A New Class of Two-parameter Generalized Exponential Distribution and Its Statistical Inference

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***Abstract*—With the development of the generalized exponential (GE) distribution, the two-parameter generalized exponential (TPGE) distribution has been widely used for the analysis of lifetime data in recent years. In this paper, a new class of TPGE distribution is proposed on the basis of previous work. Then the relevant statistical inferences of the TPGE are also discussed. And, the parameters are estimated by using the moment estimation and the maximum likelihood estimation.**

***Index Terms*—Generalized exponential distribution, ME, MLE, Hazard function**

I. Introduction

As a sub-model of the exponentiated Weibull distribution, the generalized exponential (GE) distribution was firstly introduced by Mudholkar and Srivastava [1] and later studied by Mudholkar, Srivastava and Freimer [2] and Mudholkar and Hutson [3]. Since it can be used as a substitute for the commonly used gamma and Weibull distributions, so many scholars have studied GE distributions, especially Gupta and Kundu [4], [5], [6], [7], [8], [9], [10], Ahsanullah [11], Raqab and Ahsanullah [12], Raqab [13], Zheng [14], and Kundu, Gupta and Manglick [15]. The TPGE distribution was introduced by Gupta and Kundu [4], with the cumulative distribution function (CDF): $F(x;\alpha,\lambda)=(1-e^{-\lambda x})^{\alpha}\ ;x>0$, and the probability density function (PDF): $f(x;\alpha,\lambda)=\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1};x>0$, where α and λ are the shape and scale parameters, respectively.

By comparing the properties of generalized exponential distribution with gamma and Weibull distribution and the estimated values of different estimation methods for the same parameter, Gupta and Kundu [5], [9] found that gamma distribution and generalized exponential distribution are "almost the same" in many aspects (e.g. monotonicity of probability density function). In many cases, for a given gamma distribution, there is a generalized exponential distribution that can be replaced. For example, in the case of limited samples, the GE distribution family can be used as the replacement distribution of gamma distribution family, so the two distribution functions are almost the same. However, the mathematical structure of the GE distribution is relatively simple and easy to calculate, while the gamma distribution cannot generate random numbers that obey it due to the complexity of its own structure. Therefore, approximate gamma random numbers can be generated by using the generalized exponential distribution. Later, based on their own methods for replacing gamma distribution with GE distribution, Gupta and Kundu [16] put forward a simpler replacement method, but the precondition for using this method is that the shape parameter must be between 0 and 1.

In 2011, Gupta and Kundu [17] proposed a generalized form of GE distribution. They embed the two-parameter generalized exponential distribution into a larger distribution by introducing another shape parameter, with the CDF:

$$F\left(x;\alpha,\beta,\lambda\right)=\begin{cases}\left(1-\left(1-\beta\lambda x\right)^{\frac{1}{\beta}}\right)^{\alpha},\beta\neq 0\\ \left(1-e^{-\lambda x}\right)^{\alpha},\beta=0\end{cases},$$

$$\alpha>0,\lambda>0,-\infty<\beta<+\infty,\ x\in\begin{cases}\left(0,+\infty\right),\beta\leq 0\\ \left(0,\dfrac{1}{\beta\lambda}\right),\beta>0\end{cases}$$

Due to the addition of shape parameters β , the distribution becomes more flexible so that it can process truncated data very effectively.

Moreover, Gupta and Kundu also studied many different parameter estimation methods, including maximum likelihood estimation (MLE), moment estimation (ME) and probability map estimation based on completely random samples. Among them, Gupta and Kundu [6] used five methods of parameter estimation to analyze their generalized exponential distribution, namely MLE, quantile-based estimation (PCE), least squares estimation (LSE), weighted least squares estimation (WLSE) and linear ME (LME), and combined with numerical analysis to compare the advantages and disadvantages of several estimation methods. The results showed that PCE has the best estimation effect with small

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samples, and LME has the best estimation effect with medium and large samples.

As the GE distribution is used to analyze life data effectively, besides Gupta and Kundu, many scholars have also studied it.

Ahsanullah [11] gave the least variance linear unbiased estimation. Raqab and Ahsanullah [12] derived the expressions of mean, variance and covariance of ordered statistics with known shape parameters, and discussed the best linear unbiased estimation of position parameters and scale parameters with them. Raqab [13] discussed the best linear unbiased estimation of three-parameter generalized exponential distribution. Elkahlout [18] studied the generalized exponential distribution under asymmetric loss by Bayesian parameter estimation method under the condition of gradually deleting data, and compared the characteristics of Bayesian estimation and maximum likelihood estimation by Monte Carlo simulation method. Chen [19] estimated the parameters of generalized exponential distribution with five methods under the condition of gradual type-I interval truncation. Then Teimouri [20] pointed out that the EM algorithm is incorrect and gave the correct algorithm. According to the recorded value, Wang et al. [21] gave the MLE, Bayes estimation and empirical Bayes estimation of generalized exponential distribution and pointed out that Bayes estimation and empirical Bayes estimation are better than MLE under appropriate prior distribution. Furthermore, the TestSTAT confidence interval estimation of one parameter exponential distribution was proposed by Juthaphorn Sinsomboonthong [22], and in the paper, the efficiency comparisons in terms of the coverage probabilities and the expected lengths of the confidence intervals were investigated.

