Global Finite-Time Stabilization of Stochastic Nonlinear Systems via Output Feedback

Yanling Shang, Wenjun Tang, Jiacai Huang, and Xinxin Shi

***Abstract*—For a category of stochastic nonlinear systems (SNSs), the problem of global finite-time stabilization by output feedback is investigated in this article. By using adding a power integrator and homogeneous domination techniques, a constructive output feedback control design procedure is given for the considered systems. It is proved that, under some essential restriction on the system growth, the control design ensures the closed-loop system (CLS) is globally finite-time stable in probability. Simulation results of a numerical example are given to confirm the effectiveness of the proposed approach.**

***Index Terms*—stochastic nonlinear systems (SNSs), output feedback, finite-time stable in probability**

I. Introduction

In this article, we consider output feedback stabilization for a category of stochastic nonlinear systems (SNSs) in the form

$$\begin{aligned} dx_i &= x_{i+1}dt + f_i(t,x)dt + g_i^T(t,x)dw, \\ i &= 1,\cdots,n-1, \\ dx_n &= udt + f_n(t,x)dt + g_n^T(t,x)dw, \end{aligned} \tag{1}$$

where $x=(x_1,\cdots,x_n)^T \in R^n$, $u \in R$ and $y \in R$ are the system state, input and output, $x_2,\cdots,x_n$ are unmeasurable. ω is an r-dimensional independent standard Wiener process defined on a complete probability space $(\Omega,\mathcal{F},P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, and P being a probability measure. The uncertain nonlinear functions $f_i : R^+ \times R^n \to R$, and $g_i : R^+ \times R^n \to R^r, i=1,\cdots,n$, are assumed to be C^1 vanishing at the origin.

When $f_i = 0$ for all $i=1,\cdots,n$, system (1) degenerates into the well-known chained form, the global asymptotic stabilization of which by output feedback was first considered in [1]. Since then, by adopting different approaches, much research work has addressed the output feedback asymptotic stabilization for more general SNSs under various structures and/or growth conditions; see, for example, [2–7] and the references therein.

Compared to the asymptotic stabilization via output feedback, to investigate the finite-time stabilization of stochastic nonlinear systems by output feedback is more challenging because of the fact that the existence of a unique solution and the non-satisfaction of local Lipschitz condition are two prerequisites of studying the finite-time stability for a SNS [8], namely, *as a general condition to ensure the existence of unique solution for a SNS, local Lipschitz condition is infeasible to address stochastic finite-time stability (SFTS).* Therefore, for the SNSs, what conditions should be imposed and how to develop controller are so important and interesting should be paid careful attention. Based on SFTS theorem [9], some groundbreaking work on finite-time stabilization using state feedback have recent been made [10–12]. Naturally, a more interesting problem is: *If only partial state vector being measurable, how can one design a finite-time output feedback stabilizing controller for SNSs?*

In this article, we will deal with this problem and provide a solution to the problem of global finite-time output feedback stabilization of SNS (1). The main contribution is two-fold: First, we show that for the SNS (1), under some skilly assumption imposed on nonlinearities in the drift and diffusion vector fields, it is possible to achieve global stochastic finite-time stabilization by non-Lipschitz continuous output feedback. Second, we generalize the homogeneous domination and the adding a power integrator techniques to stochastic systems, skillfully constructing the C^2 Lyapunov function, an output feedback controller is develop to render the CLS globally finite-time stable in probability.

Notations. In this article, the notations used are fairly standard.

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Yanling Shang, Wenjun Tang, Jiacai Huang, and Xinxin Shi are School of Automation, Nanjing Institute of Technology, Nanjing 211167, P. R. China, hnnhsyl@126.com

II. Preliminary results

Consider the SNS

$$dx = f(t,x)dt + g(t,x)d\omega, \tag{2}$$

where $x \in R^n$ is the system state with the initial condition $x(0) = x_0$. The functions:$f : R^+ \times R^n \to R^n$ and $g : R^+ \times R^n \to R^{n\times r}$ are continuous and satisfy $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$. Moreover, system (2) has a pathwise unique strong solution, denoted by $x(t,x_0)$.

The following results can be found in [8-12].

Lemma 1. Suppose that $f(t,x)$ and $g(t,x)$ are continuous in x and for any $0<\delta<1$, each $N=1,2,\cdots$, and each $0 \le T < \infty$, if the following conditions hold:

(i) $|f(t,x)| \le p(t)(1+|x|)$,
(ii) $|g(t,x)|^2 \le p(t)(1+|x|^2)$,
(iii) $|f(t,x_1)-f(t,x_2)| \le p_T^N(t)|x_1-x_2|$,
$(iv)|g(t,x_1)-g(t,x_2)| \le p_T^N(t)|x_1-x_2|$,

as $0<\delta\le|x_i|\le N, i=1,2, t\in[0,T]$, where $p(t)$ and $p_T^N(t)$ are nonnegative functions such that $\int_0^T p(t)dt < \infty$ and $\int_0^T p_T^N(t)dt < \infty$. Then for all $x_0 \in R^n$, system (1) has a pathwise unique strong solution.

