Numerical Groundwater Quality Assessment Model Using Two-level Explicit Methods

Suriyun Khatbanjong, Nopparat Pochai

Abstract—**In areas of many developing countries in which landfill operations are used, leachate water is a source of the issue of surface water and groundwater contamination. To evaluate drinking water quality, groundwater quality assessment is required, and the location of the polluting concentration must be approximated. There are several approaches for assessing the quality of groundwater, such as sampling and mathematical simulation. Under various scenarios, mathematical models are used to simulate the effect on groundwater quality over long periods of time. In this research, the use of numerical technique introduces a one-dimensional mathematical model of groundwater quality assessment. Two-level explicit methods, the Lax-Wendroff method and the traditional upwind method, are used to precisely estimate a better solution to the problem than the upwind method. The proposed simulation may be used in the future to forewarn of the problems of groundwater pollution around landfills.**

Index Terms— **groundwater quality assessment, one-dimensional advection-diffusion equation, the upwind explicit formula, the Lax-Wendroff formula, two-level explicit methods**

I. Introduction

Waste generation in developing countries is increasing due to the rapid growth of urbanization and industrialization. Thailand and Indonesia report similar waste generation per person of about 0.65 kg per person per day. The waste generated in Indonesia was 1.80 billion tons per year and, in Indonesia, 1.75 billion tons. The problems are mainly due to the non-segregation of wet and dry solid wastes, the composition of solid waste dumping at landfill sites, and poor management of landfill sites [1]. A site for the disposal of waste materials by burial is a landfill site. It is also the oldest method of treatment for solid waste. Historically, the most common methods of organized waste disposal have been landfills and are still so in many nations. Landfills which include internal waste disposal sites where, at the place of processing, a waste producer performs its own waste disposal, as well as sites used by multiple producers. Most of the land is also used for waste management purposes, such as temporary storage, consolidation and transition or the processing, disposal or recycling of waste materials [1]. The serious problem of solid waste is faced by developing countries such as India, where economic development and urbanization have become faster. According to a 2006 study by the Ministry of the Environment in Japan, the amount of waste produced in 2000 was approximately 12.7 billion tons, which is expected to rise to approximately 19 billion tons worldwide by 2025 and By 2050, to about 27billion tons [2]. The environmental dangers of the generation of leachate stem from escaping to the atmosphere around landfills, and in particular to waste and groundwater courses. By planning and engineering landfill sites, these threats can be mitigated. These sites are those that are built on geologically impermeable materials, or sites that use geotextile or engineered clay impermeable liners. Within the United States and the European Union, the use of linings is now obligatory, except where waste is carefully regulated and completely inert [3]. The composition of solid waste, as well as the rainfall conditions at a location, mainly determine the characteristics and composition of the leachates from these landfill sites. In both countries, the management of the collection of the leachates from these sites affects the groundwater and, thus, is a highly important aspect to consider for minimizing the adverse effects on groundwater quality. The management of leachates from landfill sites is different for India and Indonesia; as in some countries, specialized technologies and practices are adopted for their management, while in others, they are poorly managed. In this paper, a comparative scenario is presented regarding groundwater contamination from leachates from landfill sites located India and Indonesia. From this study, it was found that the unscientific design of landfill and the absence of liners allow leachate to percolate into the ground and contaminate the groundwater. The various factors affecting the groundwater contaminations from the leachates were also examined, and it was found that the landfill sites in both countries were not managed effectively. It was also revealed that the scientific disposal of the mixture of solid waste was not practiced [4]. Most developing countries are presently facing the problem of MSW management. Urbanization and industrialization have contributed higher amounts of generation of solid waste in developing nations [5]. There are hazardous compounds in many products that end up as waste. These toxins leach into our soil and groundwater over time and become threats to the atmosphere for years. Electronic waste is a good example. Waste such as televisions, computers, and other electronic appliances contain a long list of hazardous

Manuscript received Manuscript received September 30, 2019; revised January 6, 2021. This paper is supported by Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand

N. Pochai is an Assistant Professor of Department of Mathematics, Faculty of Science, King Mongkuts Institute of Technology Ladkrabang, Bangkok, 10520, Thailand (corresponding author to provide phone: 662329-8400; fax: 662-329-8400; e-mail: nop_math@yahoo.com).

