New Oscillatory Criteria for a Class of Fractional Differential Equations

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Abstract—**In this paper, based on certain Riccati transformation, inequality and integration average technique, oscillation for a class of fractional differential equations with damping term is investigated. The fractional derivative is defined in the sense of the conformable fractional derivative. Some new oscillatory criteria for this equation are established. For illustrating the oscillatory criteria established, some examples are also presented.**

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Index Terms—**Oscillation; Fractional differential equations; Conformable fractional derivative; Damping term; Riccati transformation**

I. Introduction

In the research of the theory of differential equations, difference equations, dynamic equations on time scales and so on, if their solutions are unknown, then it is necessary and important to research the qualitative properties and quantitative properties of the solutions such as the existence and uniqueness of solutions [1-3], seeking for exact solutions [4-6], numerical methods [7-10]. Oscillation belongs to the range of qualitative properties analysis. In the last few decades, research for oscillation of various equations including differential equations, difference equations and dynamic equations on time scales etc. has been a hot topic in the literature, and much effort has been done to establish new oscillatory criteria for these equations so far (for example, see [11-25], and the references therein). In these investigations, we notice that relatively less attention has been paid to the research of oscillation of fractional differential equations.

In [26], Grace et al. researched oscillation of the following fractional differential equation:

$$D_a^q x + f_1(t,x) = v(t) + f_2(t,x), \lim_{t\to a+} J_a^{1-q}x(t) = b_1,$$

under the conditions

$$xf_i(t,x) > 0,\ i = 1,2,\ x \neq 0,\ t \geq a,$$

and

$$|f_1(t,x)| \geq p_1(t)|x|^\beta,\ |f_2(t,x)| \geq p_2(t)|x|^\gamma,\ x \neq 0,\ t \geq a,$$

where D_a^q and J_a^p denote the Riemann-Liouville derivative of order q and the Riemann-Liouville fractional integral operator respectively. By reducing the fractional differential equation to the equivalent Volterra fractional integral equation and by use of certain inequality techniques, some new oscillation criteria were established.

In [27,28], the authors investigated oscillation of the following two fractional differential equations:

$$D_a^\alpha x(t) + q(t)f(x(t)) = 0,$$

and

$$(D_{0+}^{1+\alpha}y)t + p(t)(D_{0+}^\alpha y)t + q(t)f(y(t)) = 0,$$

where the fractional derivative is defined by the Riemann-Liouville derivative.

In [29], Chen researched oscillation of the following fractional differential equation:

$$[r(t)(D_-^\alpha y(t))^\eta]' - q(t)f(\textstyle\int_t^\infty (v-t)^{-\alpha}y(v)dv) = 0,\ t > 0,$$

where $r,\ q$ are positive-valued functions, η is the quotient of two odd positive numbers, $\alpha \in (0,1)$, $D_-^\alpha y(t)$ denotes the Liouville right-sided fractional derivative of order α of y, and $D_-^\alpha y(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty (\xi - t)^{-\alpha}y(\xi)d\xi$. Then in [30], under similar conditions to [29], some new oscillatory criteria are established for the following fractional differential equation with damping term:

$$D_-^{1+\alpha}y(t) - p(t)D_-^\alpha y(t) + q(t)f(\textstyle\int_t^\infty (v-t)^{-\alpha}y(v)dv) = 0,\ t > 0,$$

In [31], Han et al. investigated oscillation of a class of fractional differential equations as follows

$$[r(t)g((D_-^\alpha y)(t))]' - p(t)f(\textstyle\int_t^\infty (s-t)^{-\alpha}y(s)ds) = 0,\ t > 0,\ \alpha \in (0,1).$$

Recently, Khalil et al. proposed a new definition for fractional derivative named conformable fractional derivative [32]. The fractional derivative is defined as follows

$$D^\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

and satisfies the following properties:

(i). $D^\alpha[af(t) + bg(t)] = aD^\alpha f(t) + bD^\alpha g(t)$.

(ii). $D^\alpha(t^\gamma) = \gamma t^{\gamma-\alpha}$.

(iii). $D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t)$.

(iv). $D^\alpha C = 0$, where C is a constant.

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(v). $D_t^\alpha f[g(t)] = f_g'[g(t)] D_t^\alpha g(t)$.

(vi). $D_t^\alpha (\frac{f}{g})(t) = \frac{g(t)D^\alpha f(t) - f(t)D^\alpha g(t)}{g^2(t)}$.

(vii). $D_t^\alpha f(t) = t^{1-\alpha} f'(t)$.

Note that the properties above can be easily proved due to the definition of the conformable fractional derivative. Afterwards, many authors investigated various applications of the conformable fractional derivative [33-38].

Motivated by the analysis above, in this paper, we are concerned with oscillation of a class of fractional differential equations with damping term as follows:

$$D_t^\alpha(r(t)D_t^\alpha x(t)) + p(t)D_t^\alpha x(t) + q(t)f(x(t)) = 0,\ t \geq t_0 > 0,\ 0 < \alpha < 1, \tag{1}$$

where $D_t^\alpha(.)$ denotes the conformable fractional derivative with respect to the variable t, the function $r \in C^\alpha([t_0,\infty), R_+)$, $p,\ q \in C([t_0,\infty), R_+)$, and C^α denotes continuous derivative of order α, the function f is continuous satisfying $f(x)/x \geq K$ for some positive constant K and $\forall x \neq 0$.

