The Generalized Riemann Problem for a Simplified Model in Magnetogasdynamics

Yujin Liu and Wenhua Sun

***Abstract*—We consider the generalized Riemann problem for one dimensional ideal gas in Magnetogasdynamics in a neighborhood of the origin $(t > 0)$ in the (x, t) plane. According to the different cases of the corresponding Riemann solutions, we construct uniquely the perturbed solutions. We observe that the contact discontinuity appears for some cases after perturbation while there is no contact discontinuity of the corresponding Riemann solution. For most cases, the Riemann solutions are stable under such local small perturbations on the Riemann initial data. While for some few cases, the forward (backward) rarefaction wave can be transformed into the forward (backward) shock wave which reveal the instability of the Riemann solutions.**

***Index Terms*—Generalized Riemann problem, Magnetogasdynamics, Rarefaction wave, Shock wave.**

I. INTRODUCTION

THE inviscid and perfectly conducting compressible fluid ([1], [2], [3], [4], [5] and the references cited therein), subject to a transverse magnetic field, is given as follows

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u + pI) - \mu \mathrm{rot} H \times H = 0, \\ (\rho E + \frac{1}{2}\mu H^2)_t + \mathrm{div}(\rho u E + up - \mu(u \times H) \times H) = 0, \\ H_t - \mathrm{rot}(u \times H) = 0, \\ \mathrm{div} H = 0, \\ p = f(\rho, S) \end{cases} \tag{1}$$

where $\rho \geq 0, p, S, B \geq 0$ and $E = e + \frac{u^2}{2}$ are respectively the density, pressure, specific entropy, transverse magnetic field, and specific total energy, e is the specific internal energy. $u = (u_1, u_2, u_3)$ is the velocity of the fluid in the direction of (x_1, x_2, x_3), $H = (H_1, H_2, H_3)$ is the magnetic field in the direction of (x_1, x_2, x_3) and $H = \mu B$, where μ is the magnetic permeability.

T. Raja Sekhar and V.D. Sharma [6] investigated

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p + \frac{B^2}{2})_x = 0, \end{cases} \tag{2}$$

under the assumption $B = k\rho$, where k is positive constant, τ denotes the specific volume. They constructed the Riemann solutions (the other related problems in partial differential equations were investigated by many researchers ([7], [8], [9], [10], [11], etc.) and investigated the interactions of the elementary waves.

Shen [12] studied the Riemann solutions of (2) further and observed that the Riemann solutions of (2) converge to the corresponding Riemann solutions of the transport equations when both the pressure p and the magnetic field B vanish.

Hu and Sheng [2] studied

$$\begin{cases} \tau_t - u_x = 0, \\ u_t + (p + \frac{B^2}{2\mu})_x = 0, \\ (E + \frac{B^2\tau}{2\mu})_t + (pu + \frac{B^2 u}{2\mu})_x = 0, \end{cases} \tag{3}$$

and obtained constructively the unique solution of the Riemann problem with the characteristic method.

In [13], we removed the above assumption $B = k\rho$ and mainly considered the Riemann problem for isentropic, inviscid and perfectly conducting compressible fluid expressed by

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p + \frac{B^2}{2})_x = 0, \\ (B)_t + (Bu)_x = 0, \end{cases} \tag{4}$$

with the initial data

$$(\rho, u, B)(x, 0) = (\rho^\pm, u^\pm, B^\pm), \quad \pm x > 0, \tag{5}$$

where $\rho^\pm, u^\pm, B^\pm$ are arbitrary constants, $p = A\rho^\gamma$ holds for the polytropic gas, A is a positive constant and γ is the adiabatic constant.

Notice that the contact discontinuity is a plane curve in (ρ, u, B) and the projection on (ρ, u) is a straight line parallel to the ρ-axis which cause that the Riemann solutions for Magnetogasdynamics are much more complicated than that of the conventional gas dynamics.

