Elementary Wave Interactions for a Simplified Model in Magnetogasdynamics

Yujin Liu and Wenhua Sun

Abstract—We are mainly concerned with the elementary wave interactions for Magnetogasdynamics system and construct uniquely the solution of the initial value problem when the initial data are three piecewise constant states. We find that the results are very different from that of the corresponding case of the conventional gas dynamics which shows the complexity of the solutions for Magnetogasdynamics system. Our results can be used to construct the approximate solution by Glimm's scheme and to describe the asymptotic behavior of the solution.

Index Terms—Riemann problem, Wave interaction, Magnetogasdynamics, Shock wave, Rarefaction wave, Magnetogasdynamics.

I. INTRODUCTION

AGNETOGASDYNAMICS system is important in engineering physics and many other aspects [1], [2], [3], [4], [5], [6], [7] and useful for studying the hyperbolic system's theory.

One-dimensional inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field, is expressed by

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u + pI) - \mu \operatorname{rot} H \times H = 0, \\ (\rho E + \frac{1}{2}\mu H^2)_t + \operatorname{div}(\rho u E + up - \mu(u \times H) \times H) = 0, \\ H_t - \operatorname{rot}(u \times H) = 0, \\ \operatorname{div} H = 0, \\ p = f(\rho, S) \end{cases}$$
(1)

where $\rho \ge 0$, p, S, $B \ge 0$ and $E = e + \frac{u^2}{2}$ are respectively the density, pressure, specific entropy, transverse magnetic field, and specific total energy, e is the specific internal energy. $u = (u_1, u_2, u_3)$ is the velocity of the fluid in the direction of (x_1, x_2, x_3) , $H = (H_1, H_2, H_3)$ is the magnetic field in the direction of (x_1, x_2, x_3) and $H = \mu B$, where μ is the magnetic permeability.

Hu and Sheng [8] studied the Riemann problem (the other related problems in partial differential equations were investigated by many researchers ([9], [10], [11], [12], [13],

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etc.) for

$$\begin{aligned} \tau_t - u_x &= 0, \\ u_t + (p + \frac{B^2}{2\mu})_x &= 0, \\ (E + \frac{B^2 \tau}{2\mu})_t + (pu + \frac{B^2 u}{2\mu})_x &= 0, \end{aligned}$$
(2)

under the assumption $B = k\rho$, where k is positive constant, τ denotes the specific volume. They constructed uniquely the Riemann solution. In [14], we investigated the elementary wave interactions for the system (2).

T. Raja Sekhar and V.D. Sharma [15] studied

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p + \frac{B^2}{2})_x = 0, \end{cases}$$
(3)

and they constructed the Riemann solutions and investigated the wave interactions of the elementary waves.

Shen [16] investigated the Riemann problem for (3) and obtained that the Riemann solution converges to the corresponding Riemann solution of the transport equations when both the pressure p and the magnetic field B vanish.

In [17], we removed the assumption $B = k\rho$ and studied the Riemann problem for the following Magnetogasdynamic system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p + \frac{B^2}{2})_x = 0, \\ (B)_t + (Bu)_x = 0, \end{cases}$$
(4)

with the following initial values

$$(\rho, u, B)(x, 0) = (\rho^{\pm}, u^{\pm}, B^{\pm}), \quad \pm x > 0,$$
 (5)

where $\rho^{\pm}, u^{\pm}, B^{\pm}$ are arbitrary constants, and p is given by $p = A\rho^{\gamma}$ for polytropic gas, A is positive constant and γ is the adiabatic constant.

In this paper we investigate the wave interactions of the elementary waves for (4) with the following initial data

$$(\rho, u, B)(x, 0) = \begin{cases} (\rho_l, u_l, B_l), & -\infty < x \le -\varepsilon, \\ (\rho_m, u_m, B_m), & -\varepsilon < x \le \varepsilon, \\ (\rho_r, u_r, B_r), & \varepsilon < x < \infty, \end{cases}$$

$$(6)$$

for arbitrary $\varepsilon \in R$.

By virtue of analyzing the concrete properties of the elementary waves in the phase plane (ρ, u) , we construct uniquely of the solution of the initial value problem (4) and (6) which reveals the intrinsic mechanism of our model (4).

In [17], we studied the wave interaction between the shock wave and the contact discontinuity, in this present paper, we just need to consider the collision of the two shock waves for which we construct uniquely the global solution for (4) and (6) by solving a new Riemann problem, and the wave

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interactions containing rarefaction wave for which we obtain uniquely the local solution for (4) and (6) which plays an important role in constructing the approximate solution by means of Glimm's scheme and giving a description of the asymptotic behavior of the solution when $t \to \infty$.

This paper is organized as follows. In Section II, we give the elementary waves and the Riemann solutions of (4) and (5) for our later discussions. In Section III, we investigate the wave interactions of the elementary waves by analyzing the properties of the elementary wave curves in (ρ, u) and obtain the unique solution for the initial value problem (4) and (6). A final conclusion is given in Section IV.

II. PRELIMINARIES AND ELEMENTARY COMBUSTION WAVE

First, we give briefly the results of the Riemann problem for (4) with the initial data (5), and we refer the readers to [11], [17] for more details.

There are three eigenvalues of (4) which are $\lambda = u =$ $\lambda_0, \lambda = u \pm \sqrt{p_{\rho} + \frac{B^2}{\rho}} = \lambda_{\pm}$. They are real and distinct which shows that (4) is a strictly hyperbolic system. It is easy to see that the characteristic fields λ_{\pm} are genuinely nonlinear and the characteristic field λ_0 is linearly degenerate.

The forward or backward rarefaction wave \overleftarrow{R} in the (ρ, u, B) space passing through $N_0(\rho_0, u_0, B_0)$ is expressed by

$$\vec{R}(N_0): \begin{cases} B = k_0 \rho, \\ u = u_0 \pm \int_{\rho_0}^{\rho} \sqrt{\frac{p_{\rho} + \frac{B^2}{\rho}}{\rho^2}} d\rho, \end{cases}$$
(7)

where $k_0 = \frac{B_0}{\rho_0}$. The contact discontinuity is given as follows

$$J: \begin{cases} \sigma = u, \\ [u] = [p + \frac{B^2}{2}] = 0. \end{cases}$$
(8)

It follows that J is a plane curve with u = Const. in the (ρ, u, B) space and the projection on the (ρ, u) plane is a straight line parallel to the ρ -axis which shows that it is much more complicated than that of the conventional gas dynamics.

The forward or backward shock wave in the (ρ, u, B) space passing through $N_0(\rho_0, u_0, B_0)$ is expressed by

$$\vec{\overleftarrow{S}}(N_0): \begin{cases} B = k_0 \rho, \\ u - u_0 = \pm \sqrt{\frac{(\rho - \rho_0)}{\rho \rho_0} (p + \frac{B^2}{2} - p_0 - \frac{B_0^2}{2})}. \end{cases}$$
(9)

Based on the above analysis, we construct the Riemann solution for (4) and (5). From the properties of the elementary waves, there is no asymptote for both S and S while both R and R intersect with the *u*-axis. Particularly, we deal with the contact discontinuity carefully since it is much more complex than that of our conventional gas dynamics. We construct the solution of Riemann problem (4) and (5) as follows.

