New Results on the Extinction and Permanence of a Two Species Nonautonomous Nonlinear Competitive System

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***Abstract*—The extinction and persistent property of a two species noautonomous nonlinear competitive system is revisited in this paper. By constructing some suitable Lyapunov type extinction function, several set of new conditions which ensure that one of the components will be driven to extinction are established. By applying the differential inequality theory, two set of sufficient conditions which ensure the permanence of the system are obtained. It seems very amazing that the dynamic behaviors of the system is similar to the system without nonlinear constants, that is, α_i, α_2 have no influence on the extinct and persistent property of the system. Our results supplement and complement the results of Li and Chen[Extinction in two dimensional nonautonomous Lotka-Volterra systems with the effect of toxic substances, Applied Mathematics and Computation, 182(2006)684-690] and Chen et al.[Extinction in two species nonautonomous nonlinear competitive system, Applied Mathematics and Computation, 274(2016)119-124].**

***Index Terms*—Extinction; Nonlinear; Competition; Toxic substance.**

I. Introduction

THROUGHOUT this paper, for a given function $g(t)$, we let g_L and g_M denote $\inf_{-\infty<t<\infty} g(t)$ and $\sup_{-\infty<t<\infty} g(t)$, respectively. For continuous T-period function $g(t)$, set $m[g] = \frac{1}{T}\int_0^T g(t)dt$.

Traditional two-species autonomous Lotka-Volterra competition system takes the form [1]:

$$\begin{aligned} x_1'(t) &= x_1(t)(a_1 - b_{11}x_1(t) - b_{12}x_2(t)),\\ x_2'(t) &= x_2(t)(a_2 - b_{21}x_1(t) - b_{22}x_2(t)), \end{aligned} \tag{1.1}$$

Concerned with the extinction property of the system (1.1), we have:
(I) If the coefficients of the system (1.1) satisfy

$$\frac{a_1}{a_2} < \frac{b_{11}}{b_{21}},\ \ \frac{a_1}{a_2} < \frac{b_{12}}{b_{22}}, \tag{1.2}$$

then the first species will be driven to extinction while the other one will stabilize at the positive equilibrium of a logistic equation.
(II) If the coefficients of the system (1.1) satisfy

$$\frac{a_1}{a_2} > \frac{b_{11}}{b_{21}},\ \ \frac{a_1}{a_2} > \frac{b_{12}}{b_{22}}, \tag{1.3}$$

then the second species will be driven to extinction while the other one will be stable.

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In[2], Shair Ahamd considered the following nonautonomous system of differential equations:

$$\begin{aligned} u'(t) &= u(t)\big[a(t) - b(t)u(t) - c(t)v(t)\big],\\ v'(t) &= v(t)\big[d(t) - e(t)u(t) - f(t)v(t)\big], \end{aligned} \tag{1.4}$$

where $a(t), b(t), c(t), d(t), e(t)$ and $f(t)$ are assumed to be continuous and bounded above and below by positive constants, and $u(t), v(t)$ are population density of species u, v at time t, respectively. Ahmad[2] showed that if the coefficients of system (1.4) satisfies

$$a_L f_L > c_M d_M \text{ and } b_M d_M \le a_L e_L, \tag{1.5}$$

then species u will be permanent and species v will be extinct.

The effect of toxic substances on ecological communities is an important problem from the environmental point of view, in [3], Li and Chen studied the following two species competition system with toxic substance

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)x_2(t)\\ &\quad -c_1(t)x_1(t)x_2(t)],\\ \dot{x}_2(t) &= x_2(t)[r_2(t) - a_2(t)x_1(t) - b_2(t)x_2(t)\\ &\quad -c_2(t)x_1(t)x_2(t)], \end{aligned} \tag{1.6}$$

where $r_i(t), a_i(t), b_i(t), c_i(t), i = 1,2$ are assumed to be continuous and bounded above and below by positive constants, and $x_1(t), x_2(t)$ are population density of species x_1 and x_2 at time t, respectively. Li and Chen [3] showed that if the coefficients of system (1.1) satisfy

$$\begin{aligned} r_{1L}b_{2L} &> r_{2M}b_{1M},\\ r_{1L}a_{2L} &\ge r_{2M}a_{1M},\\ r_{1L}c_{2L} &\ge r_{2M}c_{1M}. \end{aligned} \tag{1.7}$$

Then second species will be driven to extinction while the first one will stabilize at a certain solution of a logistic equation.

Recently, with the aim of relaxing the condition (1.7) and considering a more suitable system, Chen et al [4] proposed the following two species competitive model:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1(t) - a_1(t)x_1^{\alpha_1}(t) - b_1(t)x_2^{\alpha_2}(t)\\ &\quad -c_1(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)],\\ \dot{x}_2(t) &= x_2(t)[r_2(t) - a_2(t)x_1^{\alpha_1}(t) - b_2(t)x_2^{\alpha_2}(t)\\ &\quad -c_2(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)]. \end{aligned} \tag{1.8}$$

Assume that

(H) $r_i(t), a_i(t)$, $b_i(t)$ and $c_i(t)$ are continuous bounded functions defined on $[0,+\infty)$; $r_i(t), a_i(t)$ and $b_i(t)$ are bounded above and below by positive constants; $a_i(t) \geq 0, b_i(t) \geq 0, c_i(t) \geq 0$ for all $t \in [0,+\infty)$; $\alpha_i, i=1,2$ are positive constants.