As usual, a solution $x(t)$ of Eq. (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Eq. (1) is called oscillatory if all its solutions are oscillatory.

We organize the next of this paper as follows. In Section 2, using Riccati transformation, inequality and integration average technique, we establish some new oscillatory criteria for Eq. (1), while we present some applications for them in Section 3.

For the sake of convenience, in the next of this paper, we denote $\xi = \frac{t^\alpha}{\alpha}$, $\xi_i = \frac{t_i^\alpha}{\alpha}$, $i = 0,1,2,3$, $R_+ = (0,\infty)$, $r(t) = \widetilde{r}(\xi)$, $p(t) = \widetilde{p}(\xi)$, $q(t) = \widetilde{q}(\xi)$, and $A(\xi) = exp(\int_{\xi_0}^{\xi} \frac{\widetilde{p}(\tau)}{\widetilde{r}(\tau)} d\tau)$.

II. Oscillatory criteria for Eq. (1)

Lemma 1. Assume $x(t)$ is a eventually positive solution of Eq. (1), and

$$\int_{\xi_0}^{\infty} \frac{1}{A(s)\widetilde{r}(s)} ds = \infty. \tag{2}$$

Then there exists a sufficiently large T such that $D_t^\alpha x(t) > 0$ for $t \in [T,\infty)$.

Proof. Let $x(t) = \widetilde{x}(\xi)$, where $\xi = \frac{t^\alpha}{\alpha}$. Then by use of the property (ii) we obtain $D_t^\alpha \xi(t) = 1$, and furthermore by use of the property (v), we have

$$D_t^\alpha r(t) = D_t^\alpha \widetilde{r}(\xi) = \widetilde{r}'(\xi) D_t^\alpha \xi(t) = \widetilde{r}'(\xi).$$

Similarly we have $D_t^\alpha x(t) = \widetilde{x}'(\xi)$. So Eq. (1) can be transformed into the following form:

$$(\widetilde{r}(\xi)\widetilde{x}'(\xi))' + \widetilde{p}(\xi)\widetilde{x}'(\xi) + \widetilde{q}(\xi) f(\widetilde{x}(\xi)) = 0,\ \xi \geq \xi_0 > 0, \tag{3}$$

Since $x(t)$ is a eventually positive solution of (1), then $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (3), and there exists $\xi_1 > \xi_0$ such that $\widetilde{x}(\xi) > 0$ on $[\xi_1,\infty)$. Furthermore, we have

$$\begin{aligned}(A(\xi)\widetilde{r}(\xi)\widetilde{x}'(\xi))' &= A(\xi)(\widetilde{r}(\xi)\widetilde{x}'(\xi))' + A(\xi)\widetilde{p}(\xi)\widetilde{x}'(\xi) \\ &= -\widetilde{q}(\xi)A(\xi)f(\widetilde{x}(\xi)) \\ &\leq -KA(\xi)\widetilde{q}(\xi)\widetilde{x}(\xi) < 0,\ \xi \geq \xi_1. \end{aligned} \tag{4}$$

Then $A(\xi)\widetilde{r}(\xi)\widetilde{x}'(\xi)$ is strictly decreasing on $[\xi_1,\infty)$, and thus $\widetilde{x}'(\xi)$ is eventually of one sign. We claim $\widetilde{x}'(\xi) > 0$ on $[\xi_2,\infty)$, where $\xi_2 > \xi_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $\xi_3 > \xi_2$ such that $\widetilde{x}'(\xi) < 0$ on $[\xi_3,\infty)$. Then for $\xi \in [\xi_3,\infty)$ we have

$$\widetilde{x}(\xi) - \widetilde{x}(\xi_3) = \int_{\xi_3}^{\xi} \widetilde{x}'(s)ds = \int_{\xi_3}^{\xi} \frac{A(s)\widetilde{r}(s)\widetilde{x}'(s)}{A(s)\widetilde{r}(s)} ds \leq A(\xi_3)\widetilde{r}(\xi_3)\widetilde{x}'(\xi_3) \int_{\xi_3}^{\xi} \frac{1}{A(s)\widetilde{r}(s)} ds.$$

By (2) we deduce that $\lim\limits_{\xi\to\infty} \widetilde{x}(\xi) = -\infty$, which contradicts to the fact that $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (3). So $\widetilde{x}'(\xi) > 0$ on $[\xi_2,\infty)$, and furthermore $D_t^\alpha x(t) > 0$ on $[t_2,\infty)$. The proof is complete by setting $T = t_2$.

Theorem 2. Assume (2) holds, and there exist two functions $\phi \in C^1([t_0,\infty), R_+)$ and $\varphi \in C^1([t_0,\infty), [0,\infty))$ such that

$$\int_{\xi_0}^{\infty} \{KA(s)\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\} ds = \infty, \tag{5}$$

where $\widetilde{\phi}(\xi) = \phi(t)$, $\widetilde{\varphi}(\xi) = \varphi(t)$. Then every solution of Eq. (1) is oscillatory.