In recent years, some new generators of distributions based on the exponential distribution such as the odd-generalized exponential family of distributions (OGE) [23] and odd-exponential-G family of distributions (OEG) [24] were proposed and analyzed. Muhammad [25] proposed a new family of distributions called the Poisson odd generalized exponential distribution (POGE), by joining together the odd generalized exponential family of distributions and the Poisson distribution. Shakhatreh [26] studied the five-parameter generalization of the EW distribution called the generalized extended exponential-Weibull (GExtEW) distribution, which can be very useful for modeling non-monotonic failure rate function, and provided a detailed study of its structural properties. Nasiru [27] proposed the exponentiated generalized exponential Dagum distribution. This family of distribution consists of a number of sub-models such as the exponentiated generalized Dagum distribution, Dagum distribution, Fisk distribution, Burr III distribution and exponentiated generalized exponential Burr III distribution among others. Martin Ricker and Dietrich von Rosen [28] extended exponential functions to model growth and found that the generalized exponential function is useful for modeling a path of changing relative growth continuously.

Based on the predecessors, this paper present a new class of TPGE distribution, and discuss its corresponding properties and parameters estimation. The rest of the paper is organized as follows. In section 2, we introduce the definition of a new class of TPGE distribution. The moments, hazard function and some other properties of the TPGE are given in section 3. In section 4, the moment estimation and MLE are used to estimate the parameters. Finally a conclusion is given in section 5.

II. A Class of TPGE Distribution

In this section, we introduce a new kind of TPGE distribution.

Proposition: A function $F(x)$ is a CDF if and only if it satisfies the following three conditions:

(1) $F(x_1) \le F(x_2)$, $x_1 < x_2$;

(2) $0 \le F(x) \le 1$;

(3) $\forall x \in D, F(x+0) = F(x)$, D is the domain of X .

Theorem 1: If a non-negative random variable X satisfies the following function:

$$F_1(x) = 1 + k_1 \exp\left(-\frac{x-\alpha}{k_1\beta}\right) - k_2 \exp\left(-\frac{x-\alpha}{k_2\beta}\right), \ \alpha > 0, \ \beta > 0,$$

$x \in (\alpha, +\infty)$, $k_1 \ne -1$, $k_2 = k_1 + 1$, then X is said to obey the CDF $F_1(x)$ and denoted as $X \sim GE_1(x;\alpha,\beta)$, where α is location parameter, β is scale parameter. And the corresponding PDF is

$$f_1(x) = F_1'(x) = \frac{1}{\beta}\left\{\exp\left(-\frac{x-\alpha}{k_2\beta}\right) - \exp\left(-\frac{x-\alpha}{k_1\beta}\right)\right\}.$$

Proof: (1) To satisfy the first condition of the CDF, it just needs $k_2 > k_1$ and $k_1 \ne -1$.

If the random variable $X \sim F_1(x;\alpha,\beta)$, it is easy to deduce the PDF of X is

$$f_1(x) = F_1'(x) = \frac{1}{\beta}\left\{\exp\left(-\frac{x-\alpha}{k_2\beta}\right) - \exp\left(-\frac{x-\alpha}{k_1\beta}\right)\right\}, k_1 \ne -1$$

$$x \ge \alpha > 0, \beta > 0.$$

When $k_1 = -1$, $f_1(x) = F_1'(x) = -\frac{1}{\beta}\exp(\frac{x-\alpha}{\beta}) < 0$.

If $f_1(x) = F_1'(x) > 0, x \in (\alpha, +\infty)$, then

$\exp\left(-\frac{x-\alpha}{k_2\beta}\right) > \exp\left(-\frac{x-\alpha}{k_1\beta}\right)$ and $k_1 \ne -1$,

so $-\frac{x-\alpha}{k_2\beta} > -\frac{x-\alpha}{k_1\beta}$, $k_2 > k_1$, $k_1 \ne -1$.

(2)To satisfy the boundedness, it just has to $k_2 = k_1 + 1$.

$$\begin{aligned}&\lim_{x\to+\infty} F(x;m,\beta;k_1,k_2)\\ &= \lim_{x\to+\infty}\left\{1 + k_1\exp\left(-\frac{x-\alpha}{k_1\beta}\right) - k_2\exp\left(-\frac{x-\alpha}{k_2\beta}\right)\right\}\\ &= 1\end{aligned}$$

If and only if $k_2 = k_1 + 1$, $k_1 \ne -1$, $\alpha > 0$, $\beta > 0$, there is

$$\begin{aligned}&\lim_{x\to\alpha^+} F(x;m,\beta;k_1,k_2)\\ &= \lim_{x\to\alpha^+}\left\{1 + k_1\exp\left(-\frac{x-\alpha}{k_1\beta}\right) - k_2\exp\left(-\frac{x-\alpha}{k_2\beta}\right)\right\}\\ &= 0\end{aligned}$$

Therefore, for any $x \in (\alpha, +\infty)$, when $k_2 = k_1 + 1$, $k_1 \neq -1$, $\alpha > 0$, $\beta > 0$, there is $0 \le F_1(x) \le 1$.