Denote $\overline{W}_{lB}(N_{lB}) = \overline{R}_{lB}(N_{lB}) \cup \overline{S}_{lB}(N_{lB})$ and $\overline{W}_{rB}(N_{rB}) = \overline{R}_{rB}(N_{rB}) \cup \overline{S}_{rB}(\underline{N}_{rB})$. Draw $\overline{W}_{lB}(N_{lB})$ from N_{lB} in the plane (ρ, u) and $\overrightarrow{W}_{rB}(N_{rB})$ from N_{rB} . From the properties of $\overrightarrow{W}_{lB}(N_{lB})$ and $\overrightarrow{W}_{rB}(N_{rB})$, they intersect with each other at most once. It follows that there are five cases: $W_{lB}(N_{lB}) \cap W_{rB}(N_{rB}) = (R_{lB}(N_{lB}) \cap N_{rB})$ $\overrightarrow{R}_{rB}(N_{rB})) \text{ or } (\overrightarrow{S}_{lB}(N_{lB}) \cap \overrightarrow{R}_{rB}(N_{rB})) \text{ or } (\overrightarrow{R}_{lB}(N_{lB}) \cap$ $\overrightarrow{S}_{rB}(N_{rB})$) or $(\overleftarrow{S}_{lB}(N_{lB}) \cap \overrightarrow{S}_{rB}(N_{rB}))$ or \varnothing .

For the last case, it is obvious that there is a vacuum solution. Thus, we just need to consider the first case because the other cases can be discussed similarly.

Suppose $\overleftarrow{W}_{lB}(N_{lB}) \cap \overrightarrow{W}_{rB}(N_{rB}) = \overleftarrow{R}_{lB}(N_{lB}) \cap$ $\vec{R}_{rB}(N_{rB}) = \{N_{*B}\},$ it follows that there exists (ρ_*, u_*) satisfying

$$u_* = u_l - \int_{\rho_l}^{\rho_*} \frac{\sqrt{p_\rho + k_l^2 \rho}}{\rho} \mathrm{d}\rho, \qquad (10)$$

$$u_* = u_r + \int_{\rho_r}^{\rho_*} \frac{\sqrt{p_\rho + k_r^2 \rho}}{\rho} \mathrm{d}\rho, \qquad (11)$$

where $k_l = \frac{B_l}{\rho_l}$ and $k_r = \frac{B_r}{\rho_r}$. Denote

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$$f_{1}(\rho_{1}) = \begin{cases} u_{l} - \int_{\rho_{l}}^{\rho_{1}} \frac{\sqrt{p_{\rho} + k_{l}^{2}\rho}}{\rho} d\rho, \\ (\rho \leq \rho_{l}), \\ u_{l} - \sqrt{\frac{\rho_{1} - \rho_{l}}{\rho_{1}\rho_{l}}} (p_{1} + \frac{1}{2}k_{l}^{2}\rho_{1}^{2} - p_{l} - \frac{1}{2}k_{l}^{2}\rho_{l}^{2}), \\ (\rho > \rho_{l}), \end{cases}$$
(12)

$$f_{2}(\rho_{2}) = \begin{cases} u_{r} + \int_{\rho_{r}}^{\rho_{2}} \frac{\sqrt{p_{\rho} + k_{r}^{2} \rho}}{\rho} d\rho, \\ (\rho \leq \rho_{r}), \\ u_{r} + \sqrt{\frac{\rho_{2} - \rho_{r}}{\rho_{2} \rho_{r}} (p_{2} + \frac{1}{2} k_{r}^{2} \rho_{2}^{2} - p_{r} - \frac{1}{2} k_{r}^{2} \rho_{r}^{2})}, \\ (\rho > \rho_{r}), \end{cases}$$
(13)

$$g_1(\rho_1) = p_1 + \frac{1}{2}k_l^2\rho_1^2, \tag{14}$$

$$g_2(\rho_2) = p_2 + \frac{1}{2}k_r^2\rho_2^2.$$
 (15)

We consider the following problem

$$\begin{cases} f_1(\rho_1) = f_2(\rho_2), \\ g_1(\rho_1) = g_2(\rho_2), \end{cases}$$
(16)

and prove that it has a unique solution, which shows that there exists a unique contact discontinuity J joining the two states which are respectively located on \overleftarrow{R} and \overleftarrow{S} (Fig 1).

From [17], it follows that $f_1(\rho_1)$ and $f_2(\rho_2)$ are both smooth functions, and the curve $\rho_2 = \rho_2(\rho_1)$ defined by $f_1(\rho_1) = f_2(\rho_2)$ is monotonically decreasing, while the curve $\rho_2 = \rho_2(\rho_1)$ defined by $g_1(\rho_1) = g_2(\rho_2)$ is monotonically increasing. Thus, the uniqueness of the solution of (16) is obtained. Now we consider the existence of the solution of (16).

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Fig 1 The Elementary waves in the (ρ, u) plane.

Form (10) and (11) we get $f_1(\rho_*) = f_2(\rho_*)$, we discuss as follows.

Case 1. $k_l = k_r$. It is equivalent to $g_1(\rho_*) = g_2(\rho_*)$. It is obvious that $\rho_1 = \rho_2 = \rho_*$ is the solution of (16). Thus, the Riemann solution is R + R, here the symbol "+" means "followed by". We observe that for this case there is no contact discontinuity.



Case 2. $k_l > k_r$. It is equivalent to $g_1(\rho_*) > g_2(\rho_*)$. So we should look for solution in $\{(\rho_1, \rho_2) : \rho_1 < \rho_*, \rho_2 > \rho_*\}$.

Subcase 2.1 $u_r \ge f_1(0)$. (Fig 2)

There exists $\hat{\rho}_1$ such that $f_1(0) = f_2(\hat{\rho}_1)$, where $\rho_* < \hat{\rho}_1 < \rho_r$.

Since $g_1(0) < g_2(\hat{\rho}_1)$ and the curves $f_1(\rho_1)$, $f_2(\rho_2)$ are smooth, from the method of continuity, there exists $(\bar{\rho}_1, \bar{\rho}_2)$ satisfying $0 < \bar{\rho}_1 < \rho_*, \rho_* < \bar{\rho}_2 < \hat{\rho}_1$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. Thus, $(\bar{\rho}_1, \bar{\rho}_2)$ is the solution of (16). It follows that the Riemann solution is given by $\overleftarrow{R} + \overleftarrow{J} + \overrightarrow{R}$.

Subcase 2.2 $u_r < f_1(0)$. (Fig 3)

There exists $\hat{\rho}_2$ and $\hat{\rho}_3$ satisfying $f_1(\hat{\rho}_2) = u_r$ and $f_1(0) = f_2(\hat{\rho}_3)$, respectively, where $0 < \hat{\rho}_2 < \rho_*$ and $\hat{\rho}_3 > \rho_r$.

Subcase 2.2.1 $g_2(\rho_r) < g_1(\hat{\rho}_2)$. Since $g_1(\hat{\rho}_3) > g_1(0)$ and the curves $f_1(\rho_1)$, $f_2(\rho_2)$ are smooth, from the method of continuity, there exists $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \hat{\rho}_2, \rho_r < \bar{\rho}_2 < \hat{\rho}_3$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. Thus, $(\bar{\rho}_1, \bar{\rho}_2)$ is the solution of (16). The Riemann solution is $\overline{R} + \overline{J} + \overline{S}$.

