H-infinity Synchronization of Coupled Delay Partial Differential Systems with Hybrid Coupling via Nonsingular Transformation and Pinning Control

Lanchu Liu, Ziyu Liu and Yiping Luo

***Abstract*—The H_∞ synchronization problem of coupled delay partial differential systems (PDSs) with hybrid coupling is addressed in this paper. By using the nonsingular transformation method and pinning control, the coupled PDSs is decoupled to simplify the complex H_∞ synchronization problem, several H_∞ synchronization criteria are obtained. Furthermore, the asymptotical synchronization problem is studied when the external disturbances disappear. The relationship between the H_∞ synchronization and the asymptotical synchronization is also presented. Finally, an example of digital simulation elucidates the practicability and validity of the theoretical results.**

***Index Terms*— H_∞ synchronization, partial differential systems, hybrid coupling, nonsingular transformation method, pinning control**

I. Introduction

COMPLEX networks are ubiquitous in nature and human society. Such as ecosystems, food web, communication network. Synchronization is an important feature of complex networks, which means that two or more times varying quantities maintain certain relativity in the process of change. Synchronization can be seen everywhere, such as fireflies, yeast cells, image processing. Just a few, synchronization is skillfully used to solve many practical problems. Various synchronization patterns have been extensively studied by scholars at home and abroad[1-39].

Delays are inevitable due to the limited propagation velocity, traffic congestion, the technical level and other objective factors. Time delay exists objectively during the information transmission of complex networks. Ignoring it will result in the inconformity of the theoretical analysis and the actual situation and produce the erroneous results. Hence, Synchronization of complex dynamical networks with time-delay has received increasing attention[1-27].

The phenomena that are related to time and space can be modeled by PDSs. Recently, the research on synchronization of complex dynamic networks with PDSs has aroused great interest of scholars， many breakthroughs have occurred in this field[16-18, 20, 26, 34-36].

Noise, broadly speaking, is a signal of harmful interference other than useful signals in a communication system. It is well known that communication noise is ubiquitous in the process of signal transmission, it can cause a system to enter an unpredictable state and reduce the performance of the system, even disrupt synchronization. Therefore, it is very important to study the anti-interference ability of the complex network, namely H_∞ synchronization. In 2007, Y-Y Hou et al. first proposed the strict mathematical definition of H_∞ synchronization, they studied the H_∞ synchronization by using the Lyapunov function method [38]. Their pioneering work had made great progress in the study of H_∞ synchronization.

K-N Wu studied the H_∞ synchronization problem of the coupled time delay PDSs and obtained sufficient conditions to guarantee the H_∞ synchronization[25]. However, it is very difficult to prove the matrix inequality with $nN \times nN$ dimensions when the number of subsystems is fairly large. C. Li and G. Chen decoupled the general complex dynamic network with coupling time-delay by using the nonsingular transformation method and obtained the matrix inequality synchronous criteria with $n\times n$ dimensions[23]. Inspired by nonsingular transformation method, K-N Wu et al. studied the H_∞ synchronization problem of PDSs[26]. This method subtly reduces the dimension of the matrix and makes the proof easy.

Nevertheless, the model in [26] remains to be improved. On the one hand, the author only considered the state coupling term without taking the spatial diffusion coupling term into consideration. On the other hand, the author didn't implement any control methods for the H_∞ synchronization of the PDSs. In fact, since it is impossible to add controller to all the nodes, pinning control is an effective control strategy, especially for the large-scale network. In the last few years, pinning control techniques have been widely adopted to guarantee the synchronization of the network models [28-35].

Inspired by the above analysis, we considered the H_∞ synchronization of coupled delay partial differential systems with hybrid coupling via nonsingular transformation method and pinning control. The contributions of this paper are the following aspects:

(1) We add the spatial diffusion coupling term, which makes the model more general in comparison with the model in [26];

(2) We add pinning controller on the first l nodes, which will effectively promote synchronization;

(3) By using nonsingular transformation method, we obtained several sufficient conditions to guarantee the H_∞ synchronization of coupled delay PDSs. Obviously, our research methods are different from those of literature [26].

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Lanchu Liu is with the School of Science, Hunan Institute of Engineering, Hunan, 411104, China. E-mail: llc0202@163.com.

Ziyu Liu is with University of Electronic Science and Technology of China, Chengdu, 611731, China.

Yiping Luo is with Hunan Institute of Engineering, Hunan, 411104, China.

The remainder of this paper is organized as follows. In section 2, some preliminaries and notations are given. In section 3, the sufficient conditions on H_∞ synchronization of the coupled PDSs via nonsingular transformation method are provided. In section 4, an example of digital simulation is used to elucidate the practicability and validity of our control method and the correctness of the theorem. Finally, conclusions are drawn.

II. MODEL DESCRIPTION AND PRELIMARIES

In this paper, we consider the following disturbed coupled delay PDSs

$$\frac{\partial y_i(x,t)}{\partial t} = f(y_i(x,t)) + B\Delta y_i(x,t) + c\sum_{j=1}^{N} g_{ij} D_1 y_j(x,t-\tau)$$
$$+c\sum_{j=1}^{N} g_{ij} D_2 \Delta y_j(x,t) + \omega_i(x,t) + u_i(x,t),$$
$$x\in\Omega, t>0, i=1,2,\cdots,N. \quad (1)$$

where $y_i = (y_{i1}, y_{i2}, \cdots y_{in})^T \in R^n$ is the state variable of the ith subsystem, (x,t) are the spatial variable and time variable, and $\omega_i(x,t)\in R^n$ stands for the spatial-temporal disturbance which satisfies $\int_0^\infty \int_\Omega \omega_i^T(x,t)\omega_i(x,t)dxdt < \infty$. In a general way, we assume that the spatial domain be p dimensional, that is, $x=(x_1,x_2,\cdots,x_p)\in\Omega\subset R^P$. The function $f: R^n \to R^n$ is smoothly nonlinear. The coefficients $B\in R^{n\times n}$, $D_1 \in R^{n\times n}$ and $D_2\in R^{n\times n}$ are constant matrices. The positive constant c represents the coupling strength. $G=(g_{ij})_{N\times N}$ is the outer coupling matrix of the PDSs, which describes the coupling strength between node i and node $j(i\neq j)$. If there is a link between node i and $j(i\neq j)$, $g_{ij}=g_{ji}\neq 0,(i\neq j)$, otherwise, $g_{ij}=g_{ji}=0,(i\neq j)$, and

$$g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij} = -\sum_{j=1, j\neq i}^{N} g_{ji}, i=1,2,\cdots,N. \quad (2)$$