In the present paper, we discuss the generalized Riemann problem of (4) with the initial data

$$(\rho, u, B)(x, 0) = (\rho_0^\pm, u_0^\pm, B_0^\pm)(x), \quad \pm x > 0, \tag{6}$$

where $\rho_0^\pm(x)$, $u_0^\pm(x)$, $B_0^\pm(x)$ are arbitrary smooth functions satisfying

$$\lim_{x \to 0\pm} (\rho_0^\pm, u_0^\pm, B_0^\pm)(x) = (\rho^\pm, u^\pm, B^\pm).$$

The initial value problem (4) and (6) can be regarded as a perturbation of the corresponding Riemann problem (4) and (5). Naturally, we wonder if the solutions of (4) and (6) are similar to the corresponding Riemann solutions of (4) and (5) in the neighborhood of the origin $(t > 0)$ in the (x, t) plane. According to the different cases of the corresponding Riemann solutions of (4) and (5), we construct the perturbed solutions with the characteristic method. We find that for some cases, the contact discontinuity appears

This work is supported by the Foundation for Young Scholars of Shandong University of Technology (No. 115024).

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after perturbation while the corresponding Riemann solution has no contact discontinuity. For most cases, the Riemann solutions are stable and the perturbation can not affect the corresponding Riemann solutions. For some few cases, the forward (backward) rarefaction wave can be transformed into the forward (backward) shock wave which reveals that the Riemann solutions are unstable under such small perturbations on the Riemann initial data.

The paper is organized as follows. Section II gives the elementary waves and the Riemann solutions of (4) and (5) for our later investigations. According to different cases of the corresponding Riemann solutions of (4) and (5), in Section III we construct the perturbed Riemann solutions of (4) and (6). A final conclusion is given in Section IV.

II. Preliminaries

First, we give briefly the elementary waves and the Riemann solutions for the system (4) and list some notations which are used in subsequent sections. We refer readers to [9], [13] for details.

There are three eigenvalues of (4) which are $\lambda = u = \lambda_0, \lambda = u \pm \sqrt{p_\rho + \frac{B^2}{\rho}} = \lambda_\pm$. They are real and distinct which shows that (4) is a strictly hyperbolic system. It is easy to see that the characteristic fields $\lambda_\pm$ are genuinely nonlinear and the characteristic field λ_0 is linearly degenerate.

The forward or backward rarefaction wave $\overset{\rightleftarrows}{R}$ in the (ρ, u, B) space consisting of all the states which can be connected to the state $Q_0(\rho_0, u_0, B_0)$ on the right is given by

$$\overset{\rightleftarrows}{R}(N_0): \begin{cases} B = k_0\rho, \\ u = u_0 \pm \int_{\rho_0}^{\rho} \sqrt{\frac{p_\rho + \frac{B^2}{\rho}}{\rho^2}} \mathrm{d}\rho, \end{cases} \tag{7}$$

where $k_0 = \frac{B_0}{\rho_0}$.

The contact discontinuity can be expressed as

$$J: \begin{cases} \sigma = u, \\ [u] = [p + \frac{B^2}{2}] = 0. \end{cases} \tag{8}$$

The forward or backward shock wave in the (ρ, u, B) space consisting of all the states which can be connected to the state $Q_0(\rho_0, u_0, B_0)$ on the right is given by

$$\overset{\rightleftarrows}{S}(Q_0): \begin{cases} B = k_0\rho, \\ u - u_0 = \pm\sqrt{\frac{(\rho-\rho_0)}{\rho\rho_0}(p + \frac{B^2}{2} - p_0 - \frac{B_0^2}{2})}. \end{cases} \tag{9}$$

Based on the above results, we are ready to construct the Riemann solution for (4) and (5). From the properties of the elementary waves, there is no asymptote for both $\overleftarrow{S}$ and $\overrightarrow{S}$ while both $\overleftarrow{R}$ and $\overrightarrow{R}$ intersect with the u-axis. Since the image of J in the space (ρ, u, B) is a plane curve and its projection on the plane (ρ, u) is a straight line parallel to the ρ-axis, we construct the solution of Riemann problem as follows.