Subcase 2.2.2 $g_2(\rho_r) \ge g_1(\hat{\rho}_2)$. Similarly, we know that there exists $(\bar{\rho}_1, \bar{\rho}_2) : \hat{\rho}_2 < \bar{\rho}_1 < \rho_*, \rho_* < \bar{\rho}_2 < \rho_r$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. The Riemann solution is $\overleftarrow{R} + \overleftarrow{J} + \overrightarrow{R}$.

Case 3. $k_l < k_r$. It is equivalent to $g_1(\rho_*) < g_2(\rho_*)$. So we should look for solution in $\{(\rho_1, \rho_2): \rho_1 > \rho_*, \rho_2 < \rho_*\}$.



Subcase 3.1 $u_l \leq f_2(0)$. (Fig 4)

It is obvious that there exists $\hat{\rho}_4$ such that $f_1(\hat{\rho}_4) = f_2(0)$, where $\rho_* < \hat{\rho}_4 < \rho_l$. Similar discussions as above, we obtain $(\bar{\rho}_1, \bar{\rho}_2) : \rho_* < \bar{\rho}_1 < \hat{\rho}_4, 0 < \bar{\rho}_2 < \rho_*$. It follows that the Riemann solution is $\overleftarrow{R} + \overrightarrow{J} + \overrightarrow{R}$.

Subcase 3.2 $u_l > f_2(0)$. (Fig 5) There exist respectively $\hat{\rho}_5$ such that $f_2(\hat{\rho}_5) = u_l$, where $0 < \hat{\rho}_5 < \rho_*$, and $\hat{\rho}_6$ such that $f_1(\hat{\rho}_6) = f_2(0)$, where $\hat{\rho}_6 > \rho_l$.

Subcase 3.2.1 $g_1(\rho_l) \ge g_2(\hat{\rho}_5)$. Similarly, we know that there exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_* < \bar{\rho}_1 < \rho_l, \hat{\rho}_5 < \bar{\rho}_2 < \rho_*$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. The Riemann solution is $\overleftarrow{R} + \overrightarrow{J} + \overrightarrow{R}$.

Subcase 3.2.2 $g_1(\rho_l) < g_2(\hat{\rho}_5)$. Similarly, we know that there exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_l < \bar{\rho}_1 < \hat{\rho}_6, 0 < \bar{\rho}_2 < \hat{\rho}_5$ such that $f_1(\bar{\rho}_1) = f_2(\bar{\rho}_2)$ and $g_1(\bar{\rho}_1) = g_2(\bar{\rho}_2)$. The Riemann solution is $\overleftarrow{S} + \overrightarrow{J} + \overrightarrow{R}$.

Based on the above results, we have the following result. **Theorem 2.1** There exists uniquely solution for the Riemann problem (4) with the initial values (5).

III. WAVE INTERACTIONS OF THE ELEMENTARY WAVES

Now we consider the wave interactions of the elementary waves obtained from the Riemann problem (4) and (5). In order to solve the initial value problem (4) and (6), we divide our discussions into the following five cases: the collision of the two shock waves \overrightarrow{SS} , and the wave interactions containing rarefaction wave \overrightarrow{RJ} , \overrightarrow{RJ} , \overrightarrow{RK} and \overrightarrow{RS} .

In [17], we studied the wave interaction between the shock wave and the contact discontinuity, in this paper for the case which containing the shock wave, we just consider the collision of the two shock waves.

Case (i) The collision of the two shock waves \overline{SS} .

Similar discussions with Lemma 3.3.7. in [11], we have the following result and the proof is omitted for simplicity. **Lemma 3.1** Suppose the point $N_2 \in \overrightarrow{R}_B(N_1) \cup \overrightarrow{S}_B(N_1)$, then the curve $\overrightarrow{S}_B(N_1)$ does not intersects with $\overrightarrow{S}_B(N_2)$ on the side where ρ increases while $\overrightarrow{R}_B(N_1)$ does not intersects with $\overrightarrow{R}_B(N_2)$ on the side where ρ decreases.

It is easy to see that \vec{S} will intersect with \overleftarrow{S} in a finite time and a new Riemann problem is formed. From Lemma 3.1., we have $\overrightarrow{S}_B(N_l)$ does not intersect with $\overrightarrow{S}_B(N_m)$ and $\overrightarrow{S}_B(N_r)$ does not intersect with $\overrightarrow{S}_B(N_m)$, respectively. Thus we have $N_* \in \overrightarrow{S}_B(N_l) \cup \overrightarrow{S}_B(N_r)$ (Fig 6 and Fig 7) and discuss as follows.



Case 1. $k_l < k_r$. It is equivalent to $g_1(\rho_*) < g_2(\rho_*)$ and we should seek a solution in $\{(\bar{\rho}_1, \bar{\rho}_2) | \bar{\rho}_1 > \rho_*, 0 < 0\}$ $\bar{\rho}_2 < \rho_*$. It is obvious that there exist respectively $\tilde{\rho}_1$ and $\tilde{\rho}_2$ which satisfy that $u_r = u_{\overleftarrow{S}_{IB}}(\tilde{\rho}_1), \ \rho_* < \tilde{\rho}_1 < \tilde{\rho}_2$ and $u_{\overrightarrow{R}_{rB}}(0) = u_{\overleftarrow{S}_{1B}}(\widetilde{\rho}_2), \ \rho_* < \widetilde{\rho}_1 < \widetilde{\rho}_2.$

Subcase 1.1. $\tilde{g}_1(\tilde{\rho}_1) \geq g_2(\rho_r)$. From the continuity of the wave curves, it follows that there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ such that $\rho_* < \bar{\rho}_1 < \tilde{\rho}_1$, $\rho_r < \bar{\rho}_2 < \rho_*$, and the solution is shown by $\overrightarrow{S} \overleftarrow{S} \to \overleftarrow{S} \overrightarrow{J} \overrightarrow{S}$.

Subcase 1.2. $g_1(\tilde{\rho}_1) < g_2(\rho_r)$. Similarly, there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ satisfying $\tilde{\rho}_1 < \bar{\rho}_1 < \tilde{\rho}_2$ and $0 < \bar{\rho}_2 < \rho_r$ and the solution is expressed by $\overrightarrow{S} \overleftarrow{S} \rightarrow \overleftarrow{S} \overrightarrow{J} \overrightarrow{R}$.

Case 2. $k_l = k_r$. It is equivalent to $g_1(\rho_*) = g_2(\rho_*)$ and there is no contact discontinuity of the new Riemann solution, the state N_l is connected to the state N_r by the state N_* directly and we know the solution is given by $\overrightarrow{SJ} \rightarrow \overrightarrow{SS}$.