The positive constant τ represents the time delay of information transformation among subsystems. The Laplace operator Δ is defined as follows

$$\Delta y_i(x,t) = \sum_{k=1}^{p} \frac{\partial^2 y_i(x,t)}{\partial x_k^2}.$$

The boundary condition and initial value of the coupled delay PDSs are given as follows

$$y_i(x,t)=0, x\in\partial\Omega, y_i(x,t)=\varphi_i(x,t), t\in[-\tau,0].$$

And $u_i(x,t)$ is the pinning controller of the ith subsystem, which is defined as follows

$$u_i(x,t) = -\sigma_i e_i(x,t), \quad (3)$$

where

$$\sigma_i = \begin{cases} \sigma, i=1,2,\cdots,l. \\ 0, i=l+1,\cdots,N. \end{cases} \quad (4)$$

where σ is the pinning control gain.

Remark 1. Because the coupling configuration represents the exchange of information between nodes, for state coupling and spatial diffusion coupling, a node has the same communication as its neighbors. We assume that the outer coupling matrices of state coupling term and spatial diffusion coupling term are the same.

The disturbed coupled delay PDSs (1) achieves synchronization if

$$y_1(x,t) = y_2(x,t) = \cdots = y_N(x,t) = s(x,t), t\to\infty,$$

where $s(x,t)\in R^n$ is the solution of an isolated node, which satisfies the following equation

$$\frac{\partial s(x,t)}{\partial t} = f(s(x,t)) + B\Delta s(x,t), x\in\Omega, t>0, \quad (5)$$

In which $\Delta s(x,t) = \sum_{k=1}^{p}\frac{\partial^2 s(x,t)}{\partial x_k^2}$. The corresponding initial value and boundary conditions of system (5) are

$$s(x,t)=0, x\in\partial\Omega, s(x,t)=\varphi(x), t\in[-\tau,0].$$

The synchronization error is defined as

$$e_i(x,t) = y_i(x,t) - s(x,t).$$

A direct calculation yields the following synchronization error dynamics

$$\frac{\partial e_i}{\partial t} = f(y_i) - f(s) + B\Delta e_i + c\sum_{j=1}^{N} g_{ij} D_1 e_j(x,t-\tau)$$
$$+c\sum_{j=1}^{N} g_{ij} D_2 \Delta e_j + \omega_i - \sigma_i e_i. \quad (6)$$

Here and in the sequel, the variables (x,t) are suppressed for convenience. Clearly, the synchronization problem of N-coupled delay PDSs in (1) is equivalent to the stabilization problem of synchronization error dynamics in (6).

By the linearization of (6), we can get the following system

$$\frac{\partial e_i}{\partial t} = Je_i + B\Delta e_i + c\sum_{j=1}^{N} g_{ij} D_1 e_j(x,t-\tau)$$
$$+c\sum_{j=1}^{N} g_{ij} D_2 \Delta e_j + \omega_i - \sigma_i e_i,$$

where $J=J(x,t)$ is the Jacobian matrix of $f(y_i)$ at $s(x,t)$, that is $J=f'(s(x,t))$.

Let $e=(e_1,e_2,\cdots,e_N)\in R^{n\times N}$, $\omega=(\omega_1,\omega_2,\cdots,\omega_N)\in R^{n\times N}$, then we have

$$\frac{\partial e}{\partial t} = Je + B\Delta e + cD_1 e(x,t-\tau)G^T + cD_2\Delta e G^T + \omega - e\Gamma, \quad (7)$$

where $\Gamma = diag\{\underbrace{\sigma,\sigma,\cdots,\sigma}_{l}, \underbrace{0,\cdots,0}_{N-l}\}$.

The following lemma will be used to decouple the coupled synchronization error dynamics in (6).

Lemma 1. [39] Assume that a matrix G is irreducible and symmetric and satisfies the restriction (2). Then there is an unitary matrix $\Phi=(\varphi_1,\varphi_2,\cdots\varphi_N)$, such that

$$G^T\varphi_k = \lambda_k\varphi_k, k=1,2,\cdots,N,$$

Where $\lambda_i, i=1,2,\cdots,N,$ are the eigenvalues of matrix G.

Remark 2. The matrix Φ is an orthogonal matrix, that means $\Phi^{-1}=\Phi^T$.

According to Lemma 1, there exists unitary matrix $\Phi=(\varphi_1,\varphi_2,\cdots\varphi_N)\in R^{n\times N}$, such that $G^T\Phi=\Phi\Lambda$, where $\Lambda=diag\{\lambda_1,\lambda_2,\cdots,\lambda_N\}.$

Taking

$$v=e\Phi=(v_1,v_2,\cdots v_N)\in R^{n\times N},$$
$$\tilde{\omega}=\omega\Phi=(\tilde{\omega}_1,\tilde{\omega}_2,\cdots\tilde{\omega}_N)\in R^{n\times N},$$
$$\bar{\Gamma}=\Phi^{-1}\Gamma\Phi\in R^{n\times N}.$$

We get the following system

$$\frac{\partial v}{\partial t}=Jv+B\Delta v+cD_1v(x,t-\tau)\Lambda+cD_2\Delta v\Lambda+\tilde{\omega}-v\bar{\Gamma}, \quad (8)$$