Denote $\overleftarrow{W}_{lB}(N_{lB}) = \overleftarrow{R}_{lB}(N_{lB}) \cup \overleftarrow{S}_{lB}(N_{lB})$ and $\overrightarrow{W}_{rB}(N_{rB}) = \overrightarrow{R}_{rB}(N_{rB}) \cup \overrightarrow{S}_{rB}(N_{rB})$. Draw $\overleftarrow{W}_{lB}(N_{lB})$ from N_{lB} in the plane (ρ, u) and $\overrightarrow{W}_{rB}(N_{rB})$ from N_{rB}. According to the properties of $\overleftarrow{W}_{lB}(N_{lB})$ and $\overrightarrow{W}_{rB}(N_{rB})$, they intersect with each other at most once. Therefore, there are five cases: $\overleftarrow{W}_{lB}(N_{lB}) \cap \overrightarrow{W}_{rB}(N_{rB}) = (\overleftarrow{R}_{lB}(N_{lB}) \cap \overrightarrow{R}_{rB}(N_{rB}))$ or $(\overleftarrow{S}_{lB}(N_{lB}) \cap \overrightarrow{R}_{rB}(N_{rB}))$ or $(\overleftarrow{R}_{lB}(N_{lB}) \cap \overrightarrow{S}_{rB}(N_{rB}))$ or $(\overleftarrow{S}_{lB}(N_{lB}) \cap \overrightarrow{S}_{rB}(N_{rB}))$ or $\varnothing$.

For the last case, we easily know there is a vacuum solution. In what follows, we just need to consider the first case since the other cases can be studied similarly.

Suppose $\overleftarrow{W}_{lB}(N_{lB}) \cap \overrightarrow{W}_{rB}(N_{rB}) = \overleftarrow{R}_{lB}(N_{lB}) \cap \overrightarrow{R}_{rB}(N_{rB}) = \{N_{*B}\}$, we know there exists (ρ_*, u_*) satisfying

$$u_* = u_l - \int_{\rho_l}^{\rho_*} \frac{\sqrt{p_\rho + k_l^2\rho}}{\rho} \mathrm{d}\rho, \tag{10}$$

$$u_* = u_r + \int_{\rho_r}^{\rho_*} \frac{\sqrt{p_\rho + k_r^2\rho}}{\rho} \mathrm{d}\rho, \tag{11}$$

where $k_l = \frac{B_l}{\rho_l}$ and $k_r = \frac{B_r}{\rho_r}$.

Denote

$$f_1(\rho_1) = \begin{cases} u_l - \int_{\rho_l}^{\rho_1} \frac{\sqrt{p_\rho + k_l^2\rho}}{\rho} \mathrm{d}\rho, \ \rho \leq \rho_l, \\ u_l - \sqrt{\frac{\rho_1 - \rho_l}{\rho_1\rho_l}(p_1 + \frac{k_l^2\rho_1^2}{2} - p_l - \frac{k_l^2\rho_l^2}{2})}, \ \rho > \rho_l, \end{cases} \tag{12}$$

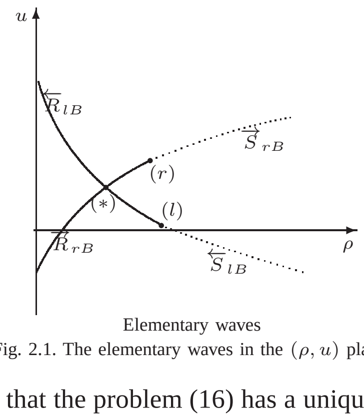
$$f_2(\rho_2) = \begin{cases} u_r + \int_{\rho_r}^{\rho_2} \frac{\sqrt{p_\rho + k_r^2\rho}}{\rho} \mathrm{d}\rho, \ \rho \leq \rho_r, \\ u_r + \sqrt{\frac{\rho_2 - \rho_r}{\rho_2\rho_r}(p_2 + \frac{k_r^2\rho_2^2}{2} - p_r - \frac{k_r^2\rho_r^2}{2})}, \ \rho > \rho_r, \end{cases} \tag{13}$$