The authors showed that in addition to (H), if

$$\limsup_{t\to+\infty} \frac{r_2(t)}{r_1(t)} < \liminf_{t\to+\infty} \Big\{ \frac{a_2(t)}{a_1(t)}, \frac{b_2(t)}{b_1(t)}, \frac{c_2(t)}{c_1(t)} \Big\} \tag{1.9}$$

hold, then species x_2 will be driven to extinction, that is, for any positive solution $(x_1(t), x_2(t))^T$ of system (1.8), $x_2(t) \to 0$ as $t \to +\infty$, and $\int_0^{+\infty} x_2(t)dt < +\infty$.

Compare condition (1.3), (1.5), (1.7) and (1.9), one could see that with the introduce of toxic substance, to ensure the extinction of the second species, Li and Chen[3] and Chen et al [4] introduced the following assumption on the coefficients of toxic substance term, respectively.

$$r_{1L}c_{2L} \geq r_{2M}c_{1M} \tag{1.10}$$

and

$$\limsup_{t\to+\infty} \frac{r_2(t)}{r_1(t)} < \liminf_{t\to+\infty} \Big\{ \frac{c_2(t)}{c_1(t)} \Big\} \tag{1.11}$$

Now, an interesting issue is proposed: what would happen if condition (1.10) or (1.11) no longer hold?

Also, as was pointed out by Chen et al[4], "the growth rate of species need not be always positive, this is realistic since the environment may change on time (e. g. seasonal effects of weather condition, temperature, mating habits and food supplies) and then on some bad time $r_i(t)$ may be negative." In this case, the following assumption on system (1.8) seems more plausible.

(H_1) $r_i(t), i=1,2$ are continuous T-period functions such that $m[r_i(t)] = \frac{1}{T}\int_0^T r_i(s)ds > 0, i=1,2$, $a_i(t)$, $b_i(t)$ and $c_i(t)$ are all continuous positive T-period functions defined on $[0,+\infty)$; $\alpha_i, i=1,2$ are positive constants.

Obviously, it is interesting to investigate the dynamic behaviors of system (1.8) under the assumption (H_1) holds, since this is not studied in [4].

To bring some hints to above two issues, let's consider the following two examples.

Example 1.1. Consider the following two-species competitive system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Big[4 - 2\cos t - (\frac{3}{2} + \cos t)x_1(t) \\ &\quad -(1 + \frac{1}{2}\cos t)x_2(t) - 0.3x_1(t)x_2(t)\Big], \\ \dot{x}_2(t) &= x_2(t)\Big[2 - \cos t - (3 + \frac{\sin t}{2})x_1(t) \\ &\quad -(\frac{3}{2} + \frac{1}{2}\sin t)x_2(t) - 0.1x_1(t)x_2(t)\Big]. \end{aligned} \tag{1.12}$$

In this case, $r_1(t) = 4 - 2\cos t, r_2(t) = 2 - \cos t, a_1(t) = \frac{3}{2} + \cos t, b_1(t) = 1 + \frac{1}{2}\cos t, c_1(t) = 0.3, a_2(t) = 3 + \frac{\sin t}{2}, b_2(t) = \frac{3}{2} + \frac{1}{2}\sin t, c_2(t) = 0.1, \alpha_1 = \alpha_2 = 1$. By simple computation, one could see that

$$\limsup_{t\to+\infty} \frac{r_2(t)}{r_1(t)} = \frac{1}{2} > \frac{1}{3} = \liminf_{t\to+\infty} \frac{c_2(t)}{c_1(t)}, \tag{1.13}$$

$$r_{1L}c_{2L} = 2 \times 0.3 < 3 \times 0.3 = r_{2M}c_{1M}. \tag{1.14}$$

That is, neither (1.10) nor (1.11) holds, however, numeric simulation (Fig.1) shows that in this case, the second species will be driven to extinction.

Example 1.2. Consider the following two-species competitive system

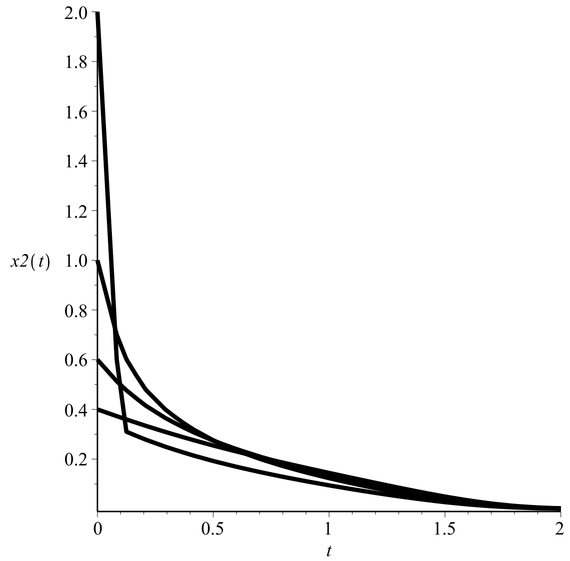

Fig. 1. Dynamics behaviors of species x_2 in system (1.12). Here, we take the initial conditions $(x_1(0), x_2(0)) = (1,1), (0.4,0.4), (0.6,0.6)$ and $(2,2)$, respectively.