(3) $F_1(x)$ satisfies right continuity.

$F_1(x)$ is a right continuous function, that is, for any $x_0 \in [\alpha, +\infty)$, $\lim_{x \to x_0^+} F_1(x) = F_1(x_0)$. So $F_1(x_0 + 0) = F_1(x_0)$.

Above all, if $k_2 = k_1 + 1$, $k_1 \neq -1$, $x \ge \alpha > 0$, $\beta > 0$, then $F_1(x)$ is a CDF.

When the parameters α and β are fixed and k_2 changes, the PDF of the distribution is shown in Fig 1.

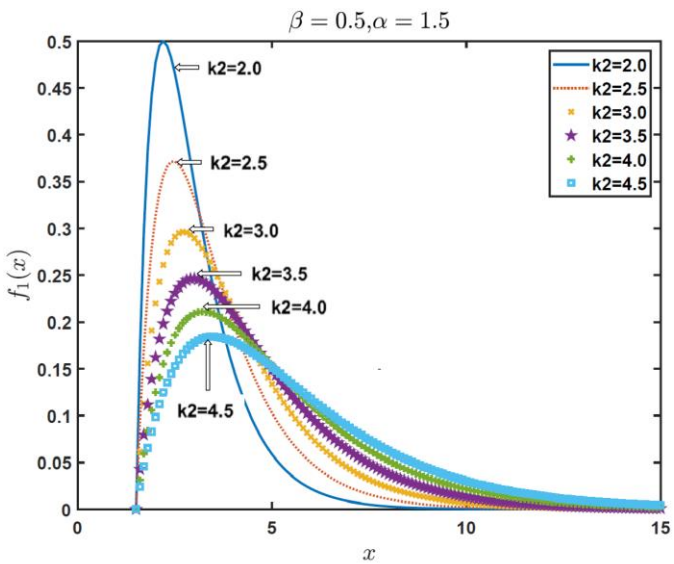

Fig.1.The figure of $f_1(x)$ when $\alpha = 0.5$, $\beta = 1.5$, $k_2 = 2, 2.5, 3, 3.5, 4, 4.5$

Special case: If $k_1 = 0$, the CDF is $F_1(x) = 1 - \exp(-\frac{x-\alpha}{\beta})$, and the PDF is $f_1(x) = \frac{1}{\beta}\exp(-\frac{x-\alpha}{\beta})$. In this case, $F_1(x)$ is a classical TPGE distribution (scale-location).

III. Statistical Properties

In this section, some statistical properties of the TPGE are discussed, such as moments, hazard function and mean residual lifetimes.

A. Moments

Theorem 2: If the random variable $X \sim GE_1(x;\alpha,\beta)$, the k-th moment $E_1(X^k)$ of X exists.

Proof:

$$E_1(X^k) = \int_\alpha^{+\infty} x^k f(x)dx$$
$$= \frac{1}{\beta}\int_\alpha^{+\infty} x^k \left\{\exp\left(-\frac{x-\alpha}{k_2\beta}\right) - \exp\left(-\frac{x-\alpha}{k_1\beta}\right)\right\}dx$$

Let $(x-\alpha)/\beta = t$, then $x = \beta t + \alpha$, $dx = \beta dt$, $t \in [0, +\infty)$.

$$E_1(X^k) = \int_0^{+\infty} (\alpha + \beta t)^k e^{-\frac{1}{k_2}t} dt - \int_0^{+\infty} (\alpha + \beta t)^k e^{-\frac{1}{k_1}t} dt$$
$$= \int_0^{+\infty} \sum_{i=0}^{k} C_k^i (\beta t)^i \alpha^{k-i} e^{-\frac{1}{k_2}t} dt - \int_0^{+\infty} \sum_{i=0}^{k} C_k^i (\beta t)^i \alpha^{k-i} e^{-\frac{1}{k_1}t} dt$$
$$= \left(\int_0^{+\infty} C_k^0 (\beta t)^0 \alpha^{k-0} e^{-\frac{1}{k_2}t} dt + \int_0^{+\infty} \sum_{i=1}^{k} C_k^i (\beta t)^i \alpha^{k-i} e^{-\frac{1}{k_2}t} dt\right)$$
$$- \left(\int_0^{+\infty} C_k^0 (\beta t)^0 \alpha^{k-0} e^{-\frac{1}{k_1}t} dt + \int_0^{+\infty} \sum_{i=1}^{k} C_k^i (\beta t)^i \alpha^{k-i} e^{-\frac{1}{k_1}t} dt\right)$$
$$= (k_2 - k_1)\alpha^k + \sum_{i=1}^{k} C_k^i \beta^i \alpha^{k-i} \int_0^{+\infty} t^i e^{-\frac{1}{k_2}t} dt - \sum_{i=1}^{k} C_k^i \beta^i \alpha^{k-i} \int_0^{+\infty} t^i e^{-\frac{1}{k_1}t} dt \tag{1}$$