In virtue of $\bar{\Gamma}$ is a symmetric matrix, so there is an orthogonal matrix $U\in R^{N\times N}$, makes the matrix $U^T\bar{\Gamma}U$ become a diagonal matrix , i.e., $U^T\bar{\Gamma}U=\tilde{\Gamma}$, where $\tilde{\Gamma}=diag\{\tilde{\sigma}_1,\tilde{\sigma}_2,\cdots,\tilde{\sigma}_N\}.$

Then we have

$$\frac{\partial v}{\partial t}=Jv+B\Delta v+cD_1v(x,t-\tau)\Lambda+cD_2\Delta v\Lambda+\tilde{\omega}-vU\tilde{\Gamma}U^T, \quad (9)$$

From (9), we can get the following N-decoupled systems

$$\frac{\partial v_i}{\partial t}=Jv_i+B\Delta v_i+c\lambda_iD_1v_i(x,t-\tau)+c\lambda_iD_2\Delta v_i+\tilde{\omega}_i-\tilde{\sigma}_iv_i, \quad (10)$$

Definition 1. The disturbed coupled delay PDSs (1) achieves the H_∞ synchronization with a disturbance attenuation $\gamma>0$, if the following inequality holds when $e_i(x,t)=0, t\in[-\tau,0],$

$$\int_0^\infty\int_\Omega(\sum_{i=1}^N e_i^Te_i-\gamma^2\sum_{i=1}^N\omega_i^T\omega_i)dxdt<0. \quad (11)$$

Lemma 2.[27] Let Ω be a cube: $|x_i|<l_i(i=1,2,\cdots,p)$, $h(x)$ be a real-valued function belonging to $C^1(\Omega)$ which vanish on the boundary $\partial\Omega$ of Ω, i.e.., $h(x)|_{\partial\Omega}=0$, then

$$\int_\Omega h^2(x)dx\le l_i^2\int_\Omega\left|\frac{\partial h}{\partial x_i}\right|^2dx.$$

Lemma 3.[26] Let Ω be a cube: $|x_i|<l_i(i=1,2,\cdots,p)$, $z(x)=(z_1(x),z_2(x),\cdots z_n(x))\in R^n$ be a function which belongs to $C^2(\Omega)$ and vanish on the boundary of Ω, then

$$\int_\Omega z^T(x)\Delta z(x)dx\le-(\sum_{k=1}^p\frac{1}{l_k^2})\int_\Omega z^T(x)z(x)dx.$$

Lemma 4.[26] The H_∞ synchronization problem of N-coupled PDSs (1) is equivalent to the H_∞ synchronization problem of N decoupled systems (10).

III. H_∞ SYNCHRONIZATION AND ASYMPTOTICAL SYNCHRONIZATION OF COUPLED DELAY PDSS

In this section, the H_∞ synchronization of coupled delay PDSs (1) is discussed via the H_∞ stabilization problem of N-decoupled systems in (10). By using the nonsingular transformation, a lower dimensional matrix inequality criterion is obtained. The asymptotical synchronization of coupled delay PDSs (1) is considered as the asymptotical stability of N-decoupled systems in (10) when the external disturbance is ignored.

Theorem 1. If there exist two series of symmetric positive definite matrices P_i and $Q_i, i=1,2,\cdots,N.$, satisfying

$$P_iB\ge0, \quad (12)$$
$$\lambda_iP_iD_2\ge0, \quad (13)$$

and the following LIMs

$$\begin{pmatrix}\Sigma & c\lambda_iP_iD_1\\ c\lambda_iD_1^TP_i & -Q_i\end{pmatrix}<0, \quad (14)$$

where $\Sigma=I+Q_i+J^TP_i+P_iJ-2\tilde{\sigma}_iP_i+\frac{1}{\gamma^2}P_i^2-2(\sum_{k=1}^p\frac{1}{l_k^2})P_iB$ $-2c(\sum_{k=1}^p\frac{1}{l_k^2})\lambda_iP_iD_2,$

Then the coupled delay PDSs (1) is in H_∞ synchronization with a given external disturbance attenuation lever $\gamma>0.$

Proof: Taking

$$V_i(v_i(\cdot,t))=\int_\Omega v_i^TP_iv_idx+\int_{t-\tau}^t\int_\Omega v_i^T(x,s)Q_iv_i(x,s)dxds$$

$$V(v(\cdot,t))=\sum_{i=1}^N V_i(v_i(\cdot,t)).$$

Obviously, $V(v(\cdot,0))=0$ when $e_i(x,t)=0, t\in[-\tau,0].$

By virtue of Lemma 4, we know what we need is to verify the H_∞ stabilization of N-decoupled systems (10). Therefore, we need the inequality (11).

Noting $V(v(\cdot,0))=0$, we have

$$\begin{aligned}&\int_0^\infty\int_\Omega\sum_{i=1}^N(v_i^Tv_i-\gamma^2\tilde{\omega}_i^T\tilde{\omega}_i)dxdt\\&=\int_0^\infty[\int_\Omega\sum_{i=1}^N(v_i^Tv_i-\gamma^2\tilde{\omega}_i^T\tilde{\omega}_i)dx+\frac{dV(v(\cdot,t))}{dt}]dt\\&\quad+V(v(\cdot,0))-V(v(\cdot,\infty))\\&\le\int_{i=1}^N\int_0^\infty\int_\Omega[v_i^Tv_i-\gamma^2\tilde{\omega}_i^T\tilde{\omega}_i+\frac{\partial v_i^T}{\partial t}P_iv_i+v_i^TP_i\frac{\partial v_i}{\partial t}\\&\quad+v_i^TQ_iv_i-v_i^T(x,t-\tau)Q_iv_i(x,t-\tau)]dxdt. \quad (15)\end{aligned}$$