Proof. Assume (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_-^\alpha x(t) > 0$ on $[t_2,\infty)$ for some sufficiently large $t_2 > t_1$. Define the generalized Riccati transformation function:

$$\omega(t) = \phi(t)\{\frac{A(\xi)r(t)D_t^\alpha x(t)}{x(t)} + \varphi(t)\}.$$

Then for $t \in [t_2,\infty)$, we have

$$\begin{aligned} D_t^\alpha \omega(t) &= D_t^\alpha \phi(t) \frac{A(\xi)r(t)D_t^\alpha x(t)}{x(t)} \\ &\quad -\phi(t)\frac{A(\xi)r(t)(D_t^\alpha x(t))^2}{x^2(t)} + \phi(t)\frac{D_t^\alpha(A(\xi)r(t)D_t^\alpha x(t))}{x(t)} \\ &\quad + D_t^\alpha \phi(t)\varphi(t) + \phi(t)D_t^\alpha \varphi(t) \\ &= \frac{D_t^\alpha \phi(t)}{\phi(t)}\omega(t) - \frac{(\omega(t) - \phi(t)\varphi(t))^2}{A(\xi)\phi(t)r(t)} \\ &\quad +\phi(t)\frac{[A(\xi)D_t^\alpha(r(t)D_t^\alpha x(t)) + r(t)D_t^\alpha x(t) D_t^\alpha(A(\xi))]}{x(t)} \\ &\quad +\phi(t)D_t^\alpha \varphi(t) \\ &= \frac{D_t^\alpha \phi(t)}{\phi(t)}\omega(t) - \frac{(\omega(t) - \phi(t)\varphi(t))^2}{A(\xi)\phi(t)r(t)} \\ &\quad +\phi(t)\frac{[A(\xi)D_t^\alpha(r(t)D_t^\alpha x(t)) + r(t)D_t^\alpha x(t) A'(\xi) D_t^\alpha \xi]}{x(t)} \end{aligned}$$

$$
\begin{aligned}
&+\phi(t)D_t^\alpha\varphi(t)\\
&=\frac{D_t^\alpha\phi(t)}{\phi(t)}\omega(t)-\frac{(\omega(t)-\phi(t)\varphi(t))^2}{A(\xi)\phi(t)r(t)}\\
&+\phi(t)\frac{[A(\xi)D_t^\alpha(r(t)D_t^\alpha x(t))+r(t)D_t^\alpha x(t)A(\xi)\frac{\widetilde{p}(\xi)}{\widetilde{r}(\xi)}]}{x(t)}\\
&+\phi(t)D_t^\alpha\varphi(t)\\
&=\frac{D_t^\alpha\phi(t)}{\phi(t)}\omega(t)-\frac{(\omega(t)-\phi(t)\varphi(t))^2}{A(\xi)\phi(t)r(t)}\\
&+\phi(t)\frac{[A(\xi)D_t^\alpha(r(t)D_t^\alpha x(t))+p(t)D_t^\alpha x(t)A(\xi)]}{x(t)}\\
&+\phi(t)D_t^\alpha\varphi(t)\\
&=\frac{D_t^\alpha\phi(t)}{\phi(t)}\omega(t)-\frac{(\omega(t)-\phi(t)\varphi(t))^2}{A(\xi)\phi(t)r(t)}\\
&c-\phi(t)\frac{A(\xi)q(t)f(x(t))}{x(t)}+\phi(t)D_t^\alpha\varphi(t)\\
&=-\phi(t)\frac{A(\xi)q(t)f(x(t))}{x(t)}+\phi(t)D_t^\alpha\varphi(t)\\
&-\frac{\phi(t)\varphi^2(t)}{A(\xi)r(t)}+\frac{2\phi(t)\varphi(t)+A(\xi)D_t^\alpha\phi(t)r(t)}{A(\xi)\phi(t)r(t)}\omega(t)\\
&-\frac{1}{A(\xi)\phi(t)r(t)}\omega^2(t)\\
&\le -KA(\xi)\phi(t)q(t)+\phi(t)D_t^\alpha\varphi(t)-\frac{\phi(t)\varphi^2(t)}{A(\xi)r(t)}\\
&+\frac{[2\phi(t)\varphi(t)+A(\xi)D_t^\alpha\phi(t)r(t)]^2}{4A(\xi)\phi(t)r(t)}. \qquad (6)
\end{aligned}
$$

Let $\omega(t)=\widetilde{\omega}(\xi)$. Then $D_t^\alpha w(t)=\widetilde{w}'(\xi)$, $D_t^\alpha\phi(t)=\widetilde{\phi}'(\xi)$, $D_t^\alpha\varphi(t)=\widetilde{\varphi}'(\xi)$, and (6) is transformed into the following form

$$
\widetilde{\omega}'(\xi)\le -KA(\xi)\widetilde{\phi}(\xi)\widetilde{q}(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)-\frac{\widetilde{\phi}(\xi)\widetilde{\varphi}^2(\xi)}{A(\xi)\widetilde{r}(\xi)}+\frac{[2\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)+A(\xi)\widetilde{\phi}'(\xi)\widetilde{r}(\xi)]^2}{4A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)},\xi\ge\xi_2. \qquad (7)
$$

Substituting ξ with s in (7), an integration for (7) with respect to s from ξ_2 to ξ yields

$$
\begin{aligned}
&\int_{\xi_2}^{\xi}\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\\
&\le\widetilde{\omega}(\xi_2)-\omega(\xi)\le\omega(\xi_2)<\infty,
\end{aligned}
$$

which contradicts to (5). So the proof is complete.