S. Khatbanjong is a PhD candidate of Mathematics Department, Faculty of Science, King Mongkuts Institute of Technology Ladkrabang, Bangkok, 10520, Thailand (e-mail: suriyun_labotron@hotmail.com).

substances, including mercury, arsenic, cadmium, PVC, solvents, acids, and lead. As waste breaks down in a landfill and water filters through the waste, leachate is the liquid created. This liquid is extremely toxic and can contaminate the pathways of soil, ground water, and water. Landfill leachate generation can cause environmental and health effects due to soil, surface, and groundwater contamination. The three main issues, the design of landfill liner systems, and the finding and valuation of the degree of contaminant that percolates in groundwater, are risks for human health and cause environment issues [6]. Groundwater quality monitoring systems help to determine the likelihood and severity of contamination problems. MSW composition contains glass pieces, metals, papers, rags, plastics, ashes, and flammable materials [7]. In addition to these, solid waste contains other substances, such as discarded chemicals, paints, scrap materials, hazardous waste generated from hospitals, dead animals, industrial, agricultural, and horticultural residues, and concrete and demolition waste [8]. In developing countries, serious environmental problems are caused due to landfill sites which are not scientifically designed and the absence of proper leachate collection and control systems. Leachate composition depends mainly on the composition of solid waste, annual rainfall, per capita waste generation, and characteristics of the total waste [9-11]. Point sources, such as landfills, can release high concentrations of contaminants into the groundwater because of the migration of leachate from its bottom [12-13].

Landfill leachate generation can cause environmental and health effects due to soil, surface, and groundwater contamination. The three main issues as seen previously, the design of landfill liner systems and the finding and valuation of the degree of contaminants that percolate in groundwater, are risks for human health and cause environmental issues [14-15]. He found that after they were fed formulas which were mixed with water from shallow wells, both infants became sick. Concentrations of nitrate-nitrogen (nitrate-N) were 90 and 150 mg/L in the associated wells. Many similar cases were documented after Comly's findings were released. Walton in 1951 [16-17]. There were computational techniques for solving the non-uniform flow of stream water comprising a one-dimensional equation of advection-dispersion-reaction in [18] and [20]. To solve groundwater contamination concerns, a fourth-order compact finite difference scheme of the two-dimensional convection-diffusion equation was suggested. To simulate the law of movement of contaminants in the medium, which was spatially precise in the fourth order and temporally precise in the second order, an effective system was created [21]. In the empirical solution of the convection diffusion equation, the two-dimensional Fourier transform and the inverse Fourier transform were taken into consideration. To obtain the numerical solution, the Crank-Nicolson finite difference method, which was second-order precise in time and space, was developed. Simulations of Numerical Simulations [22]. To solve the hydrodynamic model, the Crank-Nicolson method was also used, while the explicit Saulyev scheme was used to solve the dispersion model. For model use, simple finite difference schemes are becoming more appealing. The forward time-central space (FTCS) scheme, the MacCormack scheme, and the Saulyev scheme are simply explicit techniques. The BTCS and Tacit Schemes include and the Crank-Nicolson scheme [23] and [24]. Mathematical models describing groundwater flow and solute transport in homogeneous and heterogeneous porous media have been developed in the past and are available in literature. In Yule and Gardener [25-26]. The advection-diffusion one-dimensional equation with a constant coefficient was solved by computational techniques. The two-level finite difference approximations are based on these techniques. The results of a numerical experiment were presented and it discussed and compared the time needed for the accuracy and central processor (CPU) [27].