Proof. For any given $x_0 \in R^n \setminus \{0\}$, define

$$F_n(t,x) = \begin{cases} f(t,x), & |x| \ge \dfrac{1}{n}, \\ f(t,\dfrac{x}{n|x|}), & |x| \le \dfrac{1}{n}, \end{cases}$$

and $G_n(t,x)$ similarly. It is easy to see that F_n and G_n satisfy the all conditions in Lemma 1. Hence the stochastic nonlinear system

$$dx = F_n(t,x)dt + G_n(t,x)d\omega, \tag{3}$$

has a pathwise unique strong solution, denoted by $x_n(t,x_0)$. Define that τ_n is the first time satisfying $|x_n| \le \frac{1}{n}$, especially, $\tau_n := T$ if τ_n does not exist. Then we have $x_n(t,x_0) = x_{n'}(t,x_0)$ a.s. provided that $0 < t < \tau_n$ and $n' \ge n$. Moreover τ_n is strictly increasing in n. Let $\tau = \lim_{n\to+\infty} \tau_n(\theta)$

$$x(t,x_0) = \begin{cases} \lim\limits_{n\to+\infty} x_n(t,x_0,\theta), \ t \le \tau, \\ 0, \qquad\qquad \tau \le t < +\infty, \end{cases} \tag{4}$$

where $\theta \in \{|x_n| \le \frac{1}{n}\}$.

Obviously, $x(t,x_0)$ is well-defined and satisfies

$$x(t,x_0) = x_0 + \int_0^t f(s,x(s))ds + \int_0^t g(s,x(s))dw(s),$$

on the time period $[0,T)$.

Lemma 2. For system (2),assume that the pathwise uniqueness be satisfied, and there exists a $C^{1,2}$ function $V:[0,+\infty)\times R^n \to R^+$, class K_∞ functions μ_1 and μ_2, and a continuously differentiable $h: R^+ \to R^+$ such that for all $x \in R^n$, $t \ge t_0$, such that

(i) $\mu_1(|x|) \le V(t,x) \le \mu_2(|x|)$,
(ii) $\mathcal{L}V(t,x) \le -h(V(t,x))$,
(iii) $\int_0^\varepsilon \frac{1}{h(v)} dv < +\infty, \forall \varepsilon \in [0,+\infty)$,
(iv) $h'(v) > 0, \forall v > 0$,

where

$$\mathcal{L}V(t,x) = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f + \frac{1}{2} Tr\Big\{g^T \frac{\partial^2(t,x)}{\partial x^2} g\Big\}$$

is the infinitesimal generator associated with stochastic system (2), then the origin of system (2) is globally stochastic finite-time stable, and the stochastic settling time function $T_0(x_0,w)$ satisfies

$$E[T_0(x_0,w)] \le \int_0^{V(x_0)} \frac{1}{h(v)} dv.$$

Remark 1. If $h(V(t,x)) = -cV^\alpha(t,x)$, where $c > 0$ and $0 < \alpha < 1$ are real numbers, it is not difficult for us to get that the conditions (iii) and (iv) in Lemma 3 hold and the stochastic settling time function $T_0(x_0,w)$ satisfies

$$E[T_0(x_0,w)] \le \frac{V^{1-\alpha}(x_0)}{c(1-\alpha)}.$$

Lemma 3. For any u, $v \in \mathbb{R}$, if $m \ge 1$, then one has (i) $|u+v|^m \le 2^{m-1}|u^m + v^m|$; (ii) $(|u|+|v|)^{1/m} \le |u|^{1/m} + |v|^{1/m} \le 2^{(m-1)/m}(|u|+|v|)^{1/m}$.

Lemma 4. For any $\vartheta(u,v) > 0$ if a, $b \ge 0$, then the following inequality holds:

$$|u|^a|v|^b \le \frac{a}{a+b}\vartheta(u,v)|u|^{a+b} + \frac{b}{a+b}\vartheta^{-a/b}(u,v)|v|^{a+b}.$$

Lemma 5. For $u, u \in R$ and positive real number q, the following inequality hold:

$$|u^q - v^q| \le c|u-v||(u-v)^{q-1} + v^{q-1}|,$$

where $c = q$ for $1 < q \le 2$ and $c = q2^{q-1}$ for $q > 2$.

III. Output feedback controller design

The aim of this article is to develop a recursive design method for globally finite-time stabilizing SNS (1) via output feedback under the following assumption.