Case 3. $k_l > k_r$. It is equivalent to $g_1(\rho_*) > g_2(\rho_*)$ and we should seek solution in $\{(\bar{\rho}_1, \bar{\rho}_2)|0 < \bar{\rho}_1 < \rho_*, \bar{\rho}_2 >$ ρ_* . There exist respectively $\tilde{\rho}_3$ and $\tilde{\rho}_4$ which satisfy that $u_l = u_{\overrightarrow{S}_{rB}}(\widetilde{\rho}_3)$ and $u_{\overrightarrow{R}_{lB}}(0) = u_{\overrightarrow{S}_{rB}}(\widetilde{\rho}_4), \ \rho_* < \widetilde{\rho}_3 < \widetilde{\rho}_4.$ Subcase 3.1. $g_1(\widetilde{\rho}_3) \ge g_2(\rho_l)$. From the continuity of

the wave curves, it follows that there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ satisfying $\rho_l < \bar{\rho}_1 < \rho_*, \ \rho_* < \bar{\rho}_2 < \tilde{\rho}_3$, and the solution is shown by $\overrightarrow{S} \overleftarrow{S} \to \overleftarrow{S} \overrightarrow{J} \overrightarrow{S}$.

Subcase 3.2. $g_1(\tilde{\rho}_3) < g_2(\rho_l)$. There exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ which satisfies $0 < \bar{\rho}_1 < \rho_l$ and $\tilde{\rho}_3 < \bar{\rho}_2 < \tilde{\rho}_4$ and we know that the solution is given by $\overrightarrow{S} \overleftarrow{S} \rightarrow \overleftarrow{R} \overrightarrow{J} \overrightarrow{S}$.

Theorem 3.1 When a forward shock collides with a backward shock, the forward (backward) shock will cross immediately the backward (forward) shock, or a new forward (backward) rarefaction wave will appear. Furthermore, the contact discontinuity may appear or not after the wave interaction.

In what follows, because there is a penetration process in the interaction, we can not obtain the global solution by solving the new Riemann problem like the above discussions. However, the local solution of the new Riemann problem is still important for investigating the global solution which can be used to construct the approximate solution by Glimm's scheme and to describe the asymptotic behavior of the solution when $t \to \infty$ [11]. Next we investigate the wave interactions containing the rarefaction wave.

Case (ii) The interaction of the rarefaction wave with the contact discontinuity $\overrightarrow{R}J$.

Since

$$\vec{R}_{rB}(N_r) : u = u_r + \int_{\rho_r}^{\rho} \frac{\sqrt{p_{\rho} + k_r^2 \rho}}{\rho} d\rho, \quad \rho < \rho_r,$$
$$\vec{R}_{mB}(N_m) : u = u_m + \int_{\rho_m}^{\rho} \frac{\sqrt{p_{\rho} + k_m^2 \rho}}{\rho} d\rho, \quad \rho < \rho_m;$$

 $p_m + \frac{k_m^2 \rho_m^2}{2} = p_r + \frac{k_m^2 \rho_m^2}{2}, u_m = u_r \text{ and } \rho_m > \rho_r.$ From Lemma 3.1., we know that the curve $\overrightarrow{R}_{rB}(N_r)$ lies always above the curve $R_{mB}(N_m)$. Thus, there are two possible cases: $\overline{R}_{lB}(N_l)$ intersects with $\overline{R}_{rB}(N_r)$ at N_{*B} where a new Riemann problem is formed, or $\overline{R}_{lB}(N_l)$ intersects with $\vec{S}_{rB}(N_r)$ at N_{*B} where a new Riemann problem is formed.

Case 1. $\tilde{\rho}_1 \geq \rho_r$, where $\tilde{\rho}_1$ satisfies $u_r = u_{\overline{R}_{1,p}}(\tilde{\rho}_1)$ which shows that $\overleftarrow{R}_{lB}(N_l)$ intersects with $\overrightarrow{S}_{rB}(N_r)$ at N_{*B} (Fig 8 and Fig 9).

Subcase 1.1. $k_l = k_r$, that is $g_1(\rho_*) = g_2(\rho_*)$. There is no contact discontinuity of the new Riemann solution for this case and the solution is expressed by $\overrightarrow{R}J \rightarrow \overleftarrow{R}\overrightarrow{S}$.

Subcase 1.2. $k_l > k_r$, we know that $g_1(\rho_*) > g_2(\rho_*)$. In this case, we should seek the solution in $\{(\bar{\rho}_1, \bar{\rho}_2)|0 < \bar{\rho}_1 < 0\}$ $\rho_*, \bar{\rho}_2 > \rho_*$. It follows that there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ which satisfies $0 < \bar{\rho}_1 < \rho_*, \ \rho_* < \bar{\rho}_2$ and the solution is given by $\overrightarrow{RJ} \to \overleftarrow{RJ}\overrightarrow{S}$.

Subcase 1.3. $k_l < k_r$, we get that $g_1(\rho_*) < g_2(\rho_*)$. In this case, we should look for the solution in $\{(\bar{\rho}_1, \bar{\rho}_2) | \bar{\rho}_1 >$ $\rho_*, 0 < \bar{\rho}_2 < \rho_*$. Due to $u_l < u_r$, we divide our discussions into the following two subcases.

Subcase 1.3.1. $u_{\vec{R}_{rB}}(0) < u_l < u_r$.

If $g_2(\rho_r) \leq g_1(\tilde{
ho}_1)$, from the continuity we obtain that $\rho_r < \bar{\rho}_2 < \rho_*$. Thus, the solution is described by \overrightarrow{RJS} .

If $g_2(\rho_r) > g_1(\tilde{\rho}_1)$ and $g_1(\rho_l) \ge g_2(\tilde{\rho}_2)$, where $\tilde{\rho}_2$ is determined by $u_l = u_{\overrightarrow{R}_{rB}}(\widetilde{\rho}_2)$. Thus, there exists a point $(\overline{\rho}_1, \overline{\rho}_2)$ which satisfies $\rho_* < \widetilde{\rho}_1 < \overline{\rho}_1 < \rho_l$ and $\widetilde{\rho}_2 < \overline{\rho}_2 < \overline{\rho}$ $\rho_r < \rho_*$, and we know that the solution is given by $\overrightarrow{RJ} \rightarrow$ $\overleftarrow{R} \overrightarrow{J} \overrightarrow{R}.$

If $g_2(\rho_r) > g_1(\tilde{\rho}_1)$ and $g_1(\rho_l) < g_2(\tilde{\rho}_2)$, it is obvious that $g_1(\tilde{\rho}_3) > g_2(0)$, where $\tilde{\rho}_3 > \rho_l$ satisfies $u_{\overrightarrow{R}_{rB}}(0) =$ $u_{\overline{S}_{1B}}(\tilde{\rho}_3)$. Therefore there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ satisfying $\rho_* \stackrel{\sim}{<} \rho_l < \bar{\rho}_1 < \bar{\rho}_3$ and $0 < \bar{\rho}_2 < \bar{\rho}_2 < \rho_*$ and the solution is expressed by $\overrightarrow{R} J \to \overleftarrow{S} J \overrightarrow{R}$.



Subcase 1.3.2. $u_l < u_{\overrightarrow{R}_{rB}}(0) < u_r$. Obviously there exists a point $\tilde{\rho}_4$ < ρ_l such that $u_{\overrightarrow{R}_{lB}}(0) = u_{\overleftarrow{R}_{lB}}(\widetilde{\rho}_4).$

If $g_1(\tilde{\rho}_1) \ge g_2(\rho_r)$, there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ satisfying $ho_* < ar
ho_1 < ar
ho_1$ and $ho_r < ar
ho_2 <
ho_*$, thus the solution is expressed by $\overrightarrow{R}_{J}^{>} \rightarrow \overleftarrow{R}_{J}^{>} \overrightarrow{S}$.