In addition, the numerical method has been used for other works, as well as having been useful in scientific, technological, biological, and biomedical problems, such as a thin film flow problem on a moving belt. One nonlinear differential equation could model a thin film flow velocity on a moving belt. The model was given velocity of film flow in each layer of thickness. A finite differential method and an iterative Newton method were used to estimate the nonlinear thin fluid film velocity model solutions. Their numerical simulations of a third-grade fluid's thin film flow velocity on a moving belt with varying physical parameters were investigated. The numerical techniques proposed gave approximate solutions in many moving belt speed rates in good agreement [28]. The article provided a solution to problems with boundary values using the finite difference scheme and the Laplace transform process. Some examples were solved to demonstrate the methods; Laplace transforms gave a closed form solution, while the extended interval increased convergence in the finite difference scheme of the solution [29].

In a one-dimensional heat equation, subject to initial boundary conditions for both Neumann and Dirichlet, a Homotopy Perturbation Method was used to solve the problem. Compared with previous studies, the findings obtained were extremely reliable. In contrast to the finite difference approach, HPM also provides a continuous solution, which offers only discrete approximations. This method is considered a powerful mathematical technique and can be applied to a wide variety of linear and nonlinear problems in various fields of science and technology [30]. An analysis of a three-dimensional diffusion equation solution with non-local state using method of decomposition was efficient and provided a highly accurate solution in a series form. This also provided substantial savings in volume and times of measurement relative to conventional methods. The results obtained showed that the method of decomposition was effective and delivered a solution in a closed form [31].

In this research, accurately numerical techniques for a groundwater quality assessment model in heterogeneous soil are proposed. The two-level explicit methods, the upwind method and the Lax-Wendroff method, are employed to approximate their model solutions. The computational solutions of both methods are compared for long-term groundwater pollutant dispersion from a landfill.

II. Governing equation

A. Groundwater pollution dispersion flow through inhomogeneous soil model

In a model of groundwater efficiency, the governing equation is a partial differential equation of one-dimensional advection-diffusion as [19];

$$\frac{\partial C(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C(x,t)}{\partial x}-u(x,t)C(x,t)\right), \quad (1)$$

for all $(x,t)\in[0,L]\times[0,T]$,

where $C(x,t)$ is the dispersing concentration of groundwater pollutant at position x along the longitudinal direction at time t, D is the pollutant method's dispersion coefficient, u is a uniform flow velocity, L is the length of the considered area from the pollutant origin to the end point, and T is the rate of chance simulation time. The inhomogeneity of the soil causes variation in the groundwater flow velocity. Kumar et al. [19] proposed a variation of increasing nature. They also believed that functions were given to the dispersion parameter and the velocity parameter $f_1(x,t)$ and $f_2(x,t)$. It is possible to rewrite Eq.(1) as [32];

$$\frac{\partial C(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(D_0 f_1(x,t)\frac{\partial C(x,t)}{\partial x}-u_0 f_2(x,t)C(x,t)\right). \quad (2)$$

Eq. (2) can be written in the following form;

$$\frac{\partial C(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(D_0\frac{\partial f_1(x,t)}{\partial x}-u_0 f_2(x,t)\right)\frac{\partial C(x,t)}{\partial x}$$
$$+D_0 f_1(x,t)\frac{\partial^2 C(x,t)}{\partial x^2}-u_0\frac{\partial f_2(x,t)}{\partial x}C(x,t). \quad (3)$$

In the equation above, D_0 and u_0 are constants, the dimensions of which depend on the expression $f_1(x,t)$ and $f_2(x,t)$. The inhomogeneity of the soil allows the rate of flow to differ. A difference in the growing dispersion of groundwater contaminants in heterogeneous soil has been considered by Kumar et al. [32]. The dispersion parameter is often believed to be proportional to the velocity square. Consequently, Eq. (2) is becoming;

$$f_1(x,t)=(1+ax)^2, \quad \text{and} \quad f_2(x,t)=1+ax, \quad (4)$$

the parameter a with the $(length)^{-1}$ dimension accounts for the inhomogeneity of the soil, Eq. (3) is becoming;

$$\frac{\partial C(x,t)}{\partial t}=\left[(1+ax)(2aD_0-u_0)\right]\frac{\partial C(x,t)}{\partial x}$$
$$+D_0(1+ax)^2\frac{\partial^2 C(x,t)}{\partial x^2}-u_0 aC(x,t), \quad (5)$$