When $\alpha > 0$ and $n = 1, 2, 3...$, $\int_0^{+\infty} x^n e^{-\frac{1}{a}x} dx = a^{n+1} n!$. So

$$E_1(X^k) = (k_2 - k_1)\alpha^k + \sum_{i=1}^{k} C_k^i \beta^i \alpha^{k-i} \int_0^{+\infty} t^i e^{-\frac{1}{k_2}t} dt - \sum_{i=1}^{k} C_k^i \beta^i \alpha^{k-i} \int_0^{+\infty} t^i e^{-\frac{1}{k_1}t} dt$$
$$= (k_2 - k_1)\alpha^k + \sum_{i=1}^{k} C_k^i \beta^i \alpha^{k-i} k_2^{i+1} i! - \sum_{i=1}^{k} C_k^i \beta^i \alpha^{k-i} k_1^{i+1} i!$$
$$= (k_2 - k_1)\alpha^k + \sum_{i=1}^{k} \left(k_2^{i+1} - k_1^{i+1}\right) C_k^i \beta^i \alpha^{k-i} i!$$

$k_2 = k_1 + 1$, then:

$$E_1(X^k) = \alpha^k + \sum_{i=1}^{k} \left(k_2^{i+1} - k_1^{i+1}\right) C_k^i \beta^i \alpha^{k-i} i! \tag{2}$$

That is, the k-th moment of the random variable exists.

It is easy to deduce that $E_1(X)$, $E_1(X^2)$ and $D_1(X) = E_1(X^2) - (E_1 X)^2$ all exist, that is

$$E_1(X) = \alpha + \beta\left(k_2^2 - k_1^2\right)$$
$$E_1(X^2) = \alpha^2 + 2\alpha\beta\left(k_2^2 - k_1^2\right) + 2\beta^2\left(k_2^3 - k_1^3\right)$$
$$D_1(X) = E_1(X^2) - [E_1(X)]^2$$
$$= \alpha^2 + 2\alpha\beta\left(k_2^2 - k_1^2\right) + 2\beta^2\left(k_2^3 - k_1^3\right) - \left[\alpha + \beta\left(k_2^2 - k_1^2\right)\right]^2$$
$$= \beta^2\left[2\left(k_2^3 - k_1^3\right) - \left(k_2^2 - k_1^2\right)^2\right]$$
$$= \beta^2\left[2(k_2 - k_1)\left(k_2^2 + k_1^2 + k_1 k_2\right) - (k_2 - k_1)^2 (k_2 + k_1)^2\right]$$
$$= \beta^2\left[2\left(k_2^2 + k_1^2 + k_1 k_2\right) - (k_2 + k_1)^2\right]$$
$$= \left(k_1^2 + k_2^2\right)\beta^2$$

B. Standard Form

The standard form of the TPGE distribution is discussed below.

Let $Y = \frac{x-\alpha}{\beta}$, then: $F_{1,Y}(y) = 1 + k_1 e^{-\frac{1}{k_1}y} - k_2 e^{-\frac{1}{k_2}y}, y > 0$.

If the random variable Y obeys this distribution, it is denoted as $Y \sim GE_{1,Y}(y;0,1)$, then:

$$f_{1,Y}(y) = e^{-\frac{1}{k_2}y} - e^{-\frac{1}{k_1}y}, y > 0$$
$$E_{1,Y}(Y) = \int_0^{+\infty} y(e^{-\frac{1}{k_2}y} - e^{-\frac{1}{k_1}y})dy = k_2^2 - k_1^2$$
$$E_{1,Y}(Y^2) = \int_0^{+\infty} y^2(e^{-\frac{1}{k_2}y} - e^{-\frac{1}{k_1}y})dy = 2\left(k_2^3 - k_1^3\right)$$
$$D_{1,Y}(Y) = E_{1,Y}(Y^2) - [E_{1,Y}(Y)]^2 = k_1^2 + k_2^2$$

C. Hazard Function

The survival function (SF) and hazard function (HF) are given in this subsection.