If $g_1(\tilde{\rho}_1) < g_2(\rho_r)$, from $g_1(\tilde{\rho}_4) > g_2(0)$ and the continuity, there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ which satisfies $\rho_* < \tilde{\rho}_1 < \bar{\rho}_1 < \tilde{\rho}_4 < \rho_l$ and $0 < \bar{\rho}_2 < \rho_r < \rho_*$ which shows that the solution is given by $\overrightarrow{RJ} \rightarrow \overrightarrow{RJR}$.

Case 2. $\tilde{\rho}_1 < \rho_r$, for this case it follows that $\overline{R}_{lB}(N_l)$ intersects with $\overline{R}_{rB}(N_r)$ at N_{*B} (Fig 10). **Subcase 2.1.** $k_l = k_r$.

Since $g_1(\rho_*) = g_2(\rho_*)$, we know that the solution is $\overrightarrow{RJ} \rightarrow \overleftarrow{RR}$. Note that for this case there is no contact discontinuity.

Subcase 2.2. $k_l > k_r$. In this case, $g_1(\rho_*) > g_2(\rho_*)$ and we should seek the solution in $\{(\bar{\rho}_1, \bar{\rho}_2) | 0 \le \bar{\rho}_1 < \rho_*, \bar{\rho}_2 > \rho_*\}$. There are two possible cases as follows.



Fig 10 The interaction of \overrightarrow{R} and $\overset{>}{J}$, $\tilde{\rho}_1 < \rho_r$.

Subcase 2.2.1. $u_r \ge f_1(0)$.

There exists $\tilde{\rho}_5 \in (\rho_*, \rho_r)$ such that $u_{\overline{R}_{lB}}(0) = u_{\overline{R}_{rB}}(\tilde{\rho}_5)$ and $g_1(0) < g_2(\tilde{\rho}_5)$. It follows that there exists $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \rho_*, \ \rho_* < \bar{\rho}_2 < \tilde{\rho}_5$ and the solution is $\overrightarrow{R}_J \rightarrow \overleftarrow{R}_J \overrightarrow{R}$.

Subcase 2.2.2. $u_r < f_1(0)$.

If $g_2(\rho_r) \geq g_1(\tilde{\rho}_5)$, we know that there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$: $\tilde{\rho}_1 \leq \bar{\rho}_1 < \rho_*, \ \rho_* < \bar{\rho}_2 < \rho_r$ and the solution is given by $\overrightarrow{RJ} \rightarrow \overleftarrow{RJR}$.

If $g_2(\rho_r) < g_1(\tilde{\rho}_5)$, we obtain a point $(\bar{\rho}_1, \bar{\rho}_2)$: $0 < \bar{\rho}_1 < \tilde{\rho}_1 < \rho_*, \ \rho_* < \rho_r < \bar{\rho}_2$, and the solution is expressed by $\overrightarrow{RJ} \rightarrow \overleftarrow{RJS}$.

Subcase 2.3. $k_l < k_r$. That is to say $g_1(\rho_*) < g_2(\rho_*)$ and we should look for the solution in $\{(\bar{\rho}_1, \bar{\rho}_2) | \bar{\rho}_1 > \rho_*, 0 < \bar{\rho}_2 < \rho_*\}$.

Subcase 2.3.1. $u_l \leq f_2(0)$.

Obviously there exists a point $\tilde{\rho}_6 \in (\rho_*, \rho_l)$ such that $u_{\overline{R}_{lB}}(\tilde{\rho}_6) = u_{\overline{R}_{rB}}(0)$ and $g_1(\tilde{\rho}_6) > g_2(0)$. We know there exists a point $(\bar{\rho}_1, \bar{\rho}_2) : \rho_* < \bar{\rho}_1 < \tilde{\rho}_6, \ 0 < \bar{\rho}_2 < \rho_*$ and the solution is $\overrightarrow{R}_J \rightarrow \overleftarrow{R}_J \overrightarrow{R}$.

Subcase 2.3.2. $u_l > f_2(0)$.

Since there exists a point $\tilde{\rho}_7 \in (0, \rho_*)$ such that $u_l = u_{\overrightarrow{R}_{rB}}(\tilde{\rho}_7)$, then it follows that

if $g_1(\rho_l) \ge g_2(\tilde{\rho}_7)$, there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$: $\rho_* < \bar{\rho}_1 < \rho_l, \ \tilde{\rho}_7 \le \bar{\rho}_2 < \rho_*$ and the solution is given by as follows $\overrightarrow{RJ} \rightarrow \overleftarrow{RJR}$;

if $g_1(\rho_l) < g_2(\tilde{\rho}_7)$, since there exists a point $(\bar{\rho}_1, \bar{\rho}_2) : \rho_l < \bar{\rho}_1, \ 0 < \bar{\rho}_2 < \tilde{\rho}_7 < \rho_*$, and we get the solution is $\overrightarrow{R}_J^> \rightarrow \overleftarrow{S}_J^> \overrightarrow{R}$.

Notice that the wave interaction between J and \overline{R} can be studied similarly and omitted for simplicity.

Theorem 3.2 When a rarefaction wave collides with a contact discontinuity which is of a jump decrease in density, we observe that the rarefaction wave in the local solution of the initial value problem (4) and (6) continues to move forward in its propagating direction or a new shock wave will appear. Meanwhile, a new rarefaction wave (or shock wave) propagating in the opposite direction will appear. Furthermore, the contact discontinuity may appear or not after the wave interaction.

Case (iii) The interaction of the rarefaction wave with the contact discontinuity \overrightarrow{RJ} .

It is easy to see that $u_l < u_m = u_r$ and $\rho_m < \rho_r$. Similarly with the discussions in Case (i) of this section, we know that the curve $\overrightarrow{R}_{mB}(N_m)$ lies always above the curve $\overrightarrow{R}_{rB}(N_r)$ and the curve $\overrightarrow{S}_{lB}(N_l)$ intersects with $\overrightarrow{R}_{rB}(N_r)$ at the point N_{*B} (Fig 11 and Fig 12).

Subcase 1. $k_l = k_r$.

It holds that $g_1(\rho_*) = g_2(\rho_*)$ which shows that there is no contact discontinuity, and we obtain the solution is given by $\overrightarrow{RJ} \rightarrow \overleftarrow{SR}$.

Subcase 2. $k_l < k_r$. In this case, we know $g_1(\rho_*) < g_2(\rho_*)$ and should look for the solution in $\{(\bar{\rho}_1, \bar{\rho}_2) | \bar{\rho}_1 > \rho_*, 0 < \bar{\rho}_2 < \rho_*\}$. We define $\tilde{\rho}_1$ which satisfies $u_{\vec{R}_{rB}}(0) = u_{\vec{S}_{1B}}(\tilde{\rho}_1)$. Thus, there exists a point $(\bar{\rho}_1, \bar{\rho}_2) : \rho_* < \bar{\rho}_1 < \tilde{\rho}_* = 0 < \bar{\rho}_* < \rho_*$ and the solution is $\vec{P} < c_* < c_$

 $\tilde{\rho}_1, \ 0 < \bar{\rho}_2 < \rho_*$ and the solution is $\overrightarrow{RJ} \rightarrow \overleftarrow{SJR}$.