$$\frac{\partial C(x,t)}{\partial t}=g(x)\frac{\partial C(x,t)}{\partial x}+h(x)\frac{\partial^2 C(x,t)}{\partial x^2}-KC(x,t). \quad (6)$$

where

$$g(x)=(1+ax)(2aD_0-u_0), \quad (7)$$

$$h(x)=D_0(1+ax)^2, \quad (8)$$

$$K=au_0, \quad (9)$$

$$-\beta=g(x), \quad (10)$$

$$\alpha=h(x). \quad (11)$$

B. Initial and boundary conditions

The soil's originally groundwater-contaminated free state of concentration suggests the following initial condition;

$$C(x,0)=r(x),\ 0\le x\le L,\ t=0. \quad (12)$$

where $r(x)$ is a given initially measured groundwater pollutant function. Because of a continuous input, groundwater pollutant concentration is introduced at the origin, while the concentration gradient at the end point is defined by the average chance rate of groundwater pollutant concentration around them, obtained by the following boundary conditions;

$$C(0,t)=C_0, \quad t>0, \quad (13)$$

$$\frac{\partial C(x,t)}{\partial x}=C_s, \quad x=L, \quad t\ge 0. \quad (14)$$

where C_0 is a given average groundwater pollutant concentration at the considered landfill, and C_s is the rate of change of the pollutant concentration around the far field monitoring station.

III. Numerical techniques

A. The two-level explicit methods

A mesh of the grid line covers the solution domain of the problem. The grid point (x_i,t_n) is defined by $x_i=i\Delta x$ for all $i=0,1,2,...M$ and $t_n=n\Delta t$ for all $n=0,1,2,...N$, in which M and N are positive integers, and where x_i and t_n are parallel to the synchronized axes of space and time. The spacing of the constant spatial and temporal grid is $\Delta x=L/M$ and $\Delta t=T/N$. Take into account the following derivative approximations in advection-diffusion Equation (6). A weight that integrates θ as follows;

$$C(x,t)\cong C_i^n, \quad (15)$$

$$\frac{\partial C(x,t)}{\partial t}\cong\frac{C_i^{n+1}-C_i^n}{\Delta t}, \quad (16)$$

$$\frac{\partial C(x,t)}{\partial x}\cong\theta\frac{C_i^n-C_{i-1}^n}{\Delta x}+(1-\theta)\frac{C_{1+1}^n-C_{i-1}^n}{2\Delta x}, \quad (17)$$

$$\frac{\partial^2 C(x,t)}{\partial x^2}\cong\frac{C_{i+1}^n-2C_i^n+C_{i-1}^n}{(\Delta x)^2}. \quad (18)$$

Substituting Eqs.(15)-(18) in to Eq. (6), we can obtain that;

$$\frac{C_i^{n+1}-C_i^n}{\Delta t}=\left[(1+ai\Delta x)(2aD_0-u_0)\right]$$
$$\times\left(\theta\frac{C_i^n-C_{i-1}^n}{\Delta x}+(1-\theta)\frac{C_{i+1}^n-C_{i-1}^n}{2\Delta x}\right)$$
$$+D_0(1+ai\Delta x)^2\left(\frac{C_{i+1}^n-2C_i^n+C_{i-1}^n}{\Delta x^2}\right)-au_0C_i^n, \quad (19)$$

$$C_i^{n+1}=g(x)\left(\theta\frac{C_i^n-C_{i-1}^n}{\Delta x}+(1-\theta)\frac{C_{i+1}^n-C_{i-1}^n}{2\Delta x}\right)\Delta t$$

$$+h(x)\left(\frac{C_{i+1}^n-2C_i^n+C_{i-1}^n}{(\Delta x)^2}\right)\Delta t-K\Delta tC_i^n, \tag{20}$$