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Big[4 - 5\cos t - (\frac{3}{2} + \cos t)x_1(t) \\ &\quad -(1 + \frac{1}{2}\cos t)x_2(t) - 0.3x_1(t)x_2(t)\Big], \\ \dot{x}_2(t) &= x_2(t)\Big[2 - 3\cos t - (3 + \frac{\sin t}{2})x_1(t) \\ &\quad -(\frac{3}{2} + \frac{1}{2}\sin t)x_2(t) - 0.1x_1(t)x_2(t)\Big]. \end{aligned} \tag{1.15}$$

In this case, $r_1(t) = 4 - 5\cos t, r_2(t) = 2 - 3\cos t$. Obviously, $r_i(2k\pi) = -1, i=1,2$, that is, $r_i(t)$ maybe negative. Hence, the results of Li and Chen[3] and Chen et al [4] could not be applied to determine the dynamic behaviors of the system (1.15). Numeric simulation (Fig. 2) shows that in this case, the second species still be driven to extinction.

Above two examples enlighten us to revisit the dynamic behaviors of the system (1.8), and to find out some new conditions which ensure the extinction of the second species in system (1.8).

On the other hand, it brings to our attention that Ahmad[2], Li and Chen[3] and Chen et al [4] did not investigated the persistent property of the system they investigated, which is one of the most important topics in the study of population dynamics. For more papers on permanence and extinction of population dynamics, one could refer to [26]-[42] and the references cited therein.

In addition to this section, we arrange the paper as follows: Some basic results are presented in the next

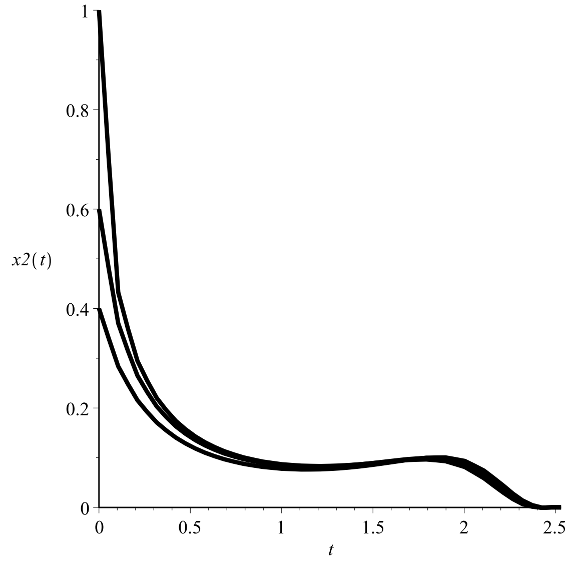

Fig. 2. Dynamics behaviors of species x_2 in system (1.15). Here, we take the initial conditions $(x_1(0), x_2(0)) = (1, 1), (0.4, 0.4), (0.6, 0.6)$ and $(2, 2)$, respectively.

section, we investigate the extinction property of system (1.8) in the third section. The persistent property of the system (1.8) is then investigated in Section 4. Some numeric simulations are presented to verify the feasibility of the main results in Section 5; We end this paper by a briefly discussion. For more works on the competition system, one could refer to [1]-[42] and the references cited therein.

II. Basic results

In this section, we shall develop some preliminary results, which will be used to prove the main result.

Lemma 2.1.[4] *Let $x(t) = (x_1(t), x_2(t))^T$ be any positive solution of system (1.8) with initial condition $x_1(0) > 0, x_2(0) > 0$, then*

$$\begin{aligned}\limsup_{t\to+\infty} x_1(t) &\le \left(\frac{r_{1M}}{a_{1L}}\right)^{\frac{1}{\alpha_1}} \stackrel{\text{def}}{=} M_1,\\ \limsup_{t\to+\infty} x_2(t) &\le \left(\frac{r_{2M}}{b_{2L}}\right)^{\frac{1}{\alpha_2}} \stackrel{\text{def}}{=} M_2.\end{aligned}$$

i.e, any positive solution of system (1.8) are ultimately bounded above by some positive constant.