Assumption 1. For $i = 1,\cdots,n$, there exist constants $b > 0$ and $\tau \in (-\frac{1}{n},0)$ such that

$$\begin{aligned} &|f_i(t,x) - f_i(t,\bar{x})| \le b(|x_1 - \bar{x}_1|^{(m_i+\tau)/m_1} \\ &+ \cdots + |x_i - \bar{x}_i|^{(m_i+\tau)/m_i}), \end{aligned} \tag{5}$$

$$\begin{aligned} &|g_i(t,x) - g_i(t,\hat{x})| \le b(|x_1 - \bar{x}_1|^{(2m_i+\tau)/2m_1} \\ &+ \cdots + |x_i - \bar{x}_i|^{(2m_i+\tau)/2m_i}), \end{aligned} \tag{6}$$

where $m_i = 1 + (n-1)\tau$, $i = 1,\cdots,n$.

For simplicity, we suppose that in this article $\tau = -\frac{p}{q}$ with p being any even integer and q being any odd integer. Then, from the definition of m_i in Assumption 1, one sees that $m_i > 0$ is an odd number.

Remark 2. Noting that when $\bar{x}_i = 0$ for $i = 1,\cdots,n$, Assumption 1 reduces to

$$|f_i(t,x)| \le b(|x_1|^{(m_i+\tau)/m_1} + \cdots + |x_i|^{(m_i+\tau)/m_i}), \tag{7}$$

$$|g_i(t,x)| \le b(|x_1|^{(2m_i+\tau)/2m_1} + \cdots + |x_i|^{(2m_i+\tau)/2m_i}), \tag{8}$$

since f_i and g_i vanishing at the origin. This growth condition will be crucial in constructing a homogeneous state feedback stabilizer.

A. Homogeneous output feedback control of the nominal system

In this subsection, we will construct an output feedback stabilizer for the following nominal system

$$\begin{aligned} &dz_i = z_{i+1}dt, \ \ i = 1,\cdots,n-1, \\ &dz_n = vdt. \end{aligned} \tag{9}$$

A1. State feedback controller design

Step 1. Let $\xi_1 = z_1^\sigma$, where $\sigma \in (2,+\infty)_{odd}$ is a constant. Choosing the Lyapunov function $V_1 = z_1^{(4l\sigma-\tau)}/(4l\sigma - \tau)$, where l is a constant satisfying $4l \in (0,+\infty)_{even}$ and $(4l-2)\sigma \ge 1$. From (1) and Assumption 1, it follows that

$$\mathcal{L}V_1 \le -n\xi_1^{4l} + \xi_1^{(4l\sigma-\tau-1)/\sigma}(z_2 - z_2^*), \tag{10}$$

where the virtual controller is chosen as

$$z_2^* = -nz_1^{1+\tau} := -\beta_1^{m_2/\sigma}\xi_1^{m_2/\sigma}. \tag{11}$$

Step k $(k = 2,\cdots,n)$**.** In this step, we can obtain the following property.

Lemma 6. Suppose at step $k-1$, there are a C^2, proper and positive definite Lyapunov function V_{k-1}, and a set of virtual controllers $z_1^*,\cdots,z_{k-1}^*$ defined by

$$\begin{aligned} z_1^* &= 0, & \xi_1 &= z_1^{\sigma/m_1} - z_1^{*\sigma/m_1}, \\ z_2^* &= -\beta_1^{m_2/\sigma}\xi_1^{m_2/\sigma}, & \xi_2 &= z_2^{\sigma/m_2} - z_2^{*\sigma/m_2}, \\ &\vdots & &\vdots \\ z_k^* &= -\beta_{k-1}^{m_k/\sigma}\xi_{k-1}^{m_k/\sigma}, & \xi_k &= z_k^{\sigma/m_k} - z_k^{*\sigma/m_k}, \end{aligned} \tag{12}$$

with constants $\beta_1 > 0$, $\cdots$, $\beta_{k-1} > 0$, such that

$$\begin{aligned} \mathcal{L}V_{k-1} \le& -(n-k+2)\sum_{i=1}^{k-1}\xi_i^{4l} \\ &+\xi_{k-1}^{(4l\sigma-\tau-m_{k-1})/\sigma}(z_k - z_k^*). \end{aligned} \tag{13}$$

Then the kth Lyapunov function

$$V_k = V_{k-1} + W_k \tag{14}$$

with

$$W_k = \int_{z_k^*}^{z_k} (s^{\sigma/m_k} - z_k^{*\sigma/m_k})^{(4l\sigma-\tau-m_k)/\sigma} ds,$$

is C^2, proper and positive definite, and there is a virtual control law

$$z_{k+1}^* = -\beta_k^{m_{k+1}/\sigma} \xi_k^{m_{k+1}/\sigma}$$

such that

$$\begin{aligned}\mathcal{L}V_k \le & -(n-k+1)\sum_{i=1}^{k} \xi_i^{4l} \\ & +\xi_k^{(4l\sigma-\tau-m_k)/\sigma}(z_{k+1} - z_{k+1}^*).\end{aligned} \tag{15}$$