$$C_i^{n+1}=g(x)\frac{\theta\Delta tC_i^n}{\Delta x}-g(x)\frac{\theta\Delta tC_{i-1}^n}{\Delta x}+g(x)\frac{(1-\theta)\Delta tC_{i+1}^n}{2\Delta x}$$
$$-g(x)\frac{(1-\theta)\Delta tC_{i-1}^n}{2\Delta x}-h(x)\frac{\Delta tC_{i+1}^n}{(\Delta x)^2}-h(x)\frac{2\Delta tC_i^n}{(\Delta x)^2}$$
$$+h(x)\frac{\Delta tC_{i-1}^n}{(\Delta x)^2}-K\Delta tC_i^n+C_i^n, \tag{21}$$

$$=\left[h(x)\frac{\Delta t}{(\Delta x)^2}-g(x)\frac{\theta\Delta t}{\Delta x}-g(x)\frac{(1-\theta)\Delta t}{2\Delta x}\right]C_{i-1}^n$$
$$+\left[g(x)\frac{\theta\Delta t}{\Delta x}-h(x)\frac{2\Delta t}{(\Delta x)^2}-K\Delta t+1\right]C_i^n$$
$$+\left[g(x)\frac{(1-\theta)\Delta t}{2\Delta x}+h(x)\frac{\Delta t}{(\Delta x)^2}\right]C_{i+1}^n, \tag{22}$$

Thus $C_i^{n+1}=(E-(1+\theta)F)C_{i-1}^n+(1+\theta F-2E-H)C_i^n$
$$+((1-\theta)F+E)C_{i+1}^n. \tag{23}$$

where

$$E=h(x)\frac{\Delta t}{(\Delta x)^2}, \tag{24}$$

$$F=g(x)\frac{\Delta t}{2\Delta x}, \tag{25}$$

$$H=K\Delta t. \tag{26}$$

for all $1\le i\le M-1$ and $1\le n\le N-1$

where

$$c=\beta\frac{\Delta t}{\Delta x}, \tag{27}$$

$$s=\alpha\frac{\Delta t}{(\Delta x)^2}. \tag{28}$$

A von Neumann stability of (22) yields [34] the condition of stability;

$$\frac{c^2-\theta c}{2}\le s\le\frac{1-\theta c}{2}. \tag{29}$$

This method's modified equivalent partial differential equation is in the following form [35];

$$\frac{\partial C(x,t)}{\partial t}+\beta\frac{\partial C(x,t)}{\partial x}-\left[\alpha+\frac{\beta\Delta x(\theta-c)}{2}\right]\frac{\partial^2C(x,t)}{\partial x^2}$$
$$+\beta\frac{(\Delta x)^2}{6}(1-3c\theta+2c^2-6s)\frac{\partial^3C(x,t)}{\partial x^3}+O((\Delta x)^3)=0. \tag{30}$$

It is noteworthy that the quantity of numerical diffusion is independent of s values, although the usable range of c varies with s values. Error of truncation for difference Eq. (19) is $O(\Delta t,\Delta x^2)$. Using a value of Δx and Δt that is small enough, the truncation error can be reduced until the accuracy achieved is within the error tolerance [33]. The initial condition Eq. (12) for Eq. (19) can be expressed in the finite difference form as;

$$C_i^0=r(x)=r(i\Delta x)=r_i,\quad x\ge 0,\quad t=0. \tag{31}$$

In the finite difference form, Boundary Condition Eq. (13) can be written as;

$$C_0^n=C_0. \tag{32}$$

If we employ the forward space method in Eq. (14) to the right boundary condition, we have;

$$C_N^n=C_{N-1}^n+\Delta xC_s. \tag{33}$$

B. The upwind explicit method

Setting $\theta=1$ in Eq(19) gives the explicit method of the following upwind-type finite difference;