For the logistic equation

$$\dot{x}(t) = x(t)\Big(r(t) - a(t)x^{\alpha}(t)\Big). \tag{2.1}$$

From Lemma 2.1 of Zhao and Chen [26], we have

Lemma 2.2. *Any positive solutions of Eq. (2.1) are defined on $[0, +\infty)$, bounded above and below by positive constants and globally attractive.*

Lemma 2.3.[22] *Assume that $r(t), a(t)$ are continuous T-periodic function. If $\int_0^T r(t)dt > 0, a(t) > 0,$ then system (2.1) has a unique strictly positive T-periodic solution $x(t)$ which is globally asymptotically stable.*

Let $x(t)$ be the positive periodic solution of system (2.1), there exists t_1 such that $x(t_1)$ obtains its maximum value and $\dot{x}(t_1) = 0$, that is

$$x(t_1)\Big(r(t_1) - a(t_1)x^{\alpha}(t_1)\Big) = 0. \tag{2.2}$$

And so

$$x(t_1) \quad \le \quad \left[\left(\frac{r}{a}\right)^u\right]^{\frac{1}{\alpha}}. \tag{2.3}$$

Let M_{11}, M_{22} be any positive numbers satisfies the following inequalities,

$$\begin{aligned}M_{11} &> M_1^* = \left[\left(\frac{r_1}{a_1}\right)^u\right]^{\frac{1}{\alpha_1}},\\ M_{22} &> M_2^* = \left[\left(\frac{r_2}{a_2}\right)^u\right]^{\frac{1}{\alpha_2}}.\end{aligned} \tag{2.4}$$

Lemma 2.4. *Assume that (H_1) hold. Let $(x_1(t), x_2(t))$ be any positive solution of system (1.8), then there exists an enough large T_1 such that for all $t \ge T_1$*

$$x_1(t) \le M_{11}, \ \ x_2(t) \le M_{22}. \tag{2.5}$$

Proof. By using Lemma 2.3, the proof of Lemma 2.4 is similar to the proof of Theorem 2.1 in [22], and we omit the detail here.

Following Lemma 2.5 is a direct corollary of Lemma 2.2 of F. D. Chen [16].

Lemma 2.5. *If $a > 0, b > 0$ and $\dot{x} \ge x(b - ax^{\alpha})$, where α is a positive constant, when $t \ge 0$ and $x(0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \ge \left(\frac{b}{a}\right)^{1/\alpha}.$$

If $a > 0, b > 0$ and $\dot{x} \le x(b - ax^{\alpha})$, where α is a positive constant, when $t \ge 0$ and $x(0) > 0$, we have

$$\limsup_{t\to+\infty} x(t) \le \left(\frac{b}{a}\right)^{1/\alpha}.$$

III. Extinction of system (1.8)

In this section, we study the extinction of species x_2 in system (1.8). Following are the main results of this section.

Theorem 3.1. *In addition to (H), assume further that*

$$\limsup_{t\to+\infty}\frac{r_2(t)}{r_1(t)} < \liminf_{t\to+\infty}\left\{\frac{a_2(t)}{a_1(t) + c_1(t)M_2^{\alpha_2}}, \frac{b_2(t)}{b_1(t)}\right\} \tag{3.1}$$

holds, then the species x_2 will be driven to extinction, that is, for any positive solution $(x_1(t), x_2(t))^T$ of system (1.8), $x_2(t) \to 0$ as $t \to +\infty$.

Proof. By (3.1) we can choose $\alpha, \tau, \varepsilon > 0$ and enough large T_2 such that for all $t \ge T_2$

$$\begin{aligned}&\frac{r_2(t)}{r_1(t)} < \frac{\alpha}{\tau} - \varepsilon < \frac{\alpha}{\tau} < \frac{a_2(t)}{a_1(t) + c_1(t)(M_2 + \varepsilon)^{\alpha_2}},\\ &\frac{r_2(t)}{r_1(t)} < \frac{\alpha}{\tau} - \varepsilon < \frac{\alpha}{\tau} < \frac{b_2(t)}{b_1(t)}.\end{aligned} \tag{3.2}$$

Let $X(t) = (x_1(t), x_2(t))^T$ be a solution of system (1.8) with $x_i(0) > 0, i = 1, 2$. It follows from Lemma 2.1 that there exists an enough large $T_3 > T_2$ such that

$$x_i(t) < M_i + \varepsilon \quad \text{for all} \quad t \geq T_3. \tag{3.3}$$

And so, for $t \geq T_3$, we have

$$r_2(t)\tau - r_1(t)\alpha < -\varepsilon\tau r_1(t) < -\varepsilon\tau r_{1L} < 0, \tag{3.4}$$

$$\alpha a_1(t) + \alpha c_1(t)(M_2 + \varepsilon)^{\alpha_2} - \tau a_2(t) < 0, \tag{3.5}$$

$$\alpha b_1(t) - \tau b_2(t) < 0. \tag{3.6}$$

From (1.8) we have

$$\begin{aligned} \frac{\dot{x}_1(t)}{x_1(t)} &= r_1(t) - a_1(t)x_1^{\alpha_1}(t) - b_1(t)x_2^{\alpha_2}(t) \\ &\quad -c_1(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t), \\ \frac{\dot{x}_2(t)}{x_2(t)} &= r_2(t) - a_2(t)x_1^{\alpha_2}(t) - b_2(t)x_2^{\alpha_2}(t) \\ &\quad -c_2(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t). \end{aligned} \tag{3.7}$$