Subcase 3. $k_l > k_r$. In this case, we know that $g_1(\rho_*) > g_2(\rho_*)$ and should look for the solution in $\{(\bar{\rho}_1, \bar{\rho}_2)|0 < \bar{\rho}_1 < \rho_*, \ \bar{\rho}_2 > \rho_*\}$. There are two possible cases as follows. **Subcase 3.1.** $u_l < u_r < u_{\overline{R}_{lB}}(0)$. Since there exists respectively $\tilde{\rho}_2 \in (0, \rho_l)$ and $\tilde{\rho}_3 \in (\rho_*, \rho_r)$ such that $u_r = u_{\overline{R}_{lB}}(\tilde{\rho}_2)$ and $u_l = u_{\overline{R}_{rB}}(\tilde{\rho}_3)$.

If $g_1(\rho_l) \leq g_2(\tilde{\rho}_3)$, similar discussions with the above, there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$: $\rho_l \leq \bar{\rho}_1 < \rho_*, \ \rho_* \leq \bar{\rho}_2 \leq \tilde{\rho}_3$ and the solution is $\vec{R} \stackrel{\frown}{J} \rightarrow \stackrel{\frown}{S} \stackrel{\frown}{J} \stackrel{\frown}{R}$.

If $g_1(\rho_l) > g_2(\tilde{\rho}_3)$ and $g_2(\rho_r) \ge g_1(\tilde{\rho}_2)$, there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$: $\tilde{\rho}_2 < \bar{\rho}_1 \le \rho_l$, $\tilde{\rho}_3 < \bar{\rho}_2 < \rho_r$, we get the solution is $\vec{R}_J^{\ j} \to \vec{R}_J^{\ j} \vec{R}$.

If $g_1(\rho_l) > g_2(\tilde{\rho}_3)$ and $g_2(\rho_r) < g_1(\tilde{\rho}_2)$, since there exists a point $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \tilde{\rho}_2, \ \bar{\rho}_2 > \rho_r$, we obtain the solution is given by $\overrightarrow{RJ} \rightarrow \overleftarrow{RJS}$.



Fig 11 Interaction of \overrightarrow{R} and \overrightarrow{J} . Find the second second

Fig 12 Wave interaction (ρ, u) .

Subcase 3.2. $u_l < u_{\overline{R}_{lB}}(0) < u_r$. Define respectively $\tilde{\rho}_4 > \rho_*$ and $\tilde{\rho}_4 < \tilde{\rho}_5 < \rho_r$ such that $u_l = u_{\overline{R}_{rB}}(\tilde{\rho}_4)$ and $u_{\overline{R}_{lB}}(0) = u_{\overline{R}_{rB}}(\tilde{\rho}_5)$. **Subcase 3.2.1.** $g_1(\rho_l) \leq g_2(\tilde{\rho}_4)$. There exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_l < \bar{\rho}_1 < \rho_*, \ \rho_* < \bar{\rho}_2 < \tilde{\rho}_4$ and the solution is $\overrightarrow{RJ} \rightarrow \overleftarrow{SJR}$. **Subcase 3.2.2.** $g_1(\rho_l) > g_2(\tilde{\rho}_4)$. In view of $g_2(\tilde{\rho}_5) > g_1(0)$, we get that there exists $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \rho_l < \rho_*, \ \rho_* < \bar{\rho}_2 < \bar{\rho}_4$

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 $\tilde{\rho}_4 < \bar{\rho}_2 \leq \tilde{\rho}_5 < \rho_r$ and the solution is expressed by $\overrightarrow{R}_J^{\leq} \rightarrow \overrightarrow{R}_J \overrightarrow{R}$.

The wave interaction between \hat{J} and \overleftarrow{R} can be studied similarly and omitted for simplicity.

Theorem 3.3 When a rarefaction wave collides with a contact discontinuity which is of a jump increase in density, we obtain that the rarefaction wave in the local solution of the initial value problem (4) and (6) continues to move forward in its propagating direction or a new shock wave will appear. Meanwhile, a new rarefaction wave (or shock wave) propagating in the opposite direction will appear. Furthermore, the contact discontinuity may appear or not after the wave interaction.

Case (iv) The collision of the two rarefaction waves \overrightarrow{RR} .

In this case, it hold that $u_l < u_m$ and $u_m < u_r$. Similar with Case (i) of this section, we obtain that the curve $\overrightarrow{R}_{rB}(N_r)$ lies always above the curve $\overrightarrow{R}_{mB}(N_m)$ and the curve $\overrightarrow{R}_{mB}(N_m)$ lies always above the curve $\overrightarrow{R}_{lB}(N_l)$. It yields that $\overrightarrow{R}_{lB}(N_l)$ intersects with $\overrightarrow{R}_{rB}(N_r)$ at N_{*B} where a new Riemann problem is formed (Fig 13 and Fig 14). In order to obtain the solution of the new Riemann problem, we discuss as follows.

Subcase 1. $k_l = k_r$.

It holds that $g_1(\rho_*) = g_2(\rho_*)$ and there is no contact discontinuity. Thus, we obtain the solution is $\overrightarrow{R} \overrightarrow{R} \to \overleftarrow{R} \overrightarrow{R}$.





Subcase 2. $k_l > k_r$. In this case, we know $g_1(\rho_*) > g_2(\rho_*)$ and should seek the solution in $\{(\bar{\rho}_1, \bar{\rho}_2) | 0 < \bar{\rho}_1 < \rho_*, \bar{\rho}_2 > \rho_*\}$. There are two possible cases as follows.

Subcase 2.1. $u_r \geq f_1(0)$. We can define $\tilde{\rho}_1$ such that $u_{\overrightarrow{R}_{rB}}(\tilde{\rho}_1) = f_1(0)$. Thus, there exists a point $(\overline{\rho}_1, \overline{\rho}_2) : 0 < \overline{\rho}_1 < \rho_*, \ \rho_* < \overline{\rho}_2 < \widetilde{\rho}_1$ and the solution is expressed by $\overrightarrow{R} \overrightarrow{R} \rightarrow \overrightarrow{R} \overrightarrow{J} \overrightarrow{R}$.

Subcase 2.2. $u_r < f_1(0)$. There exists $0 < \tilde{\rho}_2 < \rho_*$ such that $u_{\overline{R}_{lB}}(\tilde{\rho}_2) = u_r$.

Subcase 2.2.1. $g_2(\rho_r) \geq g_1(\tilde{\rho}_2)$. There exists a point $(\bar{\rho}_1, \bar{\rho}_2)$: $\tilde{\rho}_2 < \bar{\rho}_1 < \rho_*, \ \rho_* < \bar{\rho}_2 < \rho_r$ and we get that the solution is $\overrightarrow{R} \overrightarrow{R} \rightarrow \overleftarrow{R} \overrightarrow{J} \overrightarrow{R}$.

Subcase 2.2.2. $g_2(\rho_r) < g_1(\tilde{\rho}_2)$. There exists a point $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \tilde{\rho}_2 < \rho_*, \ \bar{\rho}_2 > \rho_r > \rho_*$ and the solution is $\overrightarrow{R} \overleftarrow{R} \rightarrow \overleftarrow{RJS}$.