If $X \sim F_1(x;\alpha,\beta)$, in the case of $k_1 \neq 0$, the survival

function $S(x;\alpha,\beta)$ and hazard function $h(x;\alpha,\beta)$ are given as follows:

$$S(x;\alpha,\beta)=1-F_1(x;\alpha,\beta)$$
$$=k_2\exp\left(-\frac{x-\alpha}{k_2\beta}\right)-k_1\exp\left(-\frac{x-\alpha}{k_1\beta}\right),\ x\in(\alpha,+\infty);$$
$$h(x;\alpha,\beta)=\frac{f_1(x;\alpha,\beta)}{S(x;\alpha,\beta)}$$
$$=\frac{1}{\beta}\frac{\exp\left(-\frac{x-\alpha}{k_2\beta}\right)-\exp\left(-\frac{x-\alpha}{k_1\beta}\right)}{k_2\exp\left(-\frac{x-\alpha}{k_2\beta}\right)-k_1\exp\left(-\frac{x-\alpha}{k_1\beta}\right)},\ x\in(\alpha,+\infty).$$

Differentiating both sides of the above formula

$$h'(x;\alpha,\beta)=\frac{1}{\beta^3}\frac{\left(\frac{k_2}{k_1}+\frac{k_1}{k_2}-2\right)\exp\left(-\frac{x-\alpha}{k_1\beta}\right)\exp\left(-\frac{x-\alpha}{k_2\beta}\right)}{\left[k_2\exp\left(-\frac{x-\alpha}{k_2\beta}\right)-k_1\exp\left(-\frac{x-\alpha}{k_1\beta}\right)\right]^2}$$

It is easy to show that

$$\frac{1}{\beta^3\left[k_2\exp\left(-\frac{x-\alpha}{k_2\beta}\right)-k_1\exp\left(-\frac{x-\alpha}{k_1\beta}\right)\right]^2}>0$$

(1) For $k_1>0$, $\left(\frac{k_2}{k_1}+\frac{k_1}{k_2}-2\right)>0$, then $h'(x;\alpha,\beta)>0$, which means that $h(x;\alpha,\beta)$ is an increasing function for $x>\alpha$;

(2) For $k_1<0$, $k_1\neq-1$, $\frac{k_2}{k_1}+\frac{k_1}{k_2}-2<0$, then $h'(x;\alpha,\beta)<0$, which means that $h(x;\alpha,\beta)$ is a decreasing function for $x>\alpha$;

(3) For $k_1=0$, $h(x;\alpha,\beta)=\frac{1}{\beta}$.

D. The Mean Residual Lifetime

Here the behavior of the mean residual lifetime (MRLT) of the TPGE is considered.

Let $m(t;\alpha,\beta)$ denote the MRLT of the TPGE, then

$$m(t;\alpha,\beta)=E(X-t\mid X>t)=\frac{1}{S(x;\alpha,\beta)}\int_t^\infty(x-t)f_1(x;\alpha,\beta)dx.$$

The above formula can be rewritten in the form:

$m(t;\alpha,\beta)=\int_0^\infty\exp(-\int_t^{t+x}h(\tau;\alpha,\beta)d\tau)dx$, where $h(\tau;\alpha,\beta)$ is the HF of the TPGE.

IV. Parameter Estimation

A. Moment Estimation of Parameters

The moment estimation of parameters in the case of full samples is discussed in this section. Let $X_1',X_2',\cdots,X_n'$ be an independent random sample of size n from the $GE_1(x;\alpha,\beta)$, then we discuss the parameters estimation under different cases.

1. When both α and β are unknown, it can be seen that the mathematical expectation and second moment of X' are:

$$E_1(X)=\alpha+\beta\left(k_2^2-k_1^2\right)$$
$$E_1\left(X^2\right)=\alpha^2+2\alpha\beta\left(k_2^2-k_1^2\right)+2\beta^2\left(k_2^3-k_1^3\right)$$

Then we can establish the following equation:

$$\begin{cases}\frac{1}{n}\sum_{i=1}^n X_i'=\alpha+\beta\left(k_2^2-k_1^2\right)\\ \frac{1}{n}\sum_{i=1}^n X_i'^2=\alpha^2+2\alpha\beta\left(k_2^2-k_1^2\right)+2\beta^2\left(k_2^3-k_1^3\right)\end{cases}\quad(3)$$

Thus, the moment estimates of the parameters α and β are:

$$\begin{cases}\hat{\alpha}_{1,1}=\bar{X}'-\frac{k_1+k_2}{\sqrt{k_1^2+k_2^2}}\sqrt{\overline{X'^2}-(\bar{X}')^2}\\ \hat{\beta}_{1,1}=\frac{1}{\sqrt{k_1^2+k_2^2}}\sqrt{\overline{X'^2}-(\bar{X}')^2}\end{cases}\quad(4)$$

2. When β is known (suppose $\beta=\beta_0$) and α is unknown, then we can obtain: $\frac{1}{n}\sum_{i=1}^n X_i'=\alpha+\beta_0\left(k_2^2-k_1^2\right)$.

So, the moment estimate of the parameter α is:

$$\hat{\alpha}_{1,1}=\bar{X}'-(k_1+k_2)\beta_0$$

3. When α is known (suppose $\alpha=\alpha_0$) and β is unknown, then we can obtain $\frac{1}{n}\sum_{i=1}^n X_i'=\alpha_0+\beta\left(k_2^2-k_1^2\right)$.