Let

$$V(t) = x_1^{-\alpha}(t)x_2^{\tau}(t).$$

From (3.2)-(3.6), it follows that

$$\begin{aligned} &\dot{V}(t) \\ = \;& V(t)\Big[\tau r_2(t) - \alpha r_1(t) + (\alpha a_1(t) - \tau a_2(t))x_1^{\alpha_1}(t) \\ &+(\alpha b_1(t) - \tau b_2(t))x_2^{\alpha_2}(t) \\ &+(\alpha c_1(t) - \tau c_2(t))x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)\Big] \\ \leq \;& V(t)\Big[\tau r_2(t) - \alpha r_1(t) \\ &+\big(\alpha(a_1(t) + c_1(t)(M_2 + \varepsilon)^{\alpha_2}) - \tau a_2(t)\big)x_1^{\alpha_1}(t) \\ &+(\alpha b_1(t) - \tau b_2(t))x_2^{\alpha_2}(t) - \tau c_2(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)\Big] \\ \leq \;& V(t)(\tau r_2(t) - \alpha r_1(t)) \\ \leq \;& -\varepsilon\tau r_{1L}V(t). \end{aligned}$$

Integrating this inequality from T_3 to $t (\geq T_3)$, it follows

$$V(t) \leq V(T_3)\exp(-\varepsilon\tau r_{1L}(t - T_3)). \tag{3.8}$$

and so,

$$x_2(t) < C\exp(-\varepsilon r_{1L}(t - T_3)), \tag{3.9}$$

where

$$C = (M_1 + \varepsilon)^{\alpha/\tau}(x_1(T_3))^{-\alpha/\tau}x_2(T_3) > 0.$$

Therefore, we have $x_2(t) \to 0$ exponentially as $t \to +\infty$. This ends the proof of Theorem 3.1.

Theorem 3.2. *In addition to (H_1), assume further that*

$$\frac{m[r_2]}{m[r_1]} < \min_{t\in[0,T]}\Big\{\frac{a_2(t)}{a_1(t) + c_1(t)M_{22}^{\alpha_2}}, \frac{b_2(t)}{b_1(t)}\Big\} \tag{3.10}$$

hold, then the species x_2 will be driven to extinction, that is, for any positive solution $(x_1(t), x_2(t))^T$ of system (1.8), $x_2(t) \to 0$ as $t \to +\infty$.

Proof. By (3.10) we can choose $\alpha, \tau, \varepsilon > 0$ such that

$$\begin{aligned} &\frac{m[r_2]}{m[r_1]} < \frac{\alpha}{\tau} - \varepsilon < \frac{\alpha}{\tau} \\ &< \min_{t\in[0,T]}\Big\{\frac{a_2(t)}{a_1(t) + c_1(t)(M_{22} + \varepsilon)^{\alpha_2}}, \frac{b_2(t)}{b_1(t)}\Big\}. \end{aligned} \tag{3.11}$$

Let $X(t) = (x_1(t), x_2(t))^T$ be a solution of system (1.8) with $x_i(0) > 0, i = 1, 2$. It follows from Lemma 2.4 that there exists an enough large $T_4 > 0$ such that

$$x_i(t) < M_{ii} \quad \text{for all} \quad t \geq T_4, \;\; i = 1, 2. \tag{3.12}$$

Also, for all $t \geq T_4$, it follows from (3.11) that

$$m[r_2]\tau - m[r_1]\alpha < -\varepsilon\tau m[r_1] < 0. \tag{3.13}$$

$$\alpha a_1(t) + \alpha c_1(t)(M_{22} + \varepsilon)^{\alpha_2} - \tau a_2(t) < 0, \tag{3.14}$$

$$\alpha b_1(t) - \tau b_2(t) < 0. \tag{3.15}$$

Let

$$V(t) = x_1^{-\alpha}(t)x_2^{\tau}(t).$$

By using (3.7), (3.14)-(3.15), we have

$$\dot{V}(t) \leq V(t)(\tau r_2(t) - \alpha r_1(t)).$$

Integrating this inequality from T_4 to $t (\geq T_4)$, it follows

$$V(t) \leq V(T_4)\exp\Big(\int_T^t(\tau r_2(t) - \alpha r_1(t))dt\Big). \tag{3.16}$$

Assume that $t \in [NT, (N + 1)T)$, then

$$\begin{aligned} &V(t) \\ \leq \;& V(T_4)\exp\Bigg(\int_{T_4}^{T_4+NT}(\tau r_2(t) - \alpha r_1(t))dt \\ &\int_{T_4+NT}^{t}(\tau r_2(t) - \alpha r_1(t))dt\Bigg) \\ \leq \;& V(T_4)\exp\Big(-\varepsilon\tau m[r_1]NT + (\tau r_2^u + \alpha r_1^u)T\Big) \\ \leq \;& V(T_4)\exp\Big(-\varepsilon\tau m[r_1](t - T_4) + \varepsilon\tau m[r_1]T \\ &+(\tau r_2^u + \alpha r_1^u)T\Big). \end{aligned} \tag{3.17}$$

And so,

$$x_2(t) < C_1\exp\big(-\varepsilon m[r_1](t - T_4)\big), \tag{3.18}$$

where

$$\begin{aligned} C_1 &= (M_{11})^{\alpha/\tau}(x_1(T_4))^{-\alpha/\tau}x_2(T_4)\times \\ &\quad \exp\big(\varepsilon m[r_1]T + (r_2^u + \tfrac{\alpha}{\tau}r_1^u)T\big) > 0. \end{aligned}$$

Therefore, we have $x_2(t) \to 0$ exponentially as $t \to +\infty$. This ends the proof of Theorem 3.2.