Keeping (10) in mind, we can get

$$\begin{aligned}&v_i^Tv_i-\gamma^2\tilde{\omega}_i^T\tilde{\omega}_i+\frac{\partial v_i^T}{\partial t}P_iv_i+v_i^TP_i\frac{\partial v_i}{\partial t}+v_i^TQ_iv_i\\&\quad-v_i^T(x,t-\tau)Q_iv_i(x,t-\tau)\end{aligned}$$

$$=-\gamma^2(\tilde{\omega}_i-\frac{1}{\gamma^2}P_iv_i)^T(\tilde{\omega}_i-\frac{1}{\gamma^2}P_iv_i)+\frac{1}{\gamma^2}v_i^TP_i^2v_i$$
$$+v_i^T(I+Q_i+J^TP_i+P_iJ-2\tilde{\sigma}_iP_i)v_i$$
$$+(\Delta v_i^TB^TP_iv_i+v_i^TP_iB\Delta v_i)+(c\lambda_iv_i^T(x,t-\tau)D_1^TP_iv_i$$
$$+c\lambda_iv_i^TP_iD_1v_i(x,t-\tau))+(c\lambda_i\Delta v_i^TD_2^TP_iv_i$$
$$+c\lambda_iv_i^TP_iD_2\Delta v_i)-v_i^T(x,t-\tau)Q_iv_i(x,t-\tau). \quad (16)$$

Since $P_iB\geq 0$, there exists a matrix H, such that $H^TH=2P_iB$, and

$$2v_i^TP_iB\Delta v_i=v_i^TH^TH\Delta v_i=(Hv_i)^T(Hv_i).$$

Taking $z=Hv_i$ and noticing that $z=Hv_i=0$ when $x\in\partial\Omega$, with the help of Lemma 3, we have

$$\int_\Omega 2v_i^TP_iB\Delta v_idx\leq -2(\sum_{k=1}^{p}\frac{1}{l_k^2})\int_\Omega v_i^TP_iBv_idx. \quad (17)$$

Since $\lambda_iP_iD_2\geq 0$, for the same reason, we can deduce

$$\int_\Omega 2\lambda_iv_i^TP_iD_2\Delta v_idx\leq -2(\sum_{k=1}^{p}\frac{1}{l_k^2})\int_\Omega \lambda_iv_i^TP_iD_2v_idx. \quad (18)$$

Substituting (14) and (15) into (12), we obtain

$$\int_0^\infty\int_\Omega(\sum_{i=1}^{N}e_i^Te_i-\gamma^2\sum_{i=1}^{N}\omega_i^T\omega_i)dxdt$$
$$\leq\sum_{i=1}^{N}\int_0^\infty\int_\Omega[v_i^T(I+Q_i+J^TP_i+P_iJ-2\tilde{\sigma}_iP_i+\frac{1}{\gamma^2}P_i^2$$
$$-2(\sum_{k=1}^{p}\frac{1}{l_k^2})P_iB-2c(\sum_{k=1}^{p}\frac{1}{l_k^2})\lambda_iP_iD_2)v_i$$
$$+c\lambda_iv_i^T(x,t-\tau)D_1^TP_iv_i+c\lambda_iv_i^TP_iD_1v_i(x,t-\tau)$$
$$-v_i^T(x,t-\tau)Q_iv_i(x,t-\tau)]dxdt$$
$$=\sum_{i=1}^{N}\int_0^{t_f}\begin{pmatrix}v_i\\ v_i(x,t-\tau)\end{pmatrix}^T\Psi\begin{pmatrix}v_i\\ v_i(x,t-\tau)\end{pmatrix}dxdt, \quad (19)$$

where $\Psi=\begin{pmatrix}\Sigma & c\lambda_iP_iD_1\\ c\lambda_iD_1^TP_i & -Q_i\end{pmatrix}$, $\Sigma=I+Q_i+J^TP_i+P_iJ$ $-2\tilde{\sigma}_iP_i+\frac{1}{\gamma^2}P_i^2-2(\sum_{k=1}^{p}\frac{1}{l_k^2})P_iB-2c(\sum_{k=1}^{p}\frac{1}{l_k^2})\lambda_iP_iD_2.$

In light of condition (14), we know that the inequality (11) is satisfied, and that completes the proof.

It is clear that the asymptotical synchronization of coupled delay PDSs (1) is equivalent to the asymptotical stabilization of decoupled systems in (10), that is, when $\omega_i=\tilde{\omega}_i=0, v_i(x,t)\to 0$ as $t\to\infty$ if and only if $e_i(x,t)\to 0$ as $t\to\infty$, since Φ is nonsingular.

The following theorem concerns on the asymptotical synchronization of the coupled delay PDSs (1) with $\omega_i=0$.

Theorem 2. Assume that there exist two series of symmetric positive definite matrices P_i and $Q_i, i=1,2,\cdots,N.$, such that

$$P_iB\geq 0, \quad (20)$$
$$\lambda_iP_iD_2\geq 0, \quad (21)$$

and the following LIMs

$$\begin{pmatrix}J^TP_i+P_iJ+Q_i-2\tilde{\sigma}_iP_i-2(\sum_{k=1}^{p}\frac{1}{l_k^2})P_iB-2c(\sum_{k=1}^{p}\frac{1}{l_k^2})\lambda_iP_iD_2 & c\lambda_iP_iD_1\\ c\lambda_iD_1^TP_i & -Q_i\end{pmatrix}<0, \quad (22)$$

then the coupled PDSs (1) achieves the asymptotical synchronization.

Taking the Lyapunov function and using the similar technique, in light of conditions (20), (21) and (22), we can get that $\dot{V}(v(\cdot,t))<0,$ which yields the asymptotical stabilization of decoupled systems in (10) with $\tilde{\omega}_i=0$ .We can complete the proof by the equivalence of the asymptotical synchronization of the coupled systems in (1). We omit the details.

Remark 3. From the conditions (14) and (22), we can get that the smaller $l_1,l_2,\cdots,l_p,$ the easier to achieve the H_∞ synchronization or the asymptotical synchronization.