So, the moment estimate of the parameter β is:

$$\hat{\beta}_{1,1}=\frac{\bar{X}'-\alpha_0}{k_1+k_2}$$

B. Maximum Likelihood Estimation of Parameters

We discuss the MLE of parameters in the case of full samples. Let $X_1',X_2',\cdots,X_n'$ be an independent random sample of size n from the $GE_1(x;\alpha,\beta)$ .Then, the likelihood function is:

$$L_1(\alpha,\beta)=\prod_{i=1}^n f_1(x_i)=\prod_{i=1}^n\frac{1}{\beta}\left(\exp\left\{-\frac{x_i-\alpha}{k_2\beta}\right\}-\exp\left\{-\frac{x_i-\alpha}{k_1\beta}\right\}\right)$$
$$=\frac{1}{\beta^n}\exp\left\{-\left(\sum_{i=1}^n x_i-n\alpha\right)\Big/k_2\beta\right\}\prod_{i=1}^n\left(1-\exp\left\{-\frac{(k_2-k_1)(x_i-\alpha)}{k_1k_2\beta}\right\}\right)\quad(5)$$
$$=\frac{1}{\beta^n}\exp\left\{-\left(\sum_{i=1}^n x_i-n\alpha\right)\Big/k_2\beta\right\}\prod_{i=1}^n\left(1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta}\right\}\right)$$

The corresponding log-likelihood function is:

$$\ln L_1(\alpha,\beta)$$
$$=\ln\left\langle\frac{1}{\beta^n}\exp\left\{-\left(\sum_{i=1}^n x_i-n\alpha\right)\Big/k_2\beta\right\}\prod_{i=1}^n\left(1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta}\right\}\right)\right\rangle\quad(6)$$
$$=\frac{n\alpha}{k_2\beta}-n\ln\beta-\frac{\sum_{i=1}^n x_i}{k_2\beta}+\sum_{i=1}^n\ln\left(1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta}\right\}\right)$$

1. When α and β are unknown, the partial derivatives of parameters α and β are calculated as follows:

$$\frac{\partial\ln L_1(\alpha,\beta)}{\partial\alpha}=\frac{n}{k_2\beta}-\frac{1}{k_1k_2\beta}\sum_{i=1}^n\frac{\exp\{-(x_i-\alpha)/k_1k_2\beta\}}{1-\exp\{-(x_i-\alpha)/k_1k_2\beta\}}$$

$$\frac{\partial\ln L_1(\alpha,\beta)}{\partial\beta}=-\frac{n\alpha}{k_2\beta^2}-\frac{n}{\beta}+\frac{\sum_{i=1}^n x_i}{k_2\beta^2}-\sum_{i=1}^n\frac{x_i-\alpha}{k_1k_2\beta^2}\frac{\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta}\right\}}{1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta}\right\}}$$

Let $\begin{cases} \dfrac{\partial \ln L_1(\alpha,\beta)}{\partial \alpha}=0 \\ \dfrac{\partial \ln L_1(\alpha,\beta)}{\partial \beta}=0 \end{cases}$, then

$$\begin{cases} \dfrac{n}{k_2\beta}-\dfrac{1}{k_1k_2\beta}\sum\limits_{i=1}^{n}\dfrac{\exp\{-(x_i-\alpha)/k_1k_2\beta\}}{1-\exp\{-(x_i-\alpha)/k_1k_2\beta\}}=0 \\ -\dfrac{n\alpha}{k_2\beta^2}-\dfrac{n}{\beta}+\dfrac{\sum\limits_{i=1}^{n}x_i}{k_2\beta^2}-\sum\limits_{i=1}^{n}\dfrac{x_i-\alpha}{k_1k_2\beta^2}\dfrac{\exp\left\{-\dfrac{x_i-\alpha}{k_1k_2\beta}\right\}}{1-\exp\left\{-\dfrac{x_i-\alpha}{k_1k_2\beta}\right\}}=0 \end{cases}$$

The MLEs of α and β can be obtained by solving the non-linear equation set.

2. When β is known (suppose $\beta=\beta_0$) and α is unknown, the likelihood function and the log-likelihood function are:

$$L_1(\alpha)$$
$$=\prod_{i=1}^{n}f(x_i)=\prod_{i=1}^{n}\frac{1}{\beta_0}\exp\left\{-\frac{x_i-\alpha}{k_2\beta_0}\right\}\left(1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta_0}\right\}\right)$$
$$=\frac{1}{\beta_0{}^n}\exp\left\{-\left(\sum_{i=1}^{n}x_i-n\alpha\right)\Big/k_2\beta_0\right\}\prod_{i=1}^{n}\left(1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta_0}\right\}\right) \tag{7}$$

$$\ln L_1(\alpha)$$
$$=\ln\left\langle\frac{1}{\beta_0{}^n}\exp\left\{-\left(\sum_{i=1}^{n}x_i-n\alpha\right)\Big/k_2\beta_0\right\}\prod_{i=1}^{n}\left(1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta_0}\right\}\right)\right\rangle$$
$$=\frac{n\alpha}{k_2\beta_0}-n\ln\beta_0-\frac{\sum\limits_{i=1}^{n}x_i}{k_2\beta_0}+\sum_{i=1}^{n}\ln\left(1-\exp\left\{-\frac{x_i-\alpha}{k_1k_2\beta_0}\right\}\right) \tag{8}$$