$$g_1(\rho_1) = p_1 + \frac{k_l^2\rho_1^2}{2}, \tag{14}$$

$$g_2(\rho_2) = p_2 + \frac{k_r^2\rho_2^2}{2}. \tag{15}$$

Let

$$\begin{cases} f_1(\rho_1) = f_2(\rho_2), \\ g_1(\rho_1) = g_2(\rho_2). \end{cases} \tag{16}$$



Elementary waves

Fig. 2.1. The elementary waves in the (ρ, u) plane.

We will prove that the problem (16) has a unique solution, which implies that there exists a unique contact discontinuity J joining the two states which are located on $\overset{\rightleftarrows}{R}$ and $\overset{\rightleftarrows}{S}$ respectively (Fig. 2.1).

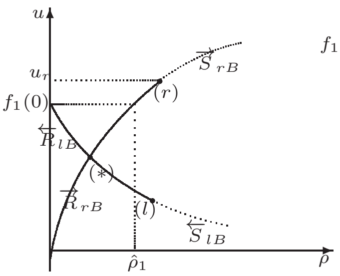
Since $f_1(\rho_1)$ and $f_2(\rho_2)$ are both smooth functions, and the curve $\rho_2 = \rho_2(\rho_1)$ defined by $f_1(\rho_1) = f_2(\rho_2)$ is monotonically decreasing, while the curve $\rho_2 = \rho_2(\rho_1)$ defined by $g_1(\rho_1) = g_2(\rho_2)$ is monotonically increasing.

Thus, we obtain the uniqueness of the solution of (16). Next we discuss the existence of the solution of (16).

Form (10) we have $f_1(\rho_*) = f_2(\rho_*)$, we proceed as follows.

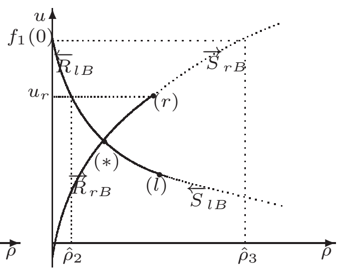
Case 1. $k_l = k_r$. It is equivalent to $g_1(\rho_*) = g_2(\rho_*)$. It is obvious that $\rho_1 = \rho_2 = \rho_*$ is the solution of (16). Thus, the Riemann solution is $\overleftarrow{R} + \overrightarrow{R}$, here the symbol " $+$ " means "followed by". We notice that for this case there is no contact discontinuity.

Case 2. $k_l > k_r$. It is equivalent to $g_1(\rho_*) > g_2(\rho_*)$. So we should look for solution in $(\rho_1, \rho_2): \ \rho_1 < \rho_*, \ \rho_2 > \rho_*$.



Subcase 2.1

Fig. 2.2. The description for Subcase 2.1



Subcase 2.2

Fig. 2.3. The description for Subcase 2.2

Subcase 2.1 $u_r \geq f_1(0)$**.** (see Fig. 2.2.)

There exists $\hat{\rho}_1$ such that $f_1(0) = f_2(\hat{\rho}_1)$, where $\rho_* < \hat{\rho}_1 < \rho_r$.

Since $g_1(0) < g_2(\hat{\rho}_1)$ and the curves $f_1(\rho_1)$, $f_2(\rho_2)$ are smooth, from the method of continuity, there exists $(\bar{\rho}_1, \bar{\rho}_2)$ satisfying $0 < \bar{\rho}_1 < \rho_*, \rho_* < \bar{\rho}_2 < \hat{\rho}_1$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. Thus, $(\bar{\rho}_1, \bar{\rho}_2)$ is the solution of (16). It follows that the Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{R}$.