Theorem 3. Assume (2) holds, and there exists a function $H\in C([\xi_0,\infty),\mathbf{R})$ such that $H(\xi,\xi)=0,\ for\ \xi\ge\xi_0,\ \ H(\xi,s)>0,\ for\ \xi>s\ge\xi_0$, and H has a nonpositive continuous partial derivative $H_s'(\xi,s)$. If

$$
\lim_{\xi\to\infty}\sup\frac{1}{H(\xi,\xi_0)}\{\int_{\xi_0}^{\xi}H(\xi,s)\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\}=\infty, \qquad (8)
$$

where $\widetilde{\phi}$, $\widetilde{\varphi}$ are defined as in Theorem 2, then every solution of Eq. (1) is oscillatory.

Proof. Assume (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^\alpha x(t)>0$ on $[t_2,\infty)$ for some sufficiently large $t_2>t_1$. Let $\omega(t)$ and $\widetilde{\omega}(\xi)$ be defined as in Theorem 2. By (7) we have

$$
\begin{aligned}
&KA(\xi)\widetilde{\phi}(\xi)\widetilde{q}(\xi)-\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)+\frac{\widetilde{\phi}(\xi)\widetilde{\varphi}^2(\xi)}{A(\xi)\widetilde{r}(\xi)}\\
&-\frac{[2\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)+A(\xi)\widetilde{\phi}'(\xi)\widetilde{r}(\xi)]^2}{4A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)}\le-\widetilde{\omega}(\xi),\ \xi\ge\xi_2. \qquad (9)
\end{aligned}
$$

Substituting ξ with s in (9), multiplying both sides by $H(\xi,s)$ and then integrating with respect to s from ξ_2 to ξ yields

$$
\begin{aligned}
&\int_{\xi_2}^{\xi}H(\xi,s)\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}\\
&-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\\
&\le-\int_{\xi_2}^{\xi}H(\xi,s)\widetilde{\omega}'(s)ds\\
&=H(\xi,\xi_2)\omega(\xi_2)+\int_{\xi_2}^{\xi}H_s'(\xi,s)\omega(s)\Delta s\\
&\le H(\xi,\xi_2)\omega(\xi_2)\le H(\xi,\xi_0)\omega(\xi_2).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{\xi_0}^{\xi}H(\xi,s)\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}\\
&-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\\
&=\int_{\xi_0}^{\xi_2}H(\xi,s)\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}\\
&-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\\
&+\int_{\xi_2}^{\xi}H(\xi,s)\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}\\
&-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\\
&\le H(\xi,\xi_0)\widetilde{\omega}(\xi_2)+H(\xi,\xi_0)\int_{\xi_0}^{\xi_2}|KA(s)\widetilde{\phi}(s)\widetilde{q}(s)\\
&-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}\\
&-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}|ds.
\end{aligned}
$$

So

$$
\begin{aligned}
&\lim_{\xi\to\infty}\sup\frac{1}{H(\xi,\xi_0)}\{\int_{\xi_0}^{\xi}H(\xi,s)\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)\\
&-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\}\\
&\le\widetilde{\omega}(\xi_2)+\int_{\xi_0}^{\xi_2}|KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}\\
&-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}|ds<\infty,
\end{aligned}
$$

which contradicts to (8). So the proof is complete.

Corollary 4. Under the conditions of Theorem 3, if

$$
\lim_{\xi\to\infty}\sup\frac{1}{(\xi-\xi_0)^\lambda}\{\int_{\xi_0}^{\xi}(\xi-s)^\lambda\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\}=\infty,
$$

then every solution of Eq. (1) is oscillatory.

Corollary 5. Under the conditions of Theorem 3, if

$$\lim_{\xi\to\infty}\sup\frac{1}{(\ln\xi-\ln\xi_0)}\{\int_{\xi_0}^{\xi}(\ln\xi-\ln s)\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s) -\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)} -\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\}=\infty,$$

then every solution of Eq. (1) is oscillatory.

The proof of Corollaries 4-5 can be completed by choosing $H(\xi,s)=(\xi-s)^{\lambda},\ \lambda>1$ or $H(\xi,s)=\ln\frac{\xi}{s}$ in Theorem 3.

Theorem 6. Let $h_1,\ h_2,\ \widehat{H}\in C([\xi_0,\infty),R)$ satisfying $\widehat{H}(\xi,\xi)=0,\widehat{H}(\xi,s)>0,\ \xi>s\geq\xi_0$, and H has continuous partial derivatives $\widehat{H}'_{\xi}(\xi,s)$ and $\widehat{H}'_s(\xi,s)$ on $[\xi_0,\infty)$ such that

$$\widehat{H}'_{\xi}(\xi,s)=-h_1(\xi,s)\sqrt{\widehat{H}(\xi,s)},$$
$$\widehat{H}'_s(\xi,s)=-h_2(\xi,s)\sqrt{\widehat{H}(\xi,s)}.$$

If for any sufficiently large $T\geq\xi_0$, there exist $a,\ b,\ c$ with $T\leq a<c<b$ satisfying

$$\begin{aligned}&\frac{1}{\widehat{H}(c,a)}\int_a^c\widehat{H}(s,a)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&\quad+\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&>\frac{1}{4\widehat{H}(c,a)}\int_a^cA(s)\widetilde{\phi}(s)\widetilde{r}(s)Q_1^2(s,a)ds\\&\quad+\frac{1}{4\widehat{H}(b,c)}\int_c^bA(s)\widetilde{\phi}(s)\widetilde{r}(s)Q_2^2(b,s)ds,\end{aligned}\tag{10}$$

where $\widetilde{\phi},\ \widetilde{\varphi}$ are defined as in Theorem 2,

$$Q_1(s,\xi)=h_1(s,\xi)-(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\sqrt{\widehat{H}(s,\xi)},$$

$$Q_2(\xi,s)=h_2(\xi,s)-(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\sqrt{\widehat{H}(\xi,s)},$$

then Eq. (1) is oscillatory.