Remark 4. When $\int_0^\infty\int_\Omega\omega_i^T\omega_idxdt<\infty,$ i. e., the spatial-temporal disturbances are of finite energy

$$\int_0^\infty\int_\Omega e_i^Te_idxdt<\gamma^2\int_0^\infty\int_\Omega\omega_i^T\omega_idxdt<\infty.$$

Then we get $\int_\Omega e_i^Te_idxdt\to 0$ as $t\to\infty,$ and the asymptotical synchronization follows. Which means the H_∞ synchronization implies the asymptotical synchronization in the case of finite-energy disturbance.

From the criteria (14) and (22), we also get the same relationship. Actually, inequality (22) means

$$J^TP_i+P_iJ+Q_i-2\tilde{\sigma}_iP_i-2(\sum_{k=1}^{p}\frac{1}{l_k^2})P_iB$$
$$-2c(\sum_{k=1}^{p}\frac{1}{l_k^2})\lambda_iP_iD_2+c^2\lambda_i^2P_iD_1Q_i^{-1}D_1^TP_i<0.$$

The above inequality can be derived by the criterion (14), then we get that the criteria are in accordance with the result of the definitions.

Remark 5. The criteria of theorem 1 and 2 are easier to be verified by using the nonsingular transformation method because the dimension of the matrix is significantly reduced from $Nn\times Nn$ to $n\times n$.

IV. Numerical Examples

In this section, we give an example to elucidate the practicability and validity of our control method and the correctness of the theorem.

Consider the following disturbed coupled delay PDSs

$$\frac{\partial y_i}{\partial t}=-y_i+B\Delta y_i+c\sum_{j=1}^{3}g_{ij}D_1y_j(x_1,x_2,t-0.6)$$

$$+c\sum_{j=1}^{3} g_{ij} D_2 \Delta y_j(x_1,x_2,t) + 0.5\sin(x_1x_2)\cos(it)$$
$$-\sigma_i e_i, \qquad i=1,2,3. \qquad (23)$$

The initial values are

$$\begin{cases} y_1(x_1,x_2,t) = 5\sin(2\pi x_1)\sin(3\pi x_2)\cos(2), \\ y_2(x_1,x_2,t) = 2\sin(2\pi x_1)\sin(3\pi x_2)\sin(2), \\ y_3(x_1,x_2,t) = \sin(2\pi x_1)\sin(3\pi x_2)\tan(2), \end{cases}$$

for $t \in [-0.6, 0]$.

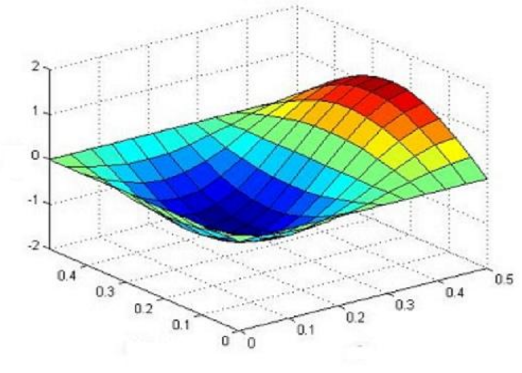
The boundary conditions are $y_i(x_1,x_2,t)=0, x\in\partial\Omega,$ where $\Omega=[0,0.5]\times[0,0.5], i=1,2,3.$ The synchronization function $s(x_1,x_2,t)$ satisfies $\frac{\partial s}{\partial t} = -s + B\Delta s,$ with the initial value $s(x_1,x_2,0) = 2\sin(2\pi x_1)\sin(3\pi x_2)\cos(2),$ let $s(x_1,x_2,t)=s(x_1,x_2,0)$ for $t\in[-0.6,0]$. The corresponding boundary condition is $s(x_1,x_2,t)=0, x\in\partial\Omega$ .Take $B=0.00004, c=0.1, D_1=0.01, D_2=-0.0001,$ $\sigma_1=\sigma_2=2, \sigma_3=0.$ and the outer coupling matrix

$$G=\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

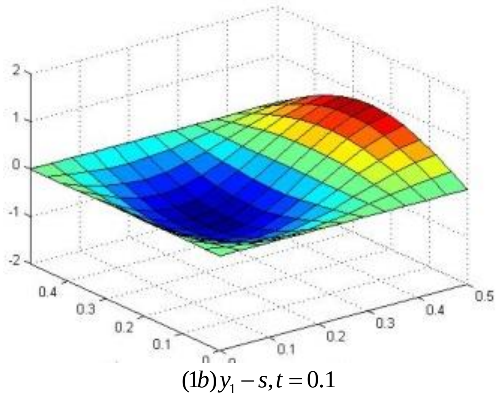
It is not difficult to see that $l_1 = l_2 = 0.5.$

Take $p_1=p_2=p_3=1, \gamma=1, Q_1=Q_2=Q_3=0.00032.$ We can verify that inequality in theorem 1 holds.

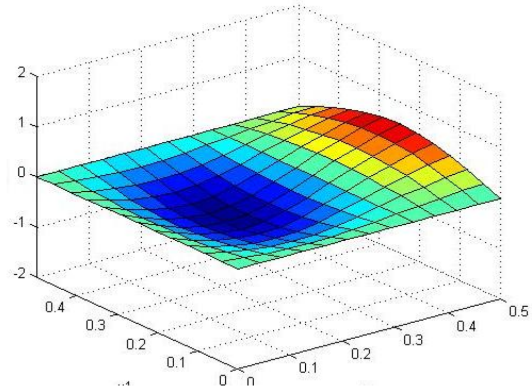
The coupled delay systems (23) achieves the asymptotical synchronization according to the relationship between the H_∞ synchronization and asymptotical synchronization when the external disturbance disappears.