Subcase 3. $k_l < k_r$. In this case, we have $g_1(\rho_*) < g_2(\rho_*)$ and should seek the solution in $\{(\bar{\rho}_1, \bar{\rho}_2) | \bar{\rho}_1 > \rho_*, 0 < \bar{\rho}_2 < \rho_*\}$. There are two possible cases as follows. **Subcase 3.1.** $u_l \leq f_2(0)$.

It is obvious that there exist $\tilde{\rho}_3 \in (\rho_*, \rho_l)$ such that $u_{\overline{R}_{l,B}}(\tilde{\rho}_3) = u_{\overline{R}_{r,B}}(0)$, and $u_l = u_{\overline{R}_{r,B}}(\tilde{\rho}_4)$, respectively.

Due to $g_1(\tilde{\rho}_3) > g_2(0)$, we get the solution is given by $\overrightarrow{RR} \rightarrow \overrightarrow{RJR}$.

Subcase 3.2. $u_l > f_2(0)$. There are two possible cases as follows.

Subcase 3.2.1. $g_1(\rho_l) \ge g_2(\tilde{\rho}_4)$, similarly there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$: $\rho_* < \bar{\rho}_1 < \rho_l, \ \tilde{\rho}_4 \le \bar{\rho}_2 < \rho_*$ and we obtain that the solution is $\overrightarrow{R} \overleftarrow{R} \rightarrow \overleftarrow{R} \overrightarrow{J} \overrightarrow{R}$.

Subcase 3.2.2. $g_1(\rho_l) < g_2(\tilde{\rho}_4)$, since there exists a point $(\bar{\rho}_1, \bar{\rho}_2) : \rho_l < \bar{\rho}_1 < \tilde{\rho}_5, \ 0 < \bar{\rho}_2 < \tilde{\rho}_4$, where $\tilde{\rho}_5$ satisfies $u_{\overline{S}_{IB}}(\tilde{\rho}_5) = u_{\overrightarrow{R}_{rB}}(0)$, and we get the solution is $\overrightarrow{R} \times \overrightarrow{R} \rightarrow \overrightarrow{S} J \overrightarrow{R}$.

Theorem 3.4 When a forward rarefaction wave collides with a backward rarefaction wave, we find that the forward (backward) rarefaction wave in the local solution of the initial value problem (4) and (6) continues to move forward in its propagating direction or a new forward (backward) shock wave will appear. Furthermore, the contact discontinuity may appear or not after the wave interaction.

Case (v) The collision of the rarefaction wave and the shock wave \overrightarrow{RS} .

Since the curve $\overline{R}_{mB}(N_m)$ lies always above the curve $\overrightarrow{R}_{rB}(N_r)$ and the curve $\overline{S}_{mB}(N_m)$ lies always above the curve $\overline{S}_{lB}(N_l)$. It yields that $\overline{S}_{lB}(N_l)$ intersects with $\overrightarrow{R}_{rB}(N_r)$ at N_{*B} where a new Riemann problem is formed (Fig 15 and Fig 16). We discuss the construction of the solution of the new Riemann problem as follows.

Subcase 1. $k_l = k_r$. We know $g_1(\rho_*) = g_2(\rho_*)$ and there is no contact discontinuity. Thus, the solution is given by $\overrightarrow{R} \xrightarrow{S} \rightarrow \overrightarrow{S} \overrightarrow{R}$.

Subcase 2. $k_l < k_r$. This shows that $g_1(\rho_*) < g_2(\rho_*)$. Since there exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_* < \bar{\rho}_1 < \tilde{\rho}_1, \ 0 < \bar{\rho}_2 < \rho_*$ where $\tilde{\rho}_1 > \rho_*$ satisfies $u_{\overrightarrow{R}_{rB}}(0) = u_{\overleftarrow{S}_{1B}}(\tilde{\rho}_1)$. It follows that the solution is given by $\overrightarrow{R} \stackrel{\scriptstyle >}{S} \rightarrow \stackrel{\scriptstyle >}{S} \stackrel{\scriptstyle >}{J} \stackrel{\scriptstyle >}{R}$.



Fig 15 Interaction of R and S. Fig 16 wave interaction in (ρ, u) . **Subcase 3.** $k_l > k_r$. This means that $g_1(\rho_*) > g_2(\rho_*)$.

There are three possible cases as follows.

Subcase 3.1. $u_l \ge u_r$. It is shown that there exists $\tilde{\rho}_2$ such that $u_r = u_{\overline{S}_{IR}}(\tilde{\rho}_2)$.

If $g_2(p_r) \ge g_1(\tilde{\rho}_2)$, since there exists $(\bar{\rho}_1, \bar{\rho}_2) : \tilde{\rho}_2 < \bar{\rho}_1 < \rho_*, \ \rho_* < \bar{\rho}_2 < \rho_r$, the solution is shown by $\overrightarrow{R} \overleftarrow{S} \to \overleftarrow{S} \overrightarrow{J} \overrightarrow{R}$.

If $g_2(p_r) < g_1(\tilde{\rho}_2)$ and $g_1(\rho_l) < g_2(\tilde{\rho}_3)$, where $\tilde{\rho}_3 > \rho_r$ satisfies $u_l = u_{\vec{R}_{rB}}(\tilde{\rho}_3)$. Since there exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_l < \bar{\rho}_1 < \tilde{\rho}_2, \ \rho_r < \bar{\rho}_2 < \tilde{\rho}_3$, the solution is given by $\vec{R} \overleftarrow{S} \rightarrow \overleftarrow{S_I S}$.

If $g_2(p_r) < g_1(\tilde{\rho}_2)$ and $g_1(\rho_l) \ge g_2(\tilde{\rho}_3)$, since there exists $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \rho_l, \ \bar{\rho}_2 > \tilde{\rho}_3$. Thus, the result is $\overrightarrow{R} : \overrightarrow{S} \to \overrightarrow{RJ} : \overrightarrow{S}$.

Subcase 3.2. $u_l < u_r < u_{\overline{R}_{lB}}(0)$. There exist $\tilde{\rho}_4 \in (\rho_*, \rho_r)$ and $\tilde{p}_5 \in (0, \rho_l)$ satisfies respectively that $u_l = u_{\overline{R}_{rB}}(\tilde{\rho}_4)$ and $u_r = u_{\overline{R}_{lB}}(\tilde{\rho}_5)$.

If $\mathcal{G}_1(\rho_l) < \mathcal{G}_2(\tilde{\rho}_4)$, since there exists $(\bar{\rho}_1, \bar{\rho}_2) : \rho_l < \bar{\rho}_1 < \rho_*$, $\rho_* < \bar{\rho}_2 < \tilde{\rho}_4$, we obtain that the solution is given by $\overrightarrow{RS} \rightarrow \overrightarrow{SJR}$.

If $g_1(\rho_l) \ge g_2(\tilde{\rho}_4)$ and $g_2(\rho_r) \ge g_1(\tilde{\rho}_5)$, since there exists $(\bar{\rho}_1, \bar{\rho}_2) : \tilde{\rho}_5 < \bar{\rho}_1 < \rho_l, \ \tilde{\rho}_4 < \bar{\rho}_2 < \rho_r$, we get that the solution is expressed by $\overrightarrow{RS} \to \overleftarrow{RJR}$.