Theorem 3.3. *In addition to (H), assume further that*

$$\limsup_{t\to+\infty}\frac{r_2(t)}{r_1(t)} < \liminf_{t\to+\infty}\Big\{\frac{a_2(t)}{a_1(t)}, \frac{b_2(t)}{b_1(t) + c_1(t)M_1^{\alpha_1}}\Big\} \tag{3.19}$$

holds, then the species x_2 will be driven to extinction, that is, for any positive solution $(x_1(t), x_2(t))^T$ of system (1.8), $x_2(t) \to 0$ as $t \to +\infty$.

Theorem 3.4. *In addition to (H_1), assume further that*

$$\frac{m[r_2]}{m[r_1]} < \min_{t\in[0,T]} \Big\{\frac{a_2(t)}{a_1(t)}, \frac{b_2(t)}{b_1(t)+c_1(t)M_{11}^{\alpha_1}}\Big\} \qquad (3.20)$$

hold, then the species x_2 will be driven to extinction, that is, for any positive solution $(x_1(t), x_2(t))^T$ of system (1.8), $x_2(t) \to 0$ as $t \to +\infty$.

Since the proof of Theorem 3.3 and 3.4 is similarly to that of the proof of Theorem 3.1 and 3.2, we omit the detail here.

IV. Permanence of system (1.8)

The aim of this section is to obtain sufficient conditions to ensure the permanence of the system, which means that the species could be coexist in a long run.

Theorem 4.1. *In addition to (H), assume further that*

$$\begin{aligned} r_{1L} &> b_{1M}\frac{r_{2M}}{b_{2L}}, \\ r_{2L} &> a_{2M}\frac{r_{1M}}{a_{1L}} \end{aligned} \qquad (4.1)$$

holds, then system (1.8) is permanent, i.e., for any positive solution $(x_1(t), x_2(t))^T$ of system (1.8), there exists positive constants $m_i, M_i, i = 1, 2$, which are independent of the solution of system (1.8), such that

$$m_i \le \liminf_{t\to+\infty} x_i(t) < \limsup_{t\to+\infty} x_i(t) \le M_i,\ i = 1, 2.$$

Proof. Condition (4.1) implies that for the given enough small positive constants ε, the following inequalities hold:

$$\begin{aligned} r_{1L} &> b_{1M}\Big[\Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}}+\varepsilon\Big]^{\alpha_2}, \\ r_{2L} &> a_{2M}\Big[\Big(\frac{r_{1M}}{a_{1L}}\Big)^{\frac{1}{\alpha_1}}+\varepsilon\Big]^{\alpha_1} \end{aligned} \qquad (4.2)$$

holds. Indeed, we could choose ε be any positive constants which satisfies the inequality

$$\varepsilon < \min\Big\{\Big(\frac{r_{1L}}{b_{1M}}\Big)^{\frac{1}{\alpha_2}} - \Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}}, \Big(\frac{r_{2L}}{a_{2M}}\Big)^{\frac{1}{\alpha_1}} - \Big(\frac{r_{1M}}{a_{1L}}\Big)^{\frac{1}{\alpha_1}}\Big\}.$$

Let $x(t) = (x_1(t), x_2(t))^T$ be any positive solution of system (1.8) with initial condition $x_1(0) > 0, x_2(0) > 0$, then from Lemma 2.1, we have

$$\begin{aligned} \limsup_{t\to+\infty} x_1(t) &\le \Big(\frac{r_{1M}}{a_{1L}}\Big)^{\frac{1}{\alpha_1}}, \\ \limsup_{t\to+\infty} x_2(t) &\le \Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}}. \end{aligned} \qquad (4.3)$$

For $\varepsilon > 0$ enough small which satisfies (4.2), it follows from (4.3) that there exists enough large T, such that for all $t \ge T$,

$$\begin{aligned} x_1(t) &\le \Big(\frac{r_{1M}}{a_{1L}}\Big)^{\frac{1}{\alpha_1}} + \varepsilon, \\ x_2(t) &\le \Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}} + \varepsilon. \end{aligned} \qquad (4.4)$$

For $t \ge T$, it follows from (4.4) and the first equation of (1.8) that

$$\begin{aligned} &\dot{x}_1(t) \\ = \ & x_1(t)\Big[r_1(t) - a_1(t)x_1^{\alpha_1}(t) - b_1(t)x_2^{\alpha_2}(t) \\ &\quad -c_1(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)\Big] \\ \ge \ & x_1(t)\Big[r_{1L} - a_{1M}x_1^{\alpha_1}(t) \\ &\quad -b_{1M}\Big(\Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}}+\varepsilon\Big)^{\alpha_2} \\ &\quad -c_{1M}\Big(\Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}}+\varepsilon\Big)^{\alpha_2}x_1^{\alpha_1}(t)\Big]. \end{aligned} \qquad (4.5)$$