Proof. Assume (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t)>0$ on $[t_2,\infty)$, where t_2 is sufficiently large. Let $\omega(t)$ and $\widetilde{\omega}(\xi)$ be defined as in Theorem 2. So for $t\in[t_2,\infty)$, we have

$$\begin{aligned}D_t^{\alpha}\omega(t)&=-\phi(t)\frac{A(\xi)q(t)f(x(t))}{x(t)}+\phi(t)D_t^{\alpha}\varphi(t)\\&\quad-\frac{\phi(t)\varphi^2(t)}{A(\xi)r(t)}+\frac{2\phi(t)\varphi(t)+A(\xi)D_t^{\alpha}\phi(t)r(t)}{A(\xi)\phi(t)r(t)}\omega(t)\\&\quad-\frac{1}{A(\xi)\phi(t)r(t)}\omega^2(t)\\&\leq-KA(\xi)\phi(t)q(t)+\phi(t)D_t^{\alpha}\varphi(t)\\&\quad-\frac{\phi(t)\varphi^2(t)}{A(\xi)r(t)}+\frac{2\phi(t)\varphi(t)+A(\xi)D_t^{\alpha}\phi(t)r(t)}{A(\xi)\phi(t)r(t)}\omega(t)\\&\quad-\frac{1}{A(\xi)\phi(t)r(t)}\omega^2(t).\end{aligned}\tag{11}$$

Furthermore, similar to (6), (11) is transformed into the following form

$$\begin{aligned}\widetilde{\omega}'(\xi)&\leq-KA(\xi)\widetilde{\phi}(\xi)\widetilde{q}(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)-\frac{\widetilde{\phi}(\xi)\widetilde{\varphi}^2(\xi)}{A(\xi)\widetilde{r}(\xi)}\\&\quad+\frac{2\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)+A(\xi)\widetilde{\phi}'(\xi)\widetilde{r}(\xi)}{A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)}\widetilde{\omega}(\xi)\\&\quad-\frac{1}{A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)}\widetilde{\omega}^2(\xi),\xi\geq\xi_2.\end{aligned}\tag{12}$$

Select $a,\ b,\ c$ arbitrarily in $[\xi_2,\infty)$ with $b>c>a$. Substituting ξ with s, multiplying both sides of (12) by $\widehat{H}(\xi,s)$ and integrating it with respect to s from c to ξ for $\xi\in[c,b)$, we get that

$$\begin{aligned}&\int_c^{\xi}\widehat{H}(\xi,s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&\leq-\int_c^{\xi}\widehat{H}(\xi,s)\widetilde{w}'(s)ds\\&\quad+\int_c^{\xi}\widehat{H}(\xi,s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\widetilde{w}(s)ds\\&\quad-\frac{\widehat{H}(\xi,s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\widetilde{\omega}^2(s)ds\\&=\widehat{H}(\xi,c)\widetilde{w}(c)-\\&\int_c^{\xi}\left[\left(\frac{\widehat{H}(\xi,s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\right)^{1/2}\widetilde{w}(s)+\frac{1}{2}(A(s)\widetilde{\phi}(s)\widetilde{r}(s))^{1/2}Q_2(\xi,s)\right]^2ds\\&+\int_c^{\xi}\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)ds\\&\leq\widehat{H}(\xi,c)\widetilde{w}(c)+\int_c^{\xi}\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)ds.\end{aligned}\tag{13}$$

Dividing both sides of the inequality (13) by $\widehat{H}(\xi,c)$ and let $\xi\to b^-$, we obtain

$$\begin{aligned}&\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&\leq\widetilde{w}(c)+\frac{1}{\widehat{H}(b,c)}\int_c^b\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(b,s)ds.\end{aligned}\tag{14}$$

On the other hand, substituting ξ with s, multiplying both sides of (12) by $\widehat{H}(s,\xi)$ and integrating it with respect to s from ξ to c for $\xi\in(a,c]$, we get that

$$\begin{aligned}&\int_{\xi}^c\widehat{H}(s,\xi)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&\leq-\int_{\xi}^c\widehat{H}(s,\xi)\widetilde{w}'(s)ds\\&\quad+\int_{\xi}^c\widehat{H}(s,\xi)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\widetilde{w}(s)ds\\&\quad-\int_{\xi}^c\widehat{H}(s,\xi)\frac{\widetilde{w}^2(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}ds\\&=-\widehat{H}(c,\xi)\widetilde{w}(c)-\\&\int_{\xi}^c\left[\left(\frac{\widehat{H}(s,\xi)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\right)^{1/2}\widetilde{w}(s)+\frac{1}{2}(A(s)\widetilde{\phi}(s)\widetilde{r}(s))^{1/2}Q_1(s,\xi)\right]^2ds\end{aligned}$$

$$+\int_{\xi}^{c}\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi)ds$$

$$\leq -\widehat{H}(c,\xi)\widetilde{w}(c)+\int_{\xi}^{c}\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi)ds. \quad (15)$$