So $\dfrac{\partial \ln L_1(\alpha)}{\partial \alpha}=\dfrac{n}{k_2\beta_0}-\dfrac{1}{k_1k_2\beta_0}\sum\limits_{i=1}^{n}\dfrac{\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}{1-\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}$

Let $\dfrac{\partial \ln L(\alpha)}{\partial \alpha}=0$ and then：

$$k_1n=\sum_{i=1}^{n}\frac{\exp\left\{-\dfrac{x_i-\alpha}{k_1k_2\beta_0}\right\}}{1-\exp\left\{-\dfrac{x_i-\alpha}{k_1k_2\beta_0}\right\}} \tag{9}$$

The solution $\hat{\alpha}_2$ of this equation is the MLE of α. Then we discuss the existence of the estimated value $\hat{\alpha}_2$.

Lemma 1: Let $g_1(\alpha)=k_1n-\sum\limits_{i=1}^{n}\dfrac{\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}{1-\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}$, $\{X_1',X_2',\cdots,X_n'\}$ is a simple random sample from the population distribution of $GE_1(x;\alpha,\beta)$, when $n\in Z^+$, $x_i>\alpha>0,\ \beta_0>0,\ k_1n>\sum\limits_{i=1}^{n}\dfrac{\exp\{-x_i/k_1k_2\beta_0\}}{1-\exp\{-x_i/k_1k_2\beta_0\}}$, then $g_1(\alpha)=0$ has a unique positive real root.

Proof: $g_1(\alpha)=k_1n-\sum\limits_{i=1}^{n}\dfrac{\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}{1-\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}$

So $g_1'(\alpha)=-\dfrac{1}{k_1k_2\beta_0}\sum\limits_{i=1}^{n}\dfrac{\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}{\left(1-\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}\right)^2}<0$, that is, when $x_i>\alpha>0,\ \beta_0>0$, $g_1(\alpha)$ decreases monotonically.

Assume $x_{(1)}=\min\left(X_1',X_2',\cdots,X_n'\right)$, when $x_i>\alpha>0,\ \beta_0>0$ there is:

$$\lim_{\alpha\to x_{(1)}}g_1(\alpha)=\lim_{\alpha\to x_{(1)}}\left\{k_1n-\sum_{i=1}^{n}\frac{\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}{1-\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}\right\}=-\infty$$

$$\lim_{\alpha\to 0}g_1(\alpha)=\lim_{\alpha\to 0}\left\{k_1n-\sum_{i=1}^{n}\frac{\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}{1-\exp\{-(x_i-\alpha)/k_1k_2\beta_0\}}\right\}$$
$$=k_1n-\sum_{i=1}^{n}\frac{\exp\{-x_i/k_1k_2\beta_0\}}{1-\exp\{-x_i/k_1k_2\beta_0\}}.$$

So: (1) when $k_1n>\sum\limits_{i=1}^{n}\dfrac{\exp\{-x_i/k_1k_2\beta_0\}}{1-\exp\{-x_i/k_1k_2\beta_0\}}$, $g_1(\alpha)$ has a unique positive real root.

(2) when $k_1n<\sum\limits_{i=1}^{n}\dfrac{\exp\{-x_i/k_1k_2\beta_0\}}{1-\exp\{-x_i/k_1k_2\beta_0\}}$, $g_1(\alpha)$ has no real root.

Theorem 3: If random variable $X\sim GE_1(x;\alpha,\beta)$, when β is known as $\beta_0(\beta_0>0)$ and α is unknown, the MLE $\hat{\alpha}_{2,1}$ of α exists only if $k_1n>\sum\limits_{i=1}^{n}\dfrac{\exp\{-x_i/k_1k_2\beta_0\}}{1-\exp\{-x_i/k_1k_2\beta_0\}}$ is satisfied and is a positive real root.

It can be proved that theorem 3 is established, and the proof process is omitted.

3. When α is known (suppose $\alpha=\alpha_0$) and β is unknown, the likelihood function and the log-likelihood function are respectively:

$$L_1(\beta)=\prod_{i=1}^{n}f(x_i)$$
$$=\prod_{i=1}^{n}\frac{1}{\beta}\exp\left\{-\frac{x_i-\alpha_0}{k_2\beta}\right\}\left(1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}\right)$$
$$=\frac{1}{\beta^n}\exp\left\{-\left(\sum_{i=1}^{n}x_i-n\alpha_0\right)\Big/k_2\beta\right\}\prod_{i=1}^{n}\left(1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}\right) \tag{10}$$

$$\ln L_1(\beta)$$
$$=\ln\left\langle\frac{1}{\beta^n}\exp\left\{-\left(\sum_{i=1}^{n}x_i-n\alpha_0\right)\Big/k_2\beta\right\}\prod_{i=1}^{n}\left(1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}\right)\right\rangle$$
$$=\frac{n\alpha_0}{k_2\beta}-n\ln\beta-\frac{\sum\limits_{i=1}^{n}x_i}{k_2\beta}+\sum_{i=1}^{n}\ln\left(1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}\right) \tag{11}$$