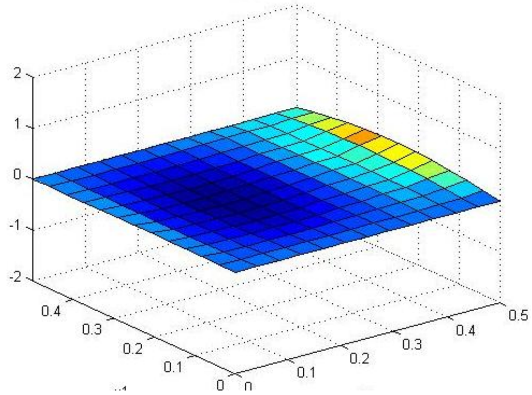
$(1a) y_1 - s, t = 0$



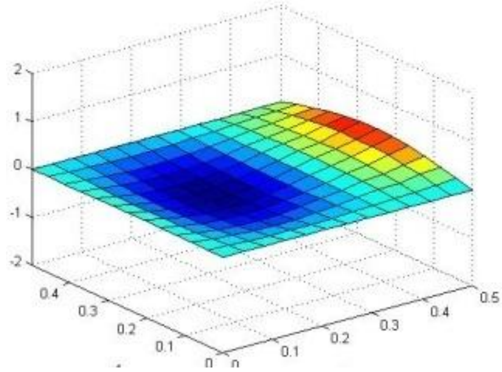
$(1b) y_1 - s, t = 0.1$



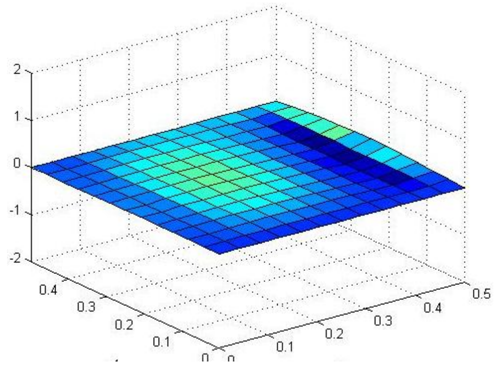
$(1c) y_1 - s, t = 0.2$



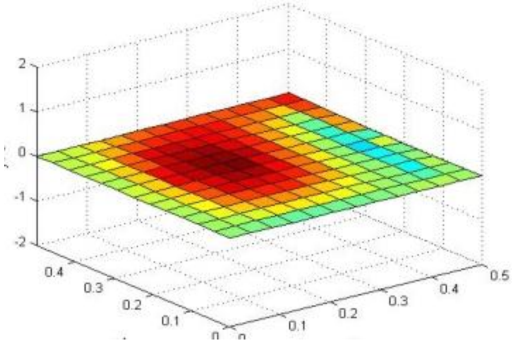
$(1d) y_1 - s, t = 0.3$



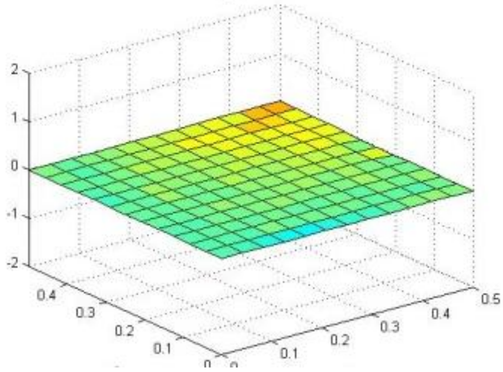
$(1e) y_1 - s, t = 0.4$



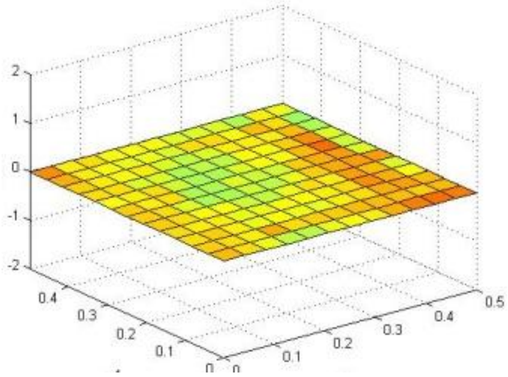
$(1f) y_1 - s, t = 0.5$



$(1g)\, y_1 - s, t = 1$

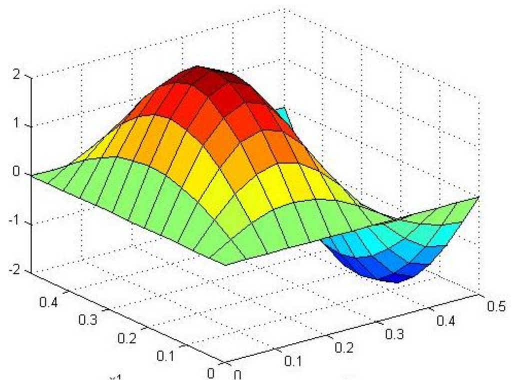


$(1h)\, y_1 - s, t = 1.5$

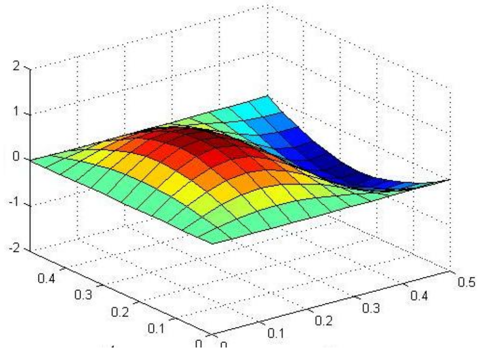


$(1i)\, y_1 - s, t = 2$

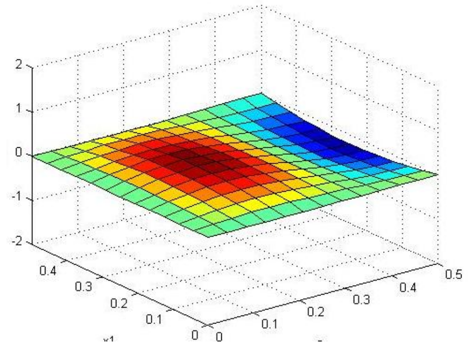
Fig1. The errors of the first node of PDSs (23) without disturbances at $t = 0, t = 0.1, t = 0.2, t = 0.3, t = 0.4, t = 0.5, t = 1, t = 1.5, t = 2.$