Dividing both sides of the inequality (15) by $\widehat{H}(c,\xi)$ and letting $\xi \to a^+$, we obtain

$$\frac{1}{\widehat{H}(c,a)}\int_a^c \widehat{H}(s,a)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds$$

$$\leq -\widetilde{w}(c)+\frac{1}{\widehat{H}(c,a)}\int_a^c\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,a)ds. \quad (16)$$

A combination of (14) and (16) yields

$$\begin{aligned}&\frac{1}{\widehat{H}(c,a)}\int_a^c \widehat{H}(s,a)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&\quad+\frac{1}{\widehat{H}(b,c)}\int_c^b \widehat{H}(b,s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&\leq \frac{1}{4\widehat{H}(c,a)}\int_a^c A(s)\widetilde{\phi}(s)\widetilde{r}(s)Q_1^2(s,a)ds\\&\quad+\frac{1}{4\widehat{H}(b,c)}\int_c^b A(s)\widetilde{\phi}(s)\widetilde{r}(s)Q_2^2(b,s)ds,\end{aligned}$$

which contradicts to (10). So the proof is complete.

Theorem 7. Under the conditions of Theorem 6, if for any $l \geq \xi_0$,

$$\lim_{\xi\to\infty}\sup\int_l^{\xi}[\widehat{H}(s,l)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,l)]ds>0 \quad (17)$$

and

$$\lim_{\xi\to\infty}\sup\int_l^{\xi}[\widehat{H}(\xi,s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)]ds>0, \quad (18)$$

then Eq. (1) is oscillatory.

Proof: For any $T \geq \xi_0$, let $a = T$. In (17) we choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c[\widehat{H}(s,a)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,a)]ds>0. \quad (19)$$

In (18) we choose $l = c > a$. Then there exists $b > c$ such that

$$\int_c^b[\widehat{H}(b,s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(b,s)]ds>0. \quad (20)$$

Combining (19) and (20) we obtain (10). The conclusion thus comes from Theorem 6, and the proof is complete.

In Theorems 6-7, if we choose $\widehat{H}(\xi,s)=(\xi-s)^{\lambda},\ \xi\geq s\geq\xi_0$, where $\lambda>1$ is a constant, then we obtain the following two corollaries.

Corollary 8. Under the conditions of Theorem 6, if for any sufficiently large $T \geq \xi_0$, there exist $a,\ b,\ c$ with $T \leq a < c < b$ satisfying

$$\begin{aligned}&\frac{1}{(c-a)^{\lambda}}\int_a^c(s-a)^{\lambda}[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&\quad+\frac{1}{(b-c)^{\lambda}}\int_c^b(b-s)^{\lambda}[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\&>\frac{1}{4(c-a)^{\lambda}}\int_a^c A(s)\widetilde{\phi}(s)\widetilde{r}(s)(s-a)^{\lambda-2}\\&\quad\left(\lambda+(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})(s-a)\right)^2ds\\&\quad+\frac{1}{4(b-c)^{\lambda}}\int_c^b A(s)\widetilde{\phi}(s)\widetilde{r}(s)(b-s)^{\lambda-2}\\&\quad\left(\lambda-(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})(b-s)\right)^2ds, \quad (21)\end{aligned}$$

then Eq. (1) is oscillatory.

Corollary 9. Under the conditions of Theorem 7, if for any $l \geq \xi_0$,

$$\begin{aligned}&\lim_{\xi\to\infty}\sup\int_l^{\xi}\{(s-l)^{\lambda}[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}(s-l)^{\lambda-2}\\&\left[\lambda+(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})(s-l)\right]^2\}ds>0 \quad (22)\end{aligned}$$

and

$$\begin{aligned}&\lim_{\xi\to\infty}\sup\int_l^{\xi}\{(\xi-s)^{\lambda}[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}(\xi-s)^{\lambda-2}\\&\left[\lambda-(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})(\xi-s)\right]^2\}ds>0, \quad (23)\end{aligned}$$

then Eq. (1) is oscillatory.

Based on the theorems established above, we prove more general oscillation criteria as below.

Theorem 10. Under the conditions of Theorem 6, furthermore, suppose (10) does not hold. If for any $T \geq \xi_0$, there exist $a,\ b$ with $b > a \geq T$ such that for any $u \in C[a,b],\ u'(t) \in L^2[a,b],\ u(a) = u(b) = 0$, the following inequality holds:

$$\int_a^b\{u^2(s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]$$

$$-A(s)\widetilde{\phi}(s)\widetilde{r}(s)$$

$$\left(u'(s)+\frac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\right)^2\}ds>0, \tag{24}$$

where $\widetilde{\phi}$, $\widetilde{\varphi}$ are defined as in Theorem 2, then Eq. (1) is oscillatory.