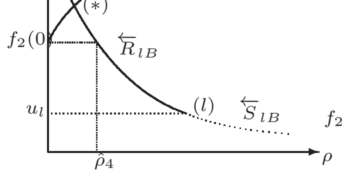
Subcase 2.2 $u_r < f_1(0)$**.** (see Fig. 2.3.)

There exists $\hat{\rho}_2$ and $\hat{\rho}_3$ satisfying $f_1(\hat{\rho}_2) = u_r$ and $f_1(0) = f_2(\hat{\rho}_3)$, respectively, where $0 < \hat{\rho}_2 < \rho_*$ and $\hat{\rho}_3 > \rho_r$.

Subcase 2.2.1 $g_2(\rho_r) < g_1(\hat{\rho}_2)$**.** Since $g_1(\hat{\rho}_3) > g_1(0)$ and the curves $f_1(\rho_1)$, $f_2(\rho_2)$ are smooth, from the method of continuity, there exists $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \hat{\rho}_2, \rho_r < \bar{\rho}_2 < \hat{\rho}_3$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. Thus, $(\bar{\rho}_1, \bar{\rho}_2)$ is the solution of (16). The Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{S}$.

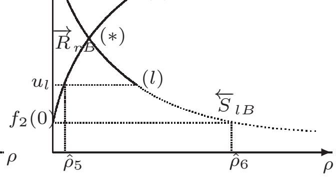
Subcase 2.2.2 $g_2(\rho_r) \geq g_1(\hat{\rho}_2)$**.** Similarly, we know that there exists $(\bar{\rho}_1, \bar{\rho}_2) : \hat{\rho}_2 < \bar{\rho}_1 < \rho_*, \rho_* < \bar{\rho}_2 < \rho_r$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. The Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{R}$.

Case 3. $k_l < k_r$. It is equivalent to $g_1(\rho_*) < g_2(\rho_*)$. So we should look for solution in $(\rho_1, \rho_2): \ \rho_1 > \rho_*, \ \rho_2 < \rho_*$.



Subcase 3.1

Fig. 2.4. The description for Subcase 3.1



Subcase 3.2

Fig. 2.5. The description for Subcase 3.2

Subcase 3.1 $u_l \leq f_2(0)$**.** (see Fig. 2.4.)

It is obvious that there exists $\hat{\rho}_4$ such that $f_1(\hat{\rho}_4) = f_2(0)$, where $\rho_* < \hat{\rho}_4 < \rho_l$. Similar discussions as above, we obtain $(\bar{\rho}_1, \bar{\rho}_2) : \rho_* < \bar{\rho}_1 < \hat{\rho}_4, 0 < \bar{\rho}_2 < \rho_*$. It follows that the Riemann solution is $\overleftarrow{R} + \overset{>}{J} + \overrightarrow{R}$.

Subcase 3.2 $u_l > f_2(0)$**.** (see Fig. 2.5.) There exist $\hat{\rho}_5$ such that $f_2(\hat{\rho}_5) = u_l$, where $0 < \hat{\rho}_5 < \rho_*$, and $\hat{\rho}_6$ such that $f_1(\hat{\rho}_6) = f_2(0)$, where $\hat{\rho}_6 > \rho_l$.

Subcase 3.2.1 $g_1(\rho_l) \geq g_2(\hat{\rho}_5)$**.** Similarly, we know that there exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_* < \bar{\rho}_1 < \rho_l, \hat{\rho}_5 < \bar{\rho}_2 < \rho_*$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. The Riemann solution is $\overleftarrow{R} + \overset{>}{J} + \overrightarrow{R}$.

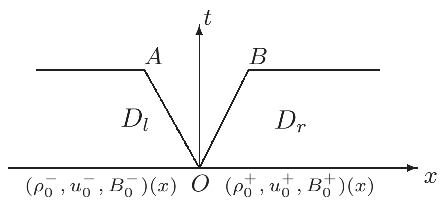
Subcase 3.2.2 $g_1(\rho_l) < g_2(\hat{\rho}_5)$**.** Similarly, we know that there exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_l < \bar{\rho}_1 < \hat{\rho}_6, 0 < \bar{\rho}_2 < \hat{\rho}_5$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. The Riemann solution is $\overleftarrow{S} + \overset{>}{J} + \overrightarrow{R}$.