By Lemma 6, hence at Step n, we can design

$$z_{n+1}^* = -(\beta_n z_n^{\sigma/m_n} + \cdots + \beta_n \cdots \beta_1 z_1^{\sigma/m_1})^{m_{n+1}/\sigma}, \tag{16}$$

such that

$$\mathcal{L}V_n \le -(\xi_1^{4l} + \cdots + \xi_n^{4l}) + \xi_n^{(4l\sigma-\tau-m_n)/\sigma}(v - z_{n+1}^*). \tag{17}$$

A2. Output feedback controller design

Since $z_2, \cdots, z_n$ are unmeasurable, we construct a homogeneous observer

$$\dot{\eta}_i = -l_{i-1}\hat{z}_i, \hat{z}_i = (\eta_i + l_{i-1}\hat{z}_{i-1})^{m_i/m_{i-1}}, i = 2, \cdots, n, \tag{18}$$

where $\hat{z}_1 = z_1$ and $l_s > 0$; $s = 1, \cdots, n-1$ are the gains to be determined. From the certainty equivalence principle, we replace z_i with $\hat{z}_i$ in (16) and gain an output feedback controller

$$v(\hat{z}) = -(\beta_n \hat{z}_n^{\sigma/m_n} + \cdots + \beta_n \cdots \beta_1 z_1^{\sigma/m_1})^{m_{n+1}/\sigma}, \tag{19}$$

where $\hat{z} = (z_1, \hat{z}_2, \cdots, \hat{z}_n)$. Considering

$$U_i = \int_{\gamma_i^{(4l\sigma-\tau-m_{i-1})/m_{i-1}}}^{z_i^{(4l\sigma-\tau-m_{i-1})/m_i}} (s^{m_{i-1}/(4l\sigma-\tau-m_{i-1})} - \gamma_i) ds, \tag{20}$$

where $\gamma_i = \eta_i + l_{i-1}z_{i-1}$ and setting $e_i = (z_i - \hat{z}_i)^{\sigma/m_i}$, for $i = 2, \cdots, n$, from (9), (18) and (20), we can obtain the following property.

Lemma 7. For $T = V_n + \sum_{i=2}^{n} U_i$, there exist constants $l_s > 0$, $s = 1, \cdots, n-1$ such that

$$\mathcal{L}T = -\frac{1}{4}\sum_{i=1}^{n} \xi_i^{4l} - \frac{1}{4}\sum_{i=2}^{n} e_i^{4l}. \tag{21}$$

Proof. The similar proof can be found in [8] and hence is omitted here.

Note that from the construction of T, it can be verified that T is positive definite and proper with respect to $Z = (z_1, \cdots, z_n, \eta_2 \cdots, \eta_n)^T$. Denoting the dilation weight

$$\Delta = (\underbrace{m_1, \cdots, m_n}_{for\ z_1,\cdots,z_n}, \underbrace{m_1, \cdots, m_{n-1}}_{for\ \eta_2\cdots,\eta_n}), \tag{22}$$

the closed-loop system which can be rewritten as

$$dZ = E(Z)dt = (z_2, \cdots, z_n, v, \dot{\eta}_2 \cdots, \dot{\eta}_n)^T dt, \tag{23}$$

is homogeneous of degree τ. It can be shown that T is homogeneous of degree $4l\sigma - \tau$ with respect to Δ.

B. Homogeneous output feedback control of system (1)

Together with the homogeneous controller and observer established above, in this subsection we are ready to use the homogeneous domination approach to globally stabilize (1) via output feedback in finite time. First, we introduce the change of coordinates

$$z_1 = x_1,\ z_i = \frac{x_i}{L^{i-1}},\ i = 2, \cdots, n,\ v = \frac{u}{L^n}, \tag{24}$$

where $L \ge 1$ is a constant to be determined later. Under (24), system (1) can be rewritten as

$$\begin{aligned} dz_i &= \Big(Lz_{i+1} + \frac{f_i}{L^{i-1}}\Big)dt + \frac{g_i^T}{L^{i-1}}dw, i = 1, \cdots, n-1, \\ dz_n &= \Big(Lv + \frac{f_n}{L^{n-1}}\Big)dt + \frac{g_i^T}{L^{n-1}}dw. \end{aligned} \tag{25}$$

Now we construct an observer with a gain L as follows

$$\dot{\eta}_i = -Ll_i\hat{z}_i, \hat{z}_i = (\eta_i + l_{i-1}\hat{z}_{i-1})^{m_i/m_{i-1}}, i = 2, \cdots, n. \tag{26}$$