Proof: Assume (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t)>0$ on $[t_2,\infty)$, where t_2 is sufficiently large. Let $\omega(t)$ and $\widetilde{\omega}(\xi)$ be defined as in Theorem 2. Similar to the proof of Theorem 6, we obtain (12). Select a, b arbitrarily in $[\xi_2,\infty)$ with $b>a$ such that $u(a)=u(b)=0$. Substituting ξ with s, multiplying both sides of (12) by $u^2(s)$, integrating it with respect to s from a to b, we get that

$$\begin{aligned}
&\int_a^b u^2(s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]ds\\
&\le -\int_a^b u^2(s)\widetilde{w}'(s)ds-\int_a^b u^2(s)\frac{\widetilde{w}^2(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}ds\\
&\quad+\int_a^b u^2(s)\widetilde{w}(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})ds\\
&=2\int_a^b u(s)u'(s)\widetilde{w}(s)ds-\int_a^b u^2(s)\frac{\widetilde{w}^2(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}ds\\
&\quad+\int_a^b u^2(s)\widetilde{w}(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})ds\\
&=-\int_a^b\{[\sqrt{\frac{1}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}}u(s)\widetilde{w}(s)-\sqrt{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\\
&\quad(u'(s)+\tfrac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}))]^2\\
&\quad+A(s)\widetilde{\phi}(s)\widetilde{r}(s)\\
&\quad\left(u'(s)+\tfrac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\right)^2\}ds.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_a^b\{u^2(s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]\\
&\quad-A(s)\widetilde{\phi}(s)\widetilde{r}(s)\\
&\quad\left(u'(s)+\frac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\right)^2\}ds\le 0,
\end{aligned} \tag{25}$$

which contradicts to (24). So every solution of Eq. (1) is oscillatory, and the proof is complete.

Remark 1. As one can see, the result in Theorem 10 is more general than that in Theorems 6, 7 in that (10) is not necessarily satisfied.

Remark 2. In Theorems 2, 3, 6, 7, 10, if we assume $f\in C^1[R,R]$ satisfying $f'(x)\ge\mu>0$ for $x\ne 0$, and modify the definition of $\omega(t)$ by $\omega(t)=\phi(t)\{\frac{A(\xi)r(t)D_t^\alpha x(t)}{f(x(t))}+\varphi(t)\}$, then following a similar process, we can obtain more extensive oscillatory criteria for Eq. (1), which are omitted here.

III. Applications

Example 1. Consider the following fractional differential equation with damping term:

$$\begin{aligned}
&D_t^\alpha(\sqrt{\frac{t^\alpha}{\alpha}}D_t^\alpha x(t))+\frac{\alpha}{t^\alpha}D_t^\alpha x(t)\\
&+(\frac{t^\alpha}{\alpha})^{-\frac{3}{2}}\frac{x(t)(2+x^2(t))}{1+x^2(t)}=0,\ t\ge 2,\ 0<\alpha<1.
\end{aligned} \tag{26}$$

In fact, if we set in Eq. (1) $t_0=2$, $r(t)=\sqrt{\frac{t^\alpha}{\alpha}}$, $p(t)=\frac{\alpha}{t^\alpha}$, $q(t)=(\frac{t^\alpha}{\alpha})^{-\frac{3}{2}}$, $f(x)=\frac{x(2+x^2)}{1+x^2}$, then we obtain (26). So $\xi_0=\frac{2^\alpha}{\alpha}$, $\widetilde{r}(\xi)=r(t)=\sqrt{\frac{t^\alpha}{\alpha}}=\sqrt{\xi}$, $\widetilde{p}(\xi)=p(t)=\frac{\alpha}{t^\alpha}=\xi^{-1}$, $\widetilde{q}(\xi)=q(t)=(\frac{t^\alpha}{\alpha})^{-\frac{3}{2}}=\xi^{-\frac{3}{2}}$, and $f(x)/x\ge 1$, which implies $K=1$. Furthermore, since $A(\xi)=exp(\int_{\xi_0}^{\xi}\tau^{-\frac{3}{2}}d\tau)=exp(2\xi_0^{-\frac{1}{2}}-2\xi^{-\frac{1}{2}})$, so $1\le A(\xi)\le exp(2\xi_0^{-\frac{1}{2}})$, and in (2),

$$\begin{aligned}
&\int_{\xi_0}^{\infty}\frac{1}{A(s)\widetilde{r}(s)}dt\ge\frac{1}{exp(2\xi_0^{-\frac{1}{2}})}\int_{\xi_0}^{\infty}\frac{1}{\widetilde{r}(s)}dt\\
&=\frac{1}{exp(2\xi_0^{-\frac{1}{2}})}\int_{\xi_0}^{\infty}\frac{1}{\sqrt{\xi}}d\xi=\infty.
\end{aligned}$$

On the other hand, so in (5), letting $\widetilde{\phi}(\xi)=\sqrt{\xi}$, $\widetilde{\varphi}(\xi)=0$, we obtain

$$\begin{aligned}
&\int_{\xi_0}^{\infty}\{KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}\\
&\quad-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4A(s)\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\\
&=\int_{\xi_0}^{\infty}\frac{15A(s)}{16s}ds\ge\int_{\xi_0}^{\infty}\frac{15}{16s}ds=\infty.
\end{aligned}$$

Therefore, Eq. (26) is oscillatory by Theorem 2.