Based on the above results, we have the following result.

Theorem 2.1 There exists uniquely solution for the Riemann problem (4) with the initial values (5).

III. The generalized Riemann problem

Now we investigate the construction of the solutions for the discontinuous initial value problem (4) with (6) in a neighborhood of the origin $(t > 0)$ on the (x, t) plane. From the results in [14] and [15], the classical solution $(\rho_l, u_l, B_l)(x, t)$ $((\rho_r, u_r, B_r)(x, t))$ can be defined in a strip domain $D_l(D_r)$ for a local time. The right boundary of D_l has characteristic $OA : x = \lambda_- t$, and the left boundary of D_r has characteristic $OB : x = \lambda_+ t$ (see Fig. 3.1).



local region

Fig. 3.1. The region of perturbed solution in (x, t) plane.

According to the different cases of the corresponding Riemann solutions of (4) and (5), we construct the solutions case by case for (4) with (6). For simplicity, we only consider some interesting phenomena. For the other cases, similar discussions can be carried out and omitted here. For simplicity, we use the same symbols after perturbation since there is no any confusion.

Case 1. When $k_l = k_r$, the corresponding Riemann solution is $\overleftarrow{R} + \overrightarrow{R}$.

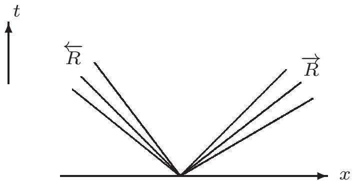
After perturbation, we obtain two subcases which are $k_l > k_r$ or $k_l < k_r$. In what follows, we discuss our problem in two subcases.

Subcase 1.1. $k_l > k_r$.

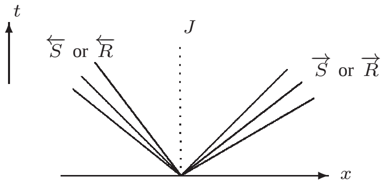
Subcase 1.1.1. If $u_r \geq f_1(0)$ or $u_r < f_1(0)$ and $g_2(\rho_r) \geq g_1(\hat{\rho}_2)$ after perturbation, and it follows that the perturbed Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{R}$. For this case, the corresponding Riemann solution is stable under such small perturbation.

Subcase 1.1.2. If $u_r < f_1(0)$and $g_2(\rho_r) < g_1(\hat{\rho}_2)$ after perturbation, we obtain the perturbed Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{S}$. For this case, the corresponding Riemann solution is unstable under such small perturbation.

Subcase 1.2. $k_l < k_r$.
Subcase 1.2.1. If $u_l \leq f_2(0)$ or $u_l > f_2(0)$ and $g_1(\rho_l) \geq g_2(\hat{\rho}_5)$ after perturbation, it follows that the perturbed Riemann solution is $\overleftarrow{R} + \overset{>}{J} + \overrightarrow{R}$.
Subcase 1.2.2. If $u_l > f_2(0)$ and $g_1(\rho_l) < g_2(\hat{\rho}_5)$ after perturbation, we obtain that the perturbed Riemann solution is $\overleftarrow{S} + \overset{>}{J} + \overrightarrow{R}$.



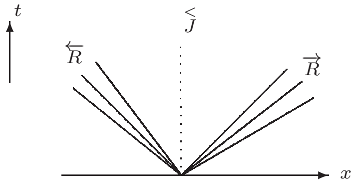
Riemann solution
Fig. 3.2. The Riemann solution in Case 1.



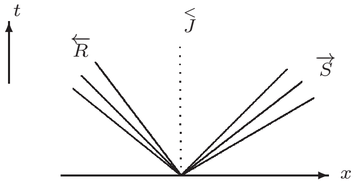
perturbed solution
Fig. 3.3. The perturbed solution in Case 1.