In addition, we design u using the same construction of (19), specifically,

$$u = -L^n(\beta_n \hat{z}_n^{\sigma/m_n} + \cdots + \beta_n \cdots \beta_1 z_1^{\sigma/m_1})^{m_{n+1}/\sigma}. \tag{27}$$

Now, the CLS (25)–(27) can be written as

$$dZ = LE(Z)dt + F(Z)dt + +G^T(Z)dw, \tag{28}$$

where $F(Z) = (f_1, \frac{f_2}{L}, \cdots, \frac{f_n}{L^{n-1}}, 0, \cdots, 0)^T$ and $G(Z) = (g_1, \frac{g_2}{L}, \cdots, \frac{g_n}{L^{n-1}}, 0, \cdots, 0)^T$.

We state the main result in this article.

Theorem 1. If Assumption 1 holds for system (1), under the output feedback controller (24), (26) and (27), then

(1) the CLS has a pathwise unique strong solution for any x_0;

(2) the equilibrium at the origin of the CLS is finite-time stable in probability.

Proof. We prove the Theorem 1 in two steps.

Step 1. We first prove that the CLS has a pathwise unique strong solution. For the simplicity of expression, let

$$\chi_1 = z_1, \quad \chi_i = \hat{z}_i, \quad i = 2, \cdots, n, \tag{29}$$

from which, the observer (26) can be rewritten as

$$\dot{\eta}_i = -Ll_i\chi_i, \chi_i = (\eta_i + l_{i-1}\chi_{i-1})^{m_i/m_{i-1}}, i = 2, \cdots, n. \tag{30}$$

Noting that $m_i/m_j < 1$, $j = 1, \cdots, i-1$, by Lemma 3, we have

$$\begin{aligned} & |\chi_i - \bar{\chi}_i| \\ &= \Big|(\eta_i + l_{i-1}\chi_{i-1})^{m_i/m_{i-1}} - (\bar{\eta}_i + l_{i-1}\bar{\chi}_{i-1})^{m_i/m_{i-1}}\Big| \\ &\le 2\Big|(\eta_i - \bar{\eta}_i) + l_{i-1}(\chi_{i-1} - \bar{\chi}_{i-1})\Big|^{m_i/m_{i-1}} \\ &\le 2|\eta_i - \bar{\eta}_i|^{m_i/m_{i-1}} + 2l_{i-1}|\chi_{i-1} - \bar{\chi}_{i-1}|^{m_i/m_{i-1}} \\ &\le c_i(|\eta_i - \bar{\eta}_i|^{m_i/m_{i-1}} + \cdots + |\eta_1 - \bar{\eta}_1|^{m_i/m_1}), \end{aligned} \tag{31}$$

where c_i is a positive constant. Furthermore

$$\begin{aligned} & |\dot{\eta}_i - \dot{\bar{\eta}}_i| = Ll_i|\chi_i - \bar{\chi}_i| \\ & \le \tilde{c}_i(|\eta_i - \bar{\eta}_i|^{m_i/m_{i-1}} + \cdots + |\eta_1 - \bar{\eta}_1|^{m_i/m_1}), \end{aligned} \tag{32}$$

where $\tilde{c}_i = Lc_il_i$ is a positive constant

From (29), the u in (27) is rewritten as

$$u = -(\bar{\beta}_1\chi_1^{\sigma/m_1} + \cdots + \bar{\beta}_n\chi_n^{\sigma/m_n})^{m_{n+1}/\sigma}, \tag{33}$$

where $\bar{\beta}_i = L^{n\sigma/m_{n+1}}\beta_n \cdots \beta_i$, $i = 1, \cdots, n$.

By using the definitions of τ, m_i's and σ, it is easy to see that $u(0) = 0$, $(m_{n+1}/\sigma) \in (0,1)_{odd}$ and $(\sigma/m_i) \in (1,\infty)_{odd}$, from which and Lemmas 3 and 5, it follows that

$$\begin{aligned}
&|u(\chi) - u(\bar{\chi})| \\
&\le 2\Big|\Big(\bar{\beta}_1\chi_1^{\sigma/m_1} + \cdots + \bar{\beta}_n\chi_n^{\sigma/m_n}\Big) \\
&\quad -\Big(\bar{\beta}_1\bar{\chi}_1^{\sigma/m_1} + \cdots + \bar{\beta}_n\bar{\chi}_n^{\sigma/m_n}\Big)\Big|^{m_{n+1}/\sigma} \\
&\le \hat{c}\Big(|\chi_1^{\sigma/m_1} - \bar{\chi}_1^{\sigma/m_1}| + \cdots + |\chi_n^{\sigma/m_n} - \bar{\chi}_n^{\sigma/m_n}|\Big)^{m_{n+1}/\sigma} \\
&\le \hat{c}\Big(\frac{\sigma}{m_1}|\chi_1 - \bar{\chi}_1|\Big|\chi_1^{(\sigma-m_1)/m_1} + (\chi_1 - \bar{\chi}_1)^{(\sigma-m_1)/m_1}\Big| \\
&\quad + \cdots + \frac{\sigma}{m_n}|\chi_n - \bar{\chi}_n|\Big|\chi_n^{(\sigma-m_n)/m_n} \\
&\quad +(\chi_n - \bar{\chi}_n)^{(\sigma-m_n)/m_n}\Big|\Big)^{m_{n+1}/\sigma},
\end{aligned} \tag{34}$$

where $\hat{c}$ is a positive constant.