Applying Lemma 2.5, we have

$$\liminf_{t\to+\infty} x_1(t) \ge \Big(\frac{A_{1\varepsilon}}{A_{2\varepsilon}}\Big)^{\frac{1}{\alpha_1}},$$

where

$$\begin{aligned} A_{1\varepsilon} &= r_{1L} - b_{1M}\Big(\Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}}+\varepsilon\Big)^{\alpha_2}, \\ A_{2\varepsilon} &= a_{1M} + c_{1M}\Big(\Big(\frac{r_{2M}}{b_{2L}}\Big)^{\frac{1}{\alpha_2}}+\varepsilon\Big)^{\alpha_2}. \end{aligned} \qquad (4.6)$$

Since ε is any enough small positive constants, setting $\varepsilon \to 0$ in (4.5) leads to

$$\liminf_{t\to+\infty} x_1(t) \ge \frac{1}{2}\Big(\frac{A_1}{A_2}\Big)^{\frac{1}{\alpha_1}}, \qquad (4.7)$$

where

$$\begin{aligned} A_1 &= r_{1L} - b_{1M}\frac{r_{2M}}{b_{2L}}, \\ A_2 &= a_{1M} + c_{1M}\frac{r_{2M}}{b_{2L}}. \end{aligned} \qquad (4.8)$$

Similarly, by using the second inequality of (4.2) and the second equation of (1.8), we could obtain

$$\liminf_{t\to+\infty} x_2(t) \ge \frac{1}{2}\Big(\frac{B_1}{B_2}\Big)^{\frac{1}{\alpha_2}}, \qquad (4.9)$$

where

$$\begin{aligned} B_1 &= r_{2L} - a_{2M}\frac{r_{1M}}{a_{1L}}, \\ B_2 &= b_{2M} + c_{2M}\frac{r_{1M}}{a_{1L}}. \end{aligned} \qquad (4.10)$$

Since A_1, A_2, B_1, B_2 are all independent of the solution of system (1.8), it follows from (4.3), (4.7) and (4.8) that the system (1.8) is permanent. This ends the proof of Theorem 4.1.

Remark 4.1. It seems amazing since the conditions (4.1) is independent of the parameters α_1, α_2. Another finding is that under the assumption (4.1), the toxic substance term $c_i(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t), i = 1, 2$ has no influence on the permanence of the system.

V. Numeric simulations

The aim of this section is to verify that Example 1.1 and 1.2 satisfy the conditions of Theorem 3.1 and 3.2, respectively. We also give a numeric example to show the feasibility of the Theorem 4.1.

Example 5.1. For Example 1.1, one could easily verify that

$$M_2^{\alpha_2} = \frac{r_{2M}}{b_{2L}} = 3.$$

Thus

$$\begin{aligned} & \liminf_{t\to+\infty} \frac{a_2(t)}{a_1(t)+c_1(t)M_2^{\alpha_2}} \\ = & \liminf_{t\to+\infty} \frac{3+\frac{\sin t}{2}}{\frac{3}{2}+\cos t+0.3\times 3} \\ \geq & \frac{\frac{5}{2}}{\frac{5}{2}+1} = \frac{5}{7}. \end{aligned} \tag{5.1}$$

Also,

$$\liminf_{t\to+\infty} \frac{b_2(t)}{1_1(t)} \geq \frac{1}{\frac{3}{2}} = \frac{2}{3}. \tag{5.2}$$

Above computation shows that the coefficients of Example 1.1 satisfies the inequality (3.1), thus, the second species will be driven to extinction.

Example 5.2. Now let's consider Example 1.2. Noting that

$$\frac{m[r_2]}{m[r_1]} = \frac{1}{2}.$$

Also, in this case, one could easily verify that

$$M_2^{\alpha_2} = \frac{r_{2M}}{b_{2L}} = 5.$$

Thus

$$\begin{aligned} & \liminf_{t\to+\infty} \frac{a_2(t)}{a_1(t)+c_1(t)M_2^{\alpha_2}} \\ = & \liminf_{t\to+\infty} \frac{3+\frac{\sin t}{2}}{\frac{3}{2}+\cos t+0.3\times 5} \\ \geq & \frac{\frac{5}{2}}{\frac{5}{2}+\frac{3}{2}} = \frac{5}{8}. \end{aligned} \tag{5.3}$$

Also,

$$\liminf_{t\to+\infty} \frac{b_2(t)}{b_1(t)} \geq \frac{1}{\frac{3}{2}} = \frac{2}{3}. \tag{5.4}$$

Above computation shows that the coefficients of Example 1.2 satisfies the inequality (3.10), thus, the second species will be driven to extinction.