Theorem 3.1 For Case 1. although there is no contact discontinuity of the corresponding Riemann solution, the contact discontinuity appears after perturbation. Furthermore, we find that the corresponding Riemann solution is stable for for Subcase 1.1.1. and Subcase 1.2.1., while for Subcase 1.1.2. and Subcase 1.2.2., the backward (forward) rarefaction wave of the corresponding Riemann solution can be transformed into the backward (forward) shock wave after perturbation which reveals that the Riemann solution is unstable under such small perturbation. (see Fig. 3.2. and Fig. 3.3.).
Case 2. $k_l > k_r$.

After perturbation, we still have $k_l > k_r$. We construct the perturbed Riemann problem as follows.
Subcase 2.1. If $u_r \geq f_1(0)$, the corresponding Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{R}$. After perturbation, we obtain that $u_r \geq f_1(0)$ or $u_r < f_1(0)$.
Subcase 2.1.1 When $u_r \geq f_1(0)$ or $u_r < f_1(0)$ and $g_2(\rho_r) \geq g_1(\hat{\rho}_2)$ after perturbation, we obtain the perturbed Riemann solution is still $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{R}$.
Subcase 2.1.2 When $u_r < f_1(0)$ and $g_2(\rho_r) < g_1(\hat{\rho}_2)$ after perturbation, we have the perturbed Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{S}$.
Subcase 2.2. If $u_r < f_1(0)$ and $g_2(\rho_r) \geq g_1(\hat{\rho}_2)$, the corresponding Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{R}$. After perturbation, we have $u_r < f_1(0)$ and $g_2(\rho_r) \geq g_1(\hat{\rho}_2)$ or $g_2(\rho_r) < g_1(\hat{\rho}_2)$, we construct the perturbed Riemann solutionis as follows.
Subcase 2.2.1 When $u_r < f_1(0)$ and $g_2(\rho_r) \geq g_1(\hat{\rho}_2)$ after perturbation, we obtain the perturbed Riemann solution is still $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{R}$.
Subcase 2.2.2 When $u_r < f_1(0)$ and $g_2(\rho_r) < g_1(\hat{\rho}_2)$ after perturbation, the perturbed Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{S}$.
Subcase 2.3. If $u_r < f_1(0)$ and $g_2(\rho_r) < g_1(\hat{\rho}_2)$, the corresponding Riemann solution is $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{S}$. After perturbation, we still have $u_r < f_1(0)$ and $g_2(\rho_r) < g_1(\hat{\rho}_2)$ and it follows that the perturbed Riemann solution is still $\overleftarrow{R} + \overset{<}{J} + \overrightarrow{S}$.



perturbed solution
Fig. 3.4. The perturbed solutions in Case 2.



perturbed solution
Fig. 3.5. The perturbed solutions in Case 2.

Theorem 3.2 The corresponding Riemann solution remain unchanged for most cases after perturbation, which shows that the corresponding Riemann solution is stable and the perturbation can not affect the corresponding Riemann solution. While for few cases such as Subcase 2.1.2 and Subcase 2.2.2, the forward rarefaction wave of the corresponding Riemann solution can be transformed into the forward shock wave after perturbation which shows that the corresponding Riemann solution is unstable under such small perturbation. (see Fig. 3.4. and Fig. 3.5.)
Case 3. $k_l < k_r$.