Note that (31) implies that

$$|\chi_i| \le c_i(|\eta_i|^{m_i/m_{i-1}} + \cdots + |\eta_1|^{m_i/m_1}). \tag{35}$$

Putting (34) and (35) together, for any $0 < \delta \le |\eta_i| \le N$, $N = 1, 2, \cdots$, by lemma 3, we have

$$\begin{aligned}
&|u(\chi) - u(\bar{\chi})| \\
&\le \tilde{c}(|\eta_1 - \bar{\eta}_1|^{m_{n+1}/\sigma} + \cdots + |\eta_n - \bar{\eta}_n|^{m_{n+1}/\sigma})
\end{aligned} \tag{36}$$

for a constant $\tilde{c} > 0$.

Based on Assumption 1, (32), (36) and (24), it is verified that all conditions in Lemma 1 are satisfied. That is to say, the CLS admits a pathwise unique strong solution.

Step 2. Then we show that the CLS is globally finite-time stable in probability. Because T is homogeneous of degree $4l\sigma - \tau$ with respect to Δ, therefore there is a constant $\bar{c}_0 > 0$ such that

$$T \le \bar{c}_0\|Z\|_\Delta^{4l\sigma-\tau}. \tag{37}$$

Similarly, since the right-hand side of (21) is homogeneous of degree $4l\sigma$, there is a constant $\bar{c}_1 > 0$ such that

$$\frac{\partial T}{\partial Z}E(Z) \le -\bar{c}_1\|Z\|_\Delta^{4l\sigma} \tag{38}$$

By (24), Assumption 1 and $L \ge 1$, one has

$$\Big|\frac{f_i(\cdot)}{L^{i-1}}\Big| \le \delta_i L^{1-\alpha_i}\|Z\|_\Delta^{m_i+\tau}, \tag{39}$$

for constants $\delta_i > 0$ and $0 < \alpha_i < 1$. Since for $i = 1, \cdots, n$, $\partial T/\partial Z_i$ is homogeneous of degree $4l\sigma - \tau - m_i$, one obtains from (39) that

$$\Big|\frac{\partial T}{\partial Z}F(Z)\Big| \le \sum_{i=1}^{n}\Big|\frac{\partial T}{\partial Z_i}\Big|\Big|\frac{f_i(\cdot)}{L^{i-1}}\Big| \le \bar{c}_2L^{1-\alpha_0}\|Z\|_\Delta^{4l\sigma}, \tag{40}$$

where $\bar{c}_2$ and $\alpha_0 = min_{1\le i\le n}\{\alpha_i\} < 1$ are positive constants.

Similarly to (39), there exist $\bar{\delta}_i$ and $\bar{\alpha}_i < \frac{1}{2}$ such that

$$\Big|\frac{g_i(\cdot)}{L^{i-1}}\Big| \le \bar{\delta}_i L^{(1-\bar{\alpha}_i)/2}\|Z\|_\Delta^{(m_i+\tau)/2}. \tag{41}$$

With the help of (41), one gets

$$\begin{aligned}
&\frac{1}{2}Tr\Big\{G\frac{\partial^2 T}{\partial Z^2}G^T\Big\} \\
&\le \frac{1}{2}r\sqrt{r}\sum_{i,j=1}^{n}\Big|\frac{\partial^2 T}{\partial Z_i\partial Z_i}\Big||G_i^T||G_j| \\
&\le \bar{c}_3L^{1-\bar{\alpha}_0}\|Z\|_\Delta^{4l\sigma},
\end{aligned} \tag{42}$$

where $\alpha_0 = min_{1\le i,j\le n}\{\alpha_i + \alpha_j\} < 1$, $\bar{c}_3$ are positive constants and the first inequality is obtained by using $TrX \le rg|X|_\infty r\sqrt{r}|X|$ (X is a r-dimension square matrix).