Example 5.3. Consider the following two-species competitive system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Big[4-2\cos t-(\frac{3}{2}+\frac{1}{2}\cos t)x_1^2(t) \\ &\quad -(\frac{1}{4}+\frac{1}{4}\cos t)x_2(t)-x_1^2(t)x_2(t)\Big], \\ \dot{x}_2(t) &= x_2(t)\Big[2-(\frac{1}{8}+\frac{\sin t}{8})x_1^2(t) \\ &\quad -(\frac{3}{2}+\frac{1}{2}\sin t)x_2(t)-x_1^2(t)x_2(t)\Big]. \end{aligned} \tag{5.5}$$

In this case, $r_1(t) = 4-2\cos t, r_2(t) = 2, a_1(t) = \frac{3}{2}+\frac{1}{2}\cos t, b_1(t) = \frac{1}{4}+\frac{1}{4}\cos t, c_1(t) = 1, a_2(t) = \frac{1}{8}+\frac{\sin t}{8}, b_2(t) = \frac{3}{2}+\frac{1}{2}\sin t, c_2(t) = 1, \alpha_1 = 2, \alpha_2 = 1.$ In (5.5), by simple computation, one could see that the coefficients of the system satisfies the following inequalities

$$\begin{aligned} r_{1L} = 2 > \frac{1}{2}\times\frac{2}{1} = b_{1M}\frac{r_{2M}}{b_{2L}}, \\ r_{2L} = 2 > \frac{1}{4}\times\frac{6}{1} = a_{2M}\frac{r_{1M}}{a_{1L}}. \end{aligned} \tag{5.6}$$

Hence, it follows from Theorem 4.1 that the system (5.5) is permanent. Numeric simulations (Fig.3 and 4) shows that in this case, the system is permanent.

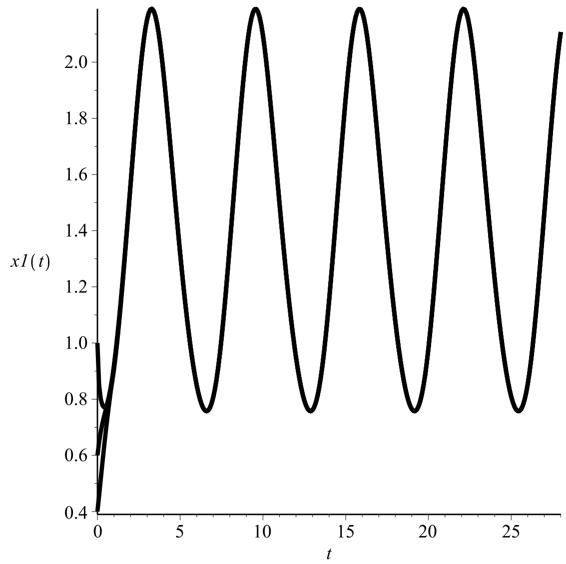

Fig. 3. Dynamics behaviors of species x_1 in system (5.5). Here, we take the initial conditions $(x_1(0), x_2(0)) = (1,1), (0.4, 0.4)$ and $(0.6, 0.6)$, respectively.

VI. Conclusion

In the previous works of Li and Chen [3] and Chen, Miao and Pu [4], the authors gave sufficient conditions which ensure the extinction of the second species in system (1.6) and (1.8), however, as we can see from the numeric examples of the introduction section, the species still maybe driven to extinction if the conditions in [3] and [4] are not hold. This motivated us to revisit the extinction property of system (1.8).

On the other hand, both Li and Chen [3] and Chen, Miao and Pu [4] did not investigate the persistent property of the system they considered, this motivated us to investigated the persistent property of the system (1.8).

By further developing the analysis technique of [4] and [3], more precisely, by constructing some suitable Lyapunov type extinction function, we are able to establish two set of new sufficient conditions which ensure the extinction of the second species. Also, by using the differential inequality theory, we obtain a set of sufficient conditions which ensure the permanence of the system. It seems amazing that both the nonlinear parameters α_1, α_2 and the toxic substance term have no influence to the persistent property of the system. Our results supplement the main results of [4] and [3].

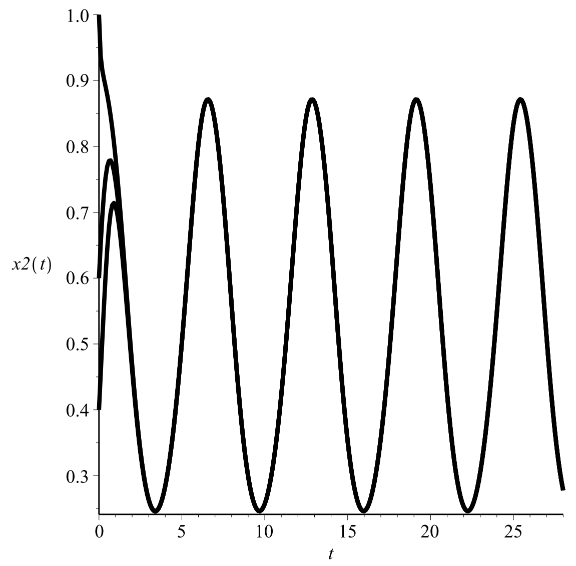

Fig. 4. Dynamics behaviors of species x_2 in system (5.5). Here, we take the initial conditions $(x_1(0), x_2(0)) = (1, 1), (0.4, 0.4)$ and $(0.6, 0.6)$, respectively.

VII. Declarations

Competing interests

The author declare that there is no conflict of interests.

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Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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