Example 2. Consider the following fractional differential equation with damping term:

$$\begin{aligned}
&D_t^\alpha(D_t^\alpha x(t))+\frac{1}{\alpha}t^\alpha D_t^\alpha x(t)+\frac{1}{\alpha}t^\alpha x(t)e^{x^2(t)}=0,\\
&t\ge 5,\ 0<\alpha<1.
\end{aligned} \tag{27}$$

In fact, if we set in Eq. (1) $t_0=5$, $r(t)\equiv 1$, $p(t)=\frac{t^\alpha}{\alpha}$, $q(t)=\frac{t^\alpha}{\alpha}$, $f(x)=xe^{x^2}$, then we obtain (27). So $\widetilde{r}(\xi)\equiv 1$, $\widetilde{p}(\xi)=p(t)=\frac{t^\alpha}{\alpha}=\xi$, $\widetilde{q}(\xi)=q(t)=\frac{t^\alpha}{\alpha}=\xi$, and $f(x)/x=e^{x^2}\ge 1$, which implies $K=1$. Furthermore, since $A(\xi)=exp(\int_{\xi_0}^{\xi}\tau d\tau)=exp(\frac{\xi^2}{2}-\frac{\xi_0^2}{2})\ge 1$, so in (22)-(23), after letting $\widetilde{\phi}(\xi)\equiv 1$, $\widetilde{\varphi}(\xi)=0$, $\lambda=2$, considering $\widetilde{q}(s)\equiv 1$, we obtain

$$\begin{aligned}
&\lim_{\xi\to\infty}\sup\int_l^{\xi}\{(s-l)^\lambda[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\
&\quad+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}(s-l)^{\lambda-2}\\
&\quad\left[\lambda+(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})(s-l)\right]^2\}ds\\
&=\lim_{\xi\to\infty}\sup\int_l^{\xi}A(s)\left[s(s-l)^2-1\right]ds\\
&\ge\lim_{\xi\to\infty}\sup\int_l^{\xi}\left[s(s-l)^2-1\right]ds=\infty
\end{aligned}$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi\{(\xi-s)^\lambda[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s) +\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}]-\frac{A(s)\widetilde{\phi}(s)\widetilde{r}(s)}{4}(\xi-s)^{\lambda-2} \left[\lambda-(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})(\xi-s)\right]^2\}ds$$
$$=\lim_{\xi\to\infty}\sup\int_l^\xi A(s)\left[s(\xi-s)^2-1\right]ds$$
$$\geq\lim_{\xi\to\infty}\sup\int_l^\xi \left[s(\xi-s)^2-1\right]ds=\infty.$$

So according to Corollary 9 we deduce that Eq. (27) is oscillatory.

Example 3. Consider the following fractional differential equation with damping term:

$$D_t^\alpha\left(\sin^2(\frac{t^\alpha}{\alpha})D_t^\alpha x(t)\right)+t^2D_t^\alpha x(t)+x(t)(1+x^2(t))=0,$$
$$t\geq 2,\ 0<\alpha<1. \tag{28}$$

If we set in Eq. (1) $t_0=2,\ r(t)=\sin^2(\frac{t^\alpha}{\alpha}),\ p(t)=t^2,\ q(t)\equiv 1,\ f(x)=x(1+x^2)$, then we obtain (28). So $\widetilde{r}(\xi)=r(t)=\sin^2(\frac{t^\alpha}{\alpha})=\sin^2\xi,\ \widetilde{q}(\xi)\equiv 1$, and $f(x)/x=1+x^2\geq 1$, which implies $K=1$. Furthermore we have $A(\xi)\geq 1$. So in (24), after letting $\widetilde{\phi}(\xi)\equiv 1,\ \widetilde{\varphi}(\xi)=0,\ a=2k\pi,\ b=2k\pi+\pi,\ u(s)=\sin s$, then $u(a)=u(b)=0$, and we obtain

$$\int_a^b\{u^2(s)[KA(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{A(s)\widetilde{r}(s)}] -A(s)\widetilde{\phi}(s)\widetilde{r}(s) \left(u'(s)+\frac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+A(s)\widetilde{\phi}'(s)\widetilde{r}(s)}{A(s)\widetilde{\phi}(s)\widetilde{r}(s)})\right)^2\}ds$$
$$=\int_{2k\pi}^{2k\pi+\pi}A(s)\left(\sin^2 s-\sin^2 s\cos^2 s\right)ds$$
$$\geq\int_{2k\pi}^{2k\pi+\pi}\left(\sin^2 s-\sin^2 s\cos^2 s\right)ds$$
$$=\int_{2k\pi}^{2k\pi+\pi}\sin^4 sds>0.$$

Therefore, Eq. (28) is oscillatory by Theorem 10.

Remark 3. We note that oscillation for the three examples above can not be obtained by existing results so far in the literature.

IV. Further Applications

As the proof process in Section II is usual in the research of oscillation criteria, we point out that this method can be applied to research many other fractional differential equations such as the follows:

$$D_t^\alpha(r(t)D_t^\alpha x(t))+q(t)f(x(t))=0,$$
$$t\geq t_0>0,\ 0<\alpha<1, \tag{29}$$

$$D_t^\alpha(r(t)D_t^\alpha x(t))+q(t)f(x(\tau(t)))=0,$$
$$t\geq t_0>0,\ 0<\alpha<1, \tag{30}$$

where (30) is a kind of delay fractional differential equation, $D_t^\alpha(.)$ denotes the conformable fractional derivative with respect to the variable t, the function $r\in C^\alpha([t_0,\infty),R_+),\ q\in C([t_0,\infty),R_+)$, and C^α denotes continuous derivative of order α, the function f is continuous satisfying $f(x)/x\geq K$ for some positive constant K and $\forall x\neq 0$, the delay term τ satisfies $\tau(t)\leq t$.

V. Conclusions

We have established some new oscillatory criteria for a class of fractional differential equations with the fractional derivative defined in the sense of the conformable fractional derivative. Some applications for these established results are also presented. We note that the method in this paper can be applied to research oscillation of fractional differential equations with more complicated forms such as with damping term or with delay term, which are expected to further research.

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