Therefore, from (38), (40) and (42), one has we obtain

$$\begin{aligned}
\mathcal{L}T &= \frac{\partial T}{\partial Z}LE(Z) + \frac{\partial T}{\partial Z}F(Z) + \frac{1}{2}Tr\Big\{G\frac{\partial^2 T}{\partial Z^2}G^T\Big\} \\
&\le -L(\bar{c}_1 - (\bar{c}_2 + \bar{c}_3)L^{-\tilde{\alpha}_0})\|Z\|_\Delta^{4l\sigma},
\end{aligned} \tag{43}$$

where $\tilde{\alpha}_0 = max\{\alpha_0, \bar{\alpha}_0\} > 0$.

Since $\bar{c}_1$ is a constant independent of $\bar{c}_2$ and $\bar{c}_3$, by choosing

$$L > max\Big\{\Big(\frac{\bar{c}_2 + \bar{c}_3}{\bar{c}_1}\Big)^{1/\tilde{\alpha}_0}, 1\Big\}, \tag{44}$$

there exists a constant c^*, such that

$$\mathcal{L}T \le -c^*\|Z\|_\Delta^{4l\sigma}. \tag{45}$$

Setting $\alpha = 4l\sigma/(4l\sigma - \tau)$, with (37) and (45) in mind, it can be deduced from Lemma 3 that

$$\mathcal{L}T \le -KT^\alpha. \tag{46}$$

where $K = c^*/2^\alpha\bar{c}_0^\alpha$. Therefore, by Lemma 2, the system consisting of (1), (24), (26) and (27) is globally finite-time stable in probability. This completes the proof of Theorem 1.

IV. Simulation Example

To validates the proposed algorithm, we consider the following low-dimensional system

$$\begin{aligned}
dx_1 &= x_2dt + \frac{1}{8}x_1^{9/11}dt + \frac{1}{4}sinx_1dw, \\
dx_2 &= udt + \frac{1}{8}x_2^{7/9}dt + \frac{1}{8}cos2x_1dw, \\
y &= x_1.
\end{aligned} \tag{47}$$

We choose $\tau = -2/11$ which together with $m_1 = 1$ implies that $m_2 = 9/11$. By Lemma 3, it is easy to get

$$|\frac{1}{8}x_1^{9/11} - \frac{1}{8}\bar{x}_1^{9/11}| \le \frac{1}{4}|x_1 - \bar{x}_1|^{9/11},$$

$$|\frac{1}{8}x_2^{7/9} - \frac{1}{8}\bar{x}_2^{7/9}| \le \frac{1}{4}|x_2 - \bar{x}_2|^{7/9},$$

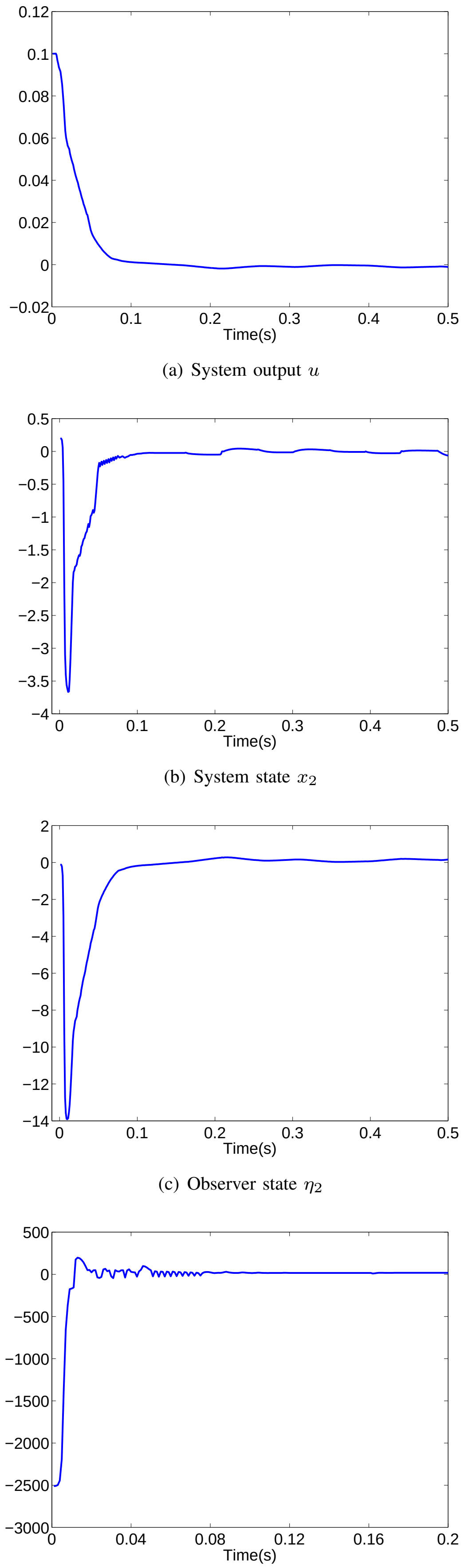
$$|\frac{1}{4}sinx_1 - \frac{1}{4}sin\bar{x}_1| \le \frac{1}{4}|x_1 - \bar{x}_1|^{10/11},$$

$$|\frac{1}{8}cos2x_1 - \frac{1}{8}cos2\bar{x}_1| \le \frac{1}{4}|x_1 - \bar{x}_1|^{8/11}.$$