If $g_1(\rho_l) \ge g_2(\tilde{\rho}_4)$ and $g_2(\rho_r) < g_1(\tilde{\rho}_5)$, since there exists $(\bar{\rho}_1, \bar{\rho}_2) : 0 < \bar{\rho}_1 < \tilde{\rho}_5, \ \rho_r < \bar{\rho}_2 < \tilde{\rho}_6$, where $\tilde{\rho}_6$ satisfies $u_{\overline{R}_{lB}}(0) = u_{\overline{S}_{rB}}(\tilde{\rho}_6)$. Therefore, we obtain that the solution is given by $\overline{R} \overleftarrow{S} \rightarrow \overline{R} \overrightarrow{J} \overrightarrow{S}$.

Subcase 3.3. $u_l < u_{\overline{R}_{1B}}(0) < u_r$. There exist respectively $\tilde{\rho}_7 \in (\rho_*, \rho_r)$ and $\tilde{\rho}_8 \in (\tilde{\rho}_7, \rho_r)$ satisfies that $u_l = u_{\overline{R}_{rB}}(\tilde{\rho}_7)$ and $u_{\overline{R}_{IR}}(0) = u_{\overline{R}_{rR}}(\tilde{\rho}_8)$.

Subcase 3.3.1. $g_1(\rho_l) < g_2(\tilde{\rho}_7)$. There exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ which satisfies $\rho_l < \bar{\rho}_1 < \rho_*$, $\rho_* < \bar{\rho}_2 < \tilde{\rho}_7 < \rho_r$ and it follows that the solution is given by $\vec{R} \overleftarrow{S} \rightarrow \overleftarrow{S} \overrightarrow{J} \vec{R}$.

Subcase 3.3.2. $g_1(\rho_l) \ge g_2(\tilde{\rho}_7)$. Since there exists a point $(\bar{\rho}_1, \bar{\rho}_2)$ such that $0 < \bar{\rho}_1 < \rho_l$, $\tilde{\rho}_7 < \bar{\rho}_2 < \tilde{\rho}_8 < \rho_r$, we know that the solution is shown by $\overrightarrow{RS} \xrightarrow{\sim} \overrightarrow{RJR}$. The wave interaction between \overrightarrow{S} and \overrightarrow{R} can be investi-

The wave interaction between S and R can be investigated similarly and omitted for simplicity.

Theorem 3.5 When a rarefaction wave collides with a shock wave, we obtain that the rarefaction wave in the local solution of the initial value problem (4) and (6) continues to move forward in its propagating direction or a new shock wave will appear. Meanwhile, the shock wave continues to move forward in its propagating direction or a new rarefaction wave will appear. Furthermore, the contact discontinuity may appear or not after the wave interaction.

IV. CONCLUSION

We have finished the wave interactions of the elementary waves and observe that the wave interactions of Magnetogasdynamics system (4) are more complicated than that of the conventional gas dynamics. The projection of the contact discontinuity on the phase plane (ρ, u) which is a straight line parallel to the ρ -axis, while the projection of the contact discontinuity for the conventional gas dynamics on the phase plane (ρ, u) which is just a point. Based on the above discussions, we obtain the following main result.

Theorem 4.1 There exists uniquely the solution of the initial value problem (4) with the initial data (6).

we have finished the discussions for all kinds of wave interactions. It is important to study the interactions of elementary waves for system (4) not only because of their significance in practical applications in Magnetogasdynamics system such as comparison with the numerical and experimental results, but also because of their basic role as building blocks for the theory of Magnetogasdynamics.

While the system (4) is one-dimensional idealized simplified system, in our next works we will consider the high dimensional corresponding system which reveals the deep mechanism of the Magnetogasdynamics system. In our coming works, we also would like to study the above problem from the numerical calculation point of view and investigate the Magnetogasdynamics system furthermore.

REFERENCES

- H. Cabannes, "Theoretical magnetofluid dynamics," Academic Press, New York, 1970.
- [2] W.X. Li, "One-dimensional unsteady flowand shock wave," Chinese Series, National Defence Industrial Press, 2003.
- [3] D.Q. Li, T.H. Qin, "Physics and partial differential equations," Chinese Series, *Higher Education Press*, 2005.
- [4] F.G. Liu, "Life-span of classical solutions for one-dimensional hydromagnetic flow," *Applied Mathematics and Mechanics (English Edition)*, vol. 28, no. 4, pp. 511-520, 2007.
- [5] R.S. Myong and P.L. Roe, "Shock waves and rarefaction waves in magnetogasdynamics," part 1. a model system, *Journal of Plasma Physics*, vol. 58, no. 3, pp. 485-519, 1997.
- [6] R.S. Myong and P.L. Roe, "Shock waves and rarefaction waves in magnetogasdynamics," part 2. the MHD system, *Journal of Plasma Physics*, vol. 58, no. 3, pp. 521-552, 1997.
- [7] R. Singh and L.P. Singh, "Solution of the Riemann Problem in magnetogasdynamics," *International Journal of Non-Linear Mechanics*, vol. 67, pp. 326-330, 2014.
- [8] Y.B. Hu and W.C. Sheng, "The Riemann problem of conservation laws in magnetogasdynamics," *Communications on Pure and Applied Analysis*, vol. 12, no. 2, pp. 755-769, 2013.
- [9] S.A. Altaie, A.F. Jameel, A. Saaban, "Homotopy Perturbation Method Approximate Analytical Solution of Fuzzy Partial Differential Equation," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 1, pp. 22-28, 2019.
- [10] J.B. Bacani, G. Peichl, "The Second-Order Eulerian Derivative of a Shape Functional of a Free Boundary Problem," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 4, pp. 425-436, 2016.
- [11] T. Chang, L. Hsiao, "The Riemann Problem and Interaction of Waves in Gas Dynamics, Pitman Monographs," no. 41, *Longman Scientific and technical*, Essex, 1989.
- [12] Y.J. Liu, W.H. Sun, "The Riemann Problem for the Simplified Combustion Model in Magnetogasdynamics," *IAENG International Journal* of Applied Mathematics, vol. 49, no. 4, pp. 513-520, 2019.
- [13] J.Q. Xie, D.W. Deng and H.S. Zheng, "A Compact Difference Scheme for One-dimensional Nonlinear Delay Reaction-diffusion Equations with Variable Coefficient," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 1, pp. 14-19, 2017.
- [14] Y.J. Liu and W.H. Sun, "Elementary wave interactions in magnetogasdynamics," *Indian Journal of Pure and Applied Mathematics*, vol. 47, no. 1, pp. 33-57, 2016.
- [15] T. Raja Sekhar and V.D. Sharma, "Riemann problem and elementary wave interactions in isentropic magnetogasdynamics," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 2, pp. 619-636, 2010.
- [16] C. Shen, "The limits of Riemann solutions to the isentropic magnetogasdynamics," *Applied Mathematics Letters*, vol. 24, no. 7, pp. 1124-1129, 2011.
- [17] Y.J. Liu and W.H. Sun, "Riemann problem and wave interactions in Magnetogasdynamics," *Journal of Mathematical Analysis and Applications*, vol. 397, no. 2, pp. 454-466, 2013.