After perturbation, it still holds that $k_l < k_r$.
Subcase 3.1. If $u_l \leq f_2(0)$, the corresponding Riemann solution is $\overleftarrow{R} + \overset{>}{J} + \overrightarrow{R}$. After perturbation, we obtain that $u_l \leq f_2(0)$ or $u_l > f_2(0)$, and construct the perturbed Riemann solutionis as follows.
Subcase 3.1.1 When $u_l \leq f_2(0)$ or $u_l > f_2(0)$ and $g_1(\rho_l) \geq g_2(\hat{\rho}_5)$ after perturbation, we get the perturbed Riemann solution is $\overleftarrow{R} + \overset{>}{J} + \overrightarrow{R}$.
Subcase 3.1.2 When $u_- > f_2(0)$ and $g_1(\rho_l) < g_2(\hat{\rho}_5)$ after perturbation, we obtain that the perturbed Riemann solution is $\overleftarrow{S} + \overset{>}{J} + \overrightarrow{R}$.
Subcase 3.2. If $u_l > f_2(0)$ and $g_1(\rho_l) \geq g_2(\hat{\rho}_5)$, the corresponding Riemann solution is $\overleftarrow{R} + \overset{>}{J} + \overrightarrow{R}$. After perturbation, we have $u_l > f_2(0)$, $g_1(\rho_l) \geq g_2(\hat{\rho}_5)$ or $g_1(\rho_l) < g_2(\hat{\rho}_5)$, we construct the perturbed Riemann solutionis as follows.
Subcase 3.2.1 When $u_l > f_2(0)$ and $g_1(\rho_l) \geq g_2(\hat{\rho}_5)$ after perturbation, we get that the perturbed Riemann solution is still $\overleftarrow{R} + \overset{>}{J} + \overrightarrow{R}$.
Subcase 3.2.2 When $u_l > f_2(0)$ and $g_1(\rho_l) < g_2(\hat{\rho}_5)$ after perturbation, we obtain that the perturbed Riemann solution is $\overleftarrow{S} + \overset{>}{J} + \overrightarrow{R}$.
Subcase 3.3. If $u_l > f_2(0)$ and $g_1(\rho_l) < g_2(\hat{\rho}_5)$, the corresponding Riemann solution is $\overleftarrow{S} + \overset{>}{J} + \overrightarrow{R}$. After perturbation, we find that it still holds that $u_l > f_2(0)$ and $g_1(\rho_l) < g_2(\hat{\rho}_5)$ and it follows that the perturbed Riemann solution is $\overleftarrow{S} + \overset{>}{J} + \overrightarrow{R}$.

Theorem 3.3 For Subcase 3.1.1, Subcase 3.2.1 and Subcase 3.3., the corresponding Riemann solution are stable. While for Subcase 3.1.2 and Subcase 3.2.2, the backward rarefaction wave of the corresponding Riemann solution can be transformed into the backward shock wave after perturbation which shows that the corresponding Riemann solution is unstable under such small perturbation. (see Fig. 3.6. and Fig. 3.7.)

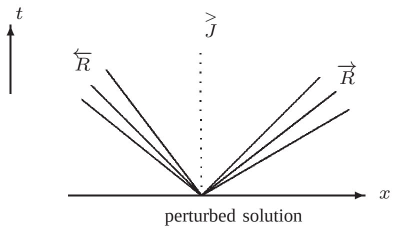


Fig. 3.6. The perturbed solutions in Case 3.

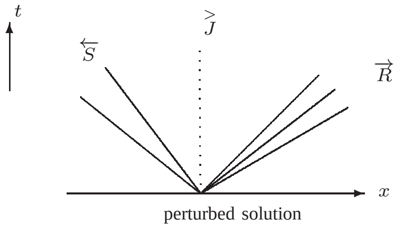


Fig. 3.7. The perturbed solutions in Case 3.

IV. Conclusion

In this work, we find that there exists a unique piecewise smooth solution of the generalized Riemann problem (4) with the initial data (6). The contact discontinuity may appear after perturbation while there is no contact discontinuity of the corresponding Riemann solution.

For most cases the perturbed Riemann solutions of (4) and (6) are stable under such a perturbation on the initial data. For some few cases, we observe the instability of the perturbed Riemann solutions of (4) and (6) under such local small perturbations on the Riemann initial values.

Since the reaction rate in our model is infinite which is an idealized hypothesis, while our model is still very important in application, we will investigate the initial value problem for the self-similar Zeldovich-von Neumann-Döring (ZND) model in magnetogasdynamic combustion with finite reaction rate in our coming works.

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