Assumption 1 is satisfied with $b = \frac{1}{4}$. By choosing $l = 2/3$ and $\sigma = 3$, according to the design procedure proposed in Section III, we can obtain the following homogeneous observer and controller

$$\begin{aligned}
&\dot{\eta}_2 = -Ll_1\hat{z}_2, \quad \hat{z}_2 = (\eta_2 + l_1z_1)^{9/11}, \\
&u = -7.3645L^2(\hat{z}_2^{11/3} + 12.6992z_1^3)^{7/33}.
\end{aligned} \tag{48}$$

In the simulation, we choose $l_1 = 150$, $L = 8$ and the initial values $x_1(0) = 0.1$, $x_2(0) = 0.2$, $\eta_2(0) = -0.1$. Fig. 1 gives the responses of the CLS (47) and (48), from which, the efficiency of the controller is demonstrated.



(a) System output u

(b) System state x_2

(c) Observer state η_2

(d) Control input u

Fig. 1. The responses of the CLS (47) and (48).

V. Conclusion

This article has solved the problem of finite-time output feedback stabilization for a category of SNSs by using homogeneous domination approach. The designed controller renders the CLS states to zero in finite time almost surely. Simulation results of a numerical example have demonstrated the effectiveness of the proposed algorithm. As future work, we consider the extension of this result to high-order SNSs.

References

[1] H. Deng and M. Krstić, "Output-feedback stochastic nonlinear stabilization," *IEEE Transactions on Automatic Controls*, vol. 44, pp. 328–333, 1999.

[2] Y. L. Shang, J. C. Huang, X. X. Shi, and X. C. Zhu, "Global finite-time Stabilization for a class of upper-triangular systems subject to input saturation," *Engineering Letters*, vol. 28, no. 1, pp. 16–21, 2020.

[3] Y. L. Shang, J. C. Huang, H. S. Li, and X. W. Wen, "Finite-time control design for nonholonomic mobile robots subject to spatial constraint," *IAENG International Journal of Applied Mathematics*, vol. 48, no. 4, pp. 449–454, 2018.

[4] F. Z. Gao, X. C. Zhu, J. C. Huang, and X. L. Wen, "Finite-time state feedback stabilization for a class of uncertain high-order nonholonomic feedforward systems," *Engineering Letters*, vol.27, no.1, pp. 108-113, 2019.

[5] F. Z. Gao, Y. Q. Wu, Z. C. Zhang, and Y. H. Liu, "Global fixed-time stabilization for a class of switched nonlinear systems with general powers and its application," *Nonlinear Analysis: Hybrid Systems*, vol.31, pp. 56-68, 2019.

[6] F. Z. Gao, Y. Q. Wu, and Z. C. Zhang, "Global fixed-time stabilization of switched nonlinear systems: a time-varying scaling transformation approach," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol.66, no.11, pp.1890-1894, 2019.

[7] F. Z. Gao, Y. L. Shang, Y.Q. Wu, and Y. H. Liu, "Global fixed-time stabilization for a class of uncertain high-order nonlinear systems with dead-zone input nonlinearity," *Transactions of the Institute of Measurement and Control*, vol.41, no.7, pp.1888–1895, 2019.

[8] J. Yin, S. Khoob, Z. Man, and X. Yu, "Finite-time stability and instability of stochastic nonlinear systems," *Automatica*, vol. 47, pp. 2671–2677, 2011.

[9] F. Z. Gao and Y. Q. Wu, "Global finite-time stabilization for a class of stochastic high order time-varying nonlinear systems," *International Journal of Control*, vol.89, no.12, pp. 2453–2465, 2016.

[10] F. Z. Gao, Y. Q. Wu, and Y. H. Liu, "Finite-time stabilization for a class of switched stochastic nonlinear systems with dead-zone input nonlinearities," *International Journal of Robust and Nonlinear Control*, vol.28, no.9, pp. 3239–3257, 2018.

[11] F. Z. Gao, Y. Q. Wu, and Y. H. Liu, "Finite-time stabilization for a class of switched stochastic nonlinear systems with dead-zone input nonlinearities," *International Journal of Robust and Nonlinear Control*, vol.28, no.9, pp. 3239–3257, 2018.

[12] F. Z. Gao, Y. Q. Wu, Z. C. Zhang, and Y. H. Liu, "Finite-time stabilisation of a class of stochastic nonlinear systems with dead-zone input," *International Journal of Control*, vol.92, no.7, pp. 1541–1550, 2019.