Differentiating (11) can obtain that:

$$\frac{\partial \ln L_1(\beta)}{\partial \beta}=-\frac{n\alpha_0}{k_2\beta^2}-\frac{n}{\beta}+\frac{\sum\limits_{i=1}^{n}x_i}{k_2\beta^2}-\sum_{i=1}^{n}\frac{x_i-\alpha_0}{k_1k_2\beta^2}\frac{\exp\left\{-\dfrac{x_i-\alpha_0}{k_1k_2\beta}\right\}}{1-\exp\left\{-\dfrac{x_i-\alpha_0}{k_1k_2\beta}\right\}}$$

Let $\dfrac{\partial \ln L(\beta)}{\partial \beta}=0$, then

$$-\frac{n\alpha_0}{k_2\beta^2}-\frac{n}{\beta}+\frac{\sum\limits_{i=1}^{n}x_i}{k_2\beta^2}-\sum_{i=1}^{n}\frac{x_i-\alpha_0}{k_1k_2\beta^2}\frac{\exp\left\{-\dfrac{x_i-\alpha_0}{k_1k_2\beta}\right\}}{1-\exp\left\{-\dfrac{x_i-\alpha_0}{k_1k_2\beta}\right\}}=0 \tag{12}$$

The solution $\hat{\beta}_2$ of this equation is the MLE of β. Then we discuss the existence of the estimated value $\hat{\beta}_2$.

Lemma 2: Let

$$g_2(\beta)=-n\alpha_0-k_2n\beta+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{(x_i-\alpha_0)}{k_1}\frac{\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}{1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}},$$

$\{X_1',X_2',\cdots,X_n'\}$ be a simple random sample from the population distribution of $GE_1(x;\alpha,\beta)$, when $n\in Z^+$, $x_i>\alpha>0,\beta_0>0$ then $g_2(\beta)=0$ has a unique positive real root.

Proof:

$$g_2(\beta)=-n\alpha_0-k_2n\beta+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{(x_i-\alpha_0)}{k_1}\frac{\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}{1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}$$

It is easy to get

$$g_2'(\beta)=-k_2n-\frac{(x_i-\alpha_0)}{k_1^2k_2\beta}\sum_{i=1}^{n}\frac{\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}{\left(1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}\right)^2}<0,$$

that is when $x_i>\alpha>0,\beta_0>0$, the function $g_2(\beta)$ decreases monotonically.

And

$$\lim_{\beta\to0^+}g_2(\beta)=\lim_{\beta\to0^+}\left(-n\alpha_0-k_2n\beta+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{x_i-\alpha_0}{k_1}\frac{\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}{1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}\right)$$

$$=\sum_{i=1}^{n}x_i-n\alpha_0>0$$

$$\lim_{\beta\to+\infty}g_2(\beta)=\lim_{\beta\to+\infty}\left(-n\alpha_0-k_2n\beta+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{x_i-\alpha_0}{k_1}\frac{\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}{1-\exp\left\{-\frac{x_i-\alpha_0}{k_1k_2\beta}\right\}}\right)$$

$$=-\infty$$

Therefore, $g_2(\beta)$ has a unique positive real root when $x_i>\alpha>0,\beta_0>0$.

It is easy to know that when $\beta_0>0$, the equation

$$-\frac{n\alpha_0}{3\beta^2}-\frac{n}{\beta}+\frac{\sum_{i=1}^{n}x_i}{3\beta^2}-\sum_{i=1}^{n}\frac{x_i-\alpha_0}{6\beta^2}\frac{\exp\left\{-\frac{x_i-\alpha_0}{6\beta}\right\}}{1-\exp\left\{-\frac{x_i-\alpha_0}{6\beta}\right\}}=0$$

has a unique positive real root when $x_i>\alpha>0$, i.e., the MLE $\hat{\beta}_{2,1}$ of the parameter β exists uniquely and is a positive real root.

Theorem 4: If random variable $X\sim GE_1(x;\alpha,\beta)$, when α is known (suppose $\alpha=\alpha_0$) and β is unknown, it can be proved that the MLE $\hat{\beta}_{2,1}$ of β exists only and is a positive real number.

It can be proved that theorem 4 is established, and the proof process is omitted.

V. Conclusion

The GE distribution can process the lifetime data more reasonably, since it overcomes the lack-of-memory of exponential distribution. Based on the predecessors, this paper proposes a new class of TPGE distribution and discusses its related properties. Firstly, this paper introduces the new class of TPGE distribution and gives it's CDF and PDF. Then, some statistical properties of the proposed distribution are discussed, such as moment, standard form, hazard function and mean residual lifetime. Finally, the estimates of the parameters are discussed by the method of moment estimation and MLE. As further work, we shall use more methods to estimate the parameters of the TPGE distribution, such as Bayesian estimation, best linear unbiased estimation and optimal covariance estimation.

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