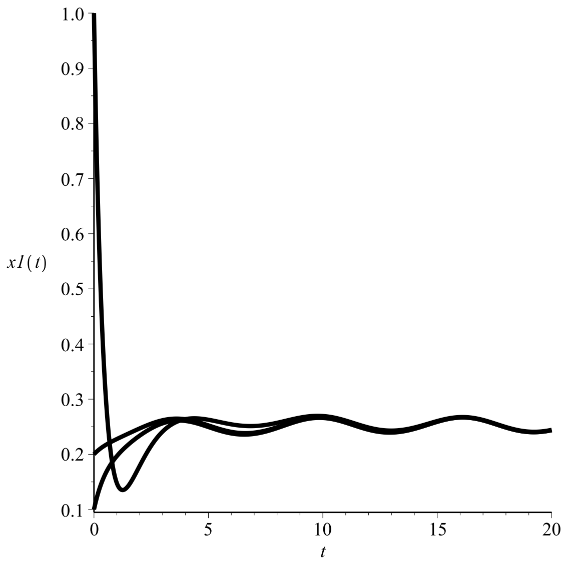

Fig. 2. Dynamics behaviors of the first component of system (4.1), the initial conditions $(x_1(0), x_2(0), y(0))=$ $(1, 2, 0.1)$,$(0.2, 1, 0.3)$ and $(0.1, 1, 0.5)$, $t \in [0, 20]$, respectively.

V. Conclusion

By numeric simulations, we found that one of the main results of Zhang and Zhang[22] is incorrect. By introducing a new lemma (Lemma 2.3) and applying the comparison theorem of the differential equation, we finally obtain a set of sufficient conditions which ensure the permanence of the system. Numeric simulations also support our finding.

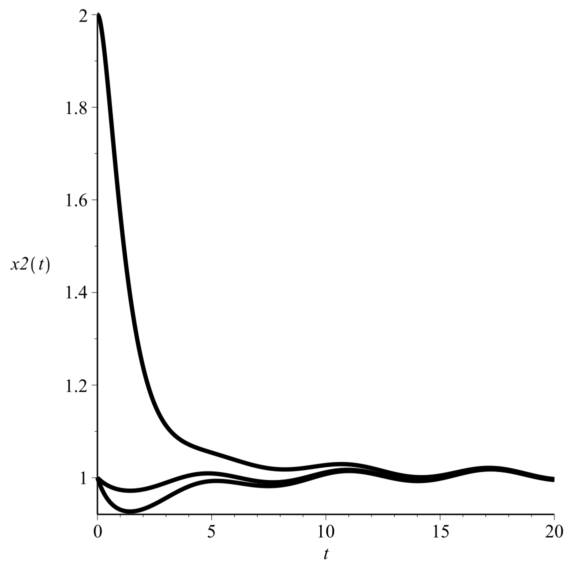

Fig. 3. Dynamics behaviors of the second component of system (4.1), the initial conditions $(x_1(0), x_2(0), y(0))=$ $(1, 2, 0.1)$,$(0.2, 1, 0.3)$ and $(0.1, 1, 0.5)$, $t \in [0, 20]$, respectively.

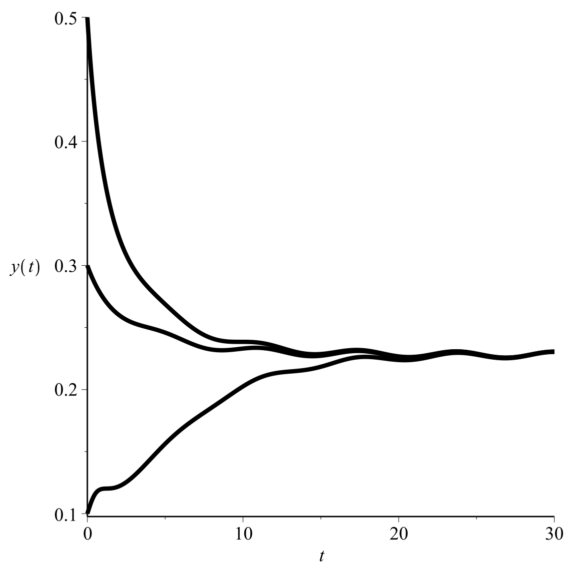

Fig. 4. Dynamics behaviors of the third component of system (4.1), the initial conditions $(x_1(0), x_2(0), y(0))=$ $(1, 2, 0.1)$,$(0.2, 1, 0.3)$ and $(0.1, 1, 0.5)$, $t \in [0, 30]$, respectively.

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