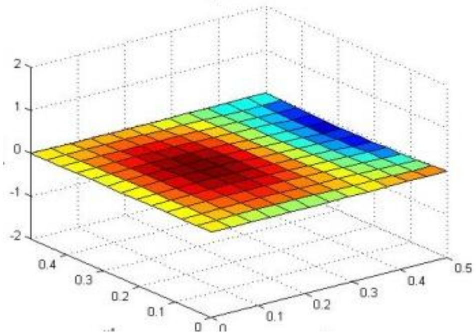
$(2a)\, y_2 - s, t = 0$



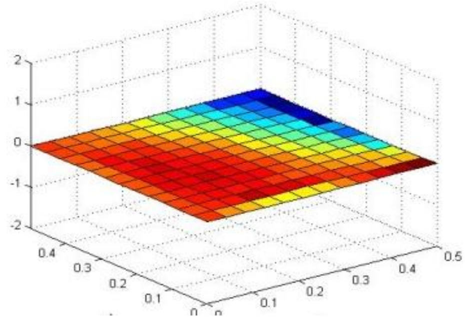
$(2b)\, y_2 - s, t = 0.5$



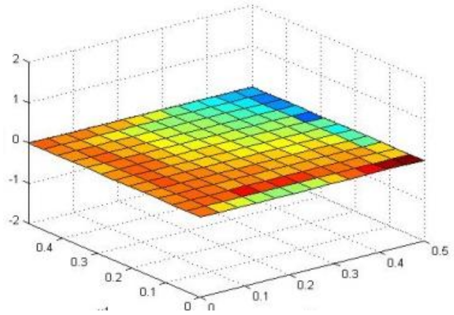
$(2c)\, y_2 - s, t = 1$



$(2d)\, y_2 - s, t = 1.5$

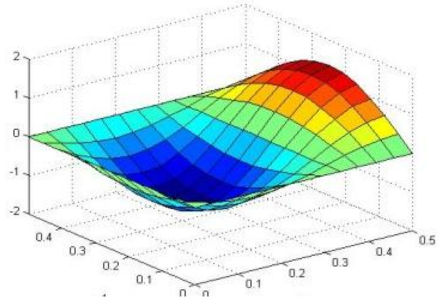


$(2e)\, y_2 - s, t = 2$

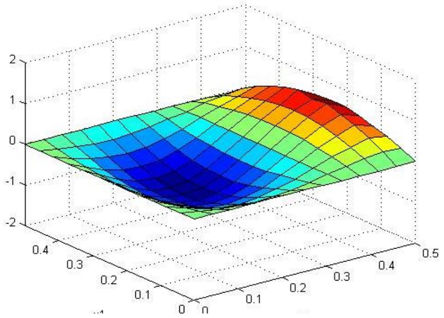


$(2f)\, y_2 - s, t = 2.5$

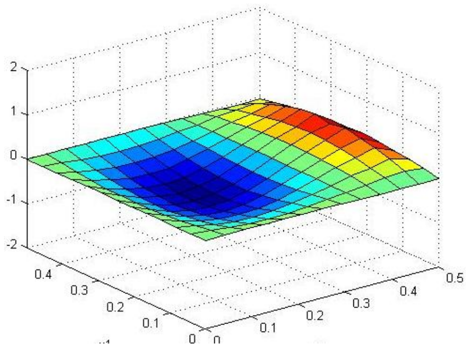
Fig2. The errors of the second node of PDSs (23) without disturbances at $t = 0, t = 0.5, t = 1, t = 1.5, t = 2, t = 2.5.$



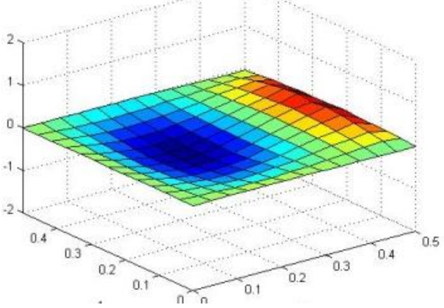
$(3a) y_3 - s, t = 0$



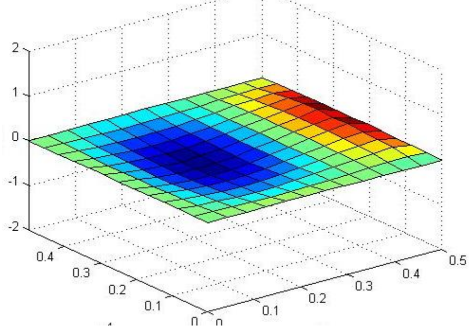
$(3b) y_3 - s, t = 0.5$



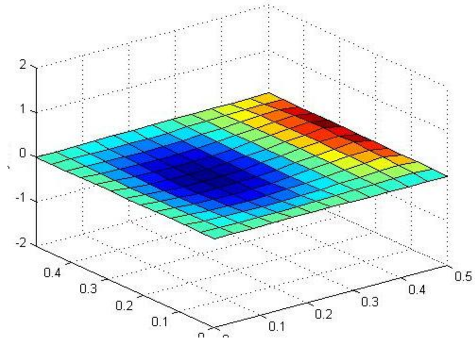
$(3c) y_3 - s, t = 1$



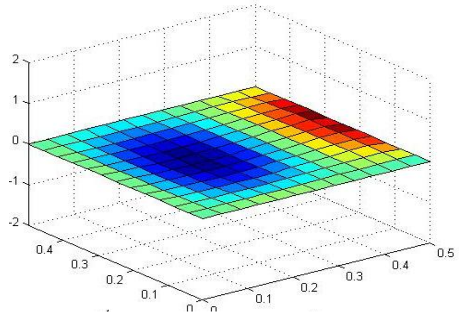
$(3d) y_3 - s, t = 1.5$



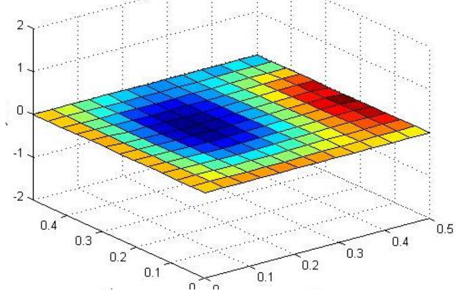
$(3e) y_3 - s, t = 2$



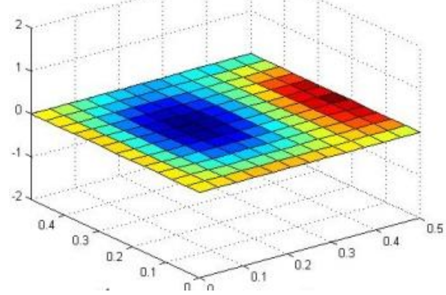
$(3f) y_3 - s, t = 2.5$



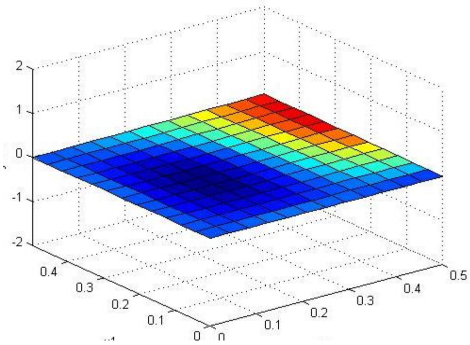
$(3g) y_3 - s, t = 3$



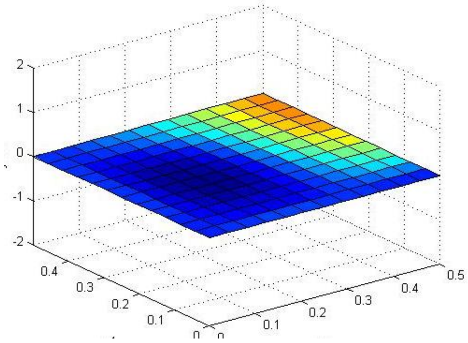
$(3h) y_3 - s, t = 3.5$



$(3i) y_3 - s, t = 4$



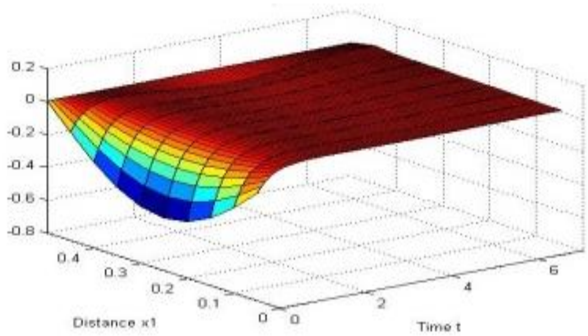
$(3j) y_3 - s, t = 4.5$



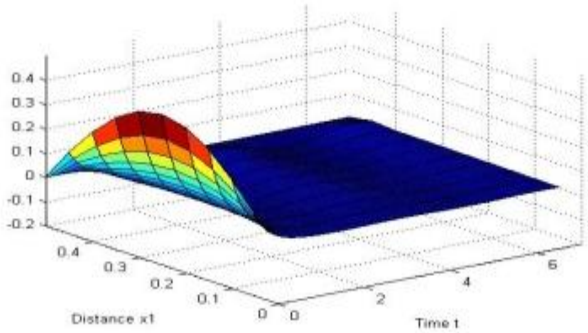
$(3k) y_3 - s, t = 5$

Fig3. The errors of the third node of PDSs (23) without disturbances at $t=0, t=0.5, t=1, t=1.5, t=2, t=2.5, t=3, t=3.5, t=4, t=4.5, t=5.$

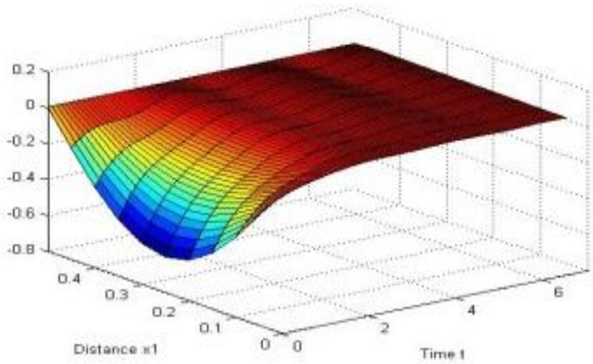
Figs. 1-3. illustrate the asymptotical synchronization .



$(4a) y_1 - s$



$(4b) y_2 - s$



$(4c) y_3 - s$

Fig. 4.The errors of PDSs (23) without disturbances along time at $x_2 = 0.3$.

Fig. 4 illustrates the asymptotical synchronization by the synchronization errors along the time t when one of the spatial variables is fixed.

Remark 6. From the Figs.1-3, we can see that the synchronization errors $y_i - s, i = 1,2$ of the first two nodes which are controlled by pinning controllers converge to zero faster than the third node which is not controlled by pinning control, the convergence effect of the first two nodes is superior to the third one, which indicates pinning control is an efficient control method for the coupled delay PDSs with hybrid coupling.

V. CONCLUSION

The H_∞ synchronization problem of coupled delay partial differential systems (PDSs) with hybrid coupling via nonsingular transformation method and pinning control is addressed in this paper. Sufficient conditions are obtained to guarantee the H_∞ synchronization by exploiting the Lyapunov functional method and some inequality techniques. The relationship between asymptotical synchronization and H_∞ synchronization is also presented for coupled delay PDSs. The effect of spatial domain on the H_∞ synchronization is pointed out and the advantage of the nonsingular matrix transformation in investigation of the H_∞ synchronization is also stated. Finally, an example of digital simulation is used to elucidate the practicability and validity of our control method and the correctness of the theorem.

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