$$C_i^{n+1}=g(x)\frac{\theta\Delta tC_i^n}{\Delta x}-g(x)\frac{\theta\Delta tC_{i-1}^n}{\Delta x}+g(x)\frac{(1-\theta)\Delta tC_{i+1}^n}{2\Delta x}$$
$$-g(x)\frac{(1-\theta)\Delta tC_{i-1}^n}{2\Delta x}-h(x)\frac{\Delta tC_{i+1}^n}{(\Delta x)^2}-h(x)\frac{2\Delta tC_i^n}{(\Delta x)^2}$$
$$+h(x)\frac{\Delta tC_{i-1}^n}{(\Delta x)^2}-K\Delta tC_i^n+C_i^n, \tag{34}$$

$$=\left[h(x)\frac{\Delta t}{(\Delta x)^2}-g(x)\frac{\theta\Delta t}{\Delta x}-g(x)\frac{(1-\theta)\Delta t}{2\Delta x}\right]C_{i-1}^n$$
$$+\left[g(x)\frac{\theta\Delta t}{\Delta x}-h(x)\frac{2\Delta t}{(\Delta x)^2}-K\Delta t+1\right]C_i^n$$
$$+\left[g(x)\frac{(1-\theta)\Delta t}{2\Delta x}+h(x)\frac{\Delta t}{(\Delta x)^2}\right]C_{i+1}^n, \tag{35}$$

$$=\left[h(x)\frac{\Delta t}{(\Delta x)^2}-g(x)\frac{\Delta t}{\Delta x}\right]C_{i-1}^n+\left[h(x)\frac{\Delta t}{(\Delta x)^2}\right]C_{i+1}^n$$
$$+\left[g(x)\frac{\Delta t}{\Delta x}-h(x)\frac{2\Delta t}{(\Delta x)^2}-K\Delta t+1\right]C_i^n, \tag{36}$$

Thus $C_i^{n+1}=(E-2F)C_{i-1}^n+(1+2F-2E-H)C_i^n+EC_{i+1}^n.$ (37)

For which is stable;

$$\frac{c^2-c}{2}\le s\le\frac{1-c}{2}. \tag{38}$$

where

$$c=-F-H, \tag{39}$$

$$s=E. \tag{40}$$

This method's modified equivalent partial differential equation is in the following form [35];

$$\frac{\partial C(x,t)}{\partial t}+\beta\frac{\partial C(x,t)}{\partial x}-\left[\alpha+\frac{\beta\Delta x(1-c)}{2}\right]\frac{\partial^2C(x,t)}{\partial x^2}$$
$$+\beta\frac{(\Delta x)^2}{6}(1-3c+2c^2-6s)\frac{\partial^3C(x,t)}{\partial x^3}+O((\Delta x)^3)=0. \tag{41}$$

The coefficients on the right side of the formula Eq. (40) in its stability range Eq.(41) are always non-negative. Usage of this upwind approach can never, therefore, create spurious negative values. This is the key reason why this formula is also used by hydrologists and oceanographers. It is only accurate in the first order, however, and generates unnecessary numerical diffusion quantities.

C. *The Lax-Wendroff method*

Putting $\theta = c$ with Eq. (19) returns the following formula for finite-difference;

$$C_i^{n+1} = \left[h(x)\frac{\Delta t}{(\Delta x)^2} - g(x)\frac{\theta\Delta t}{\Delta x} - g(x)\frac{(1-\theta)\Delta t}{2\Delta x} \right] C_{i-1}^n + \left[g(x)\frac{\theta\Delta t}{\Delta x} - h(x)\frac{2\Delta t}{(\Delta x)^2} - K\Delta t + 1 \right] C_i^n + \left[g(x)\frac{(1-\theta)\Delta t}{2\Delta x} + h(x)\frac{\Delta t}{(\Delta x)^2} \right] C_{i+1}^n, \quad (42)$$

$$= \left[h(x)\frac{\Delta t}{(\Delta x)^2} - g(x)\frac{\Delta t}{\Delta x} - g(x)\frac{(1-c)\Delta t}{2\Delta x} \right] C_{i-1}^n + \left[g(x)\frac{c\Delta t}{\Delta x} - h(x)\frac{2\Delta t}{(\Delta x)^2} - K\Delta t + 1 \right] C_i^n + \left[g(x)\frac{(1-c)\Delta t}{\Delta x} - h(x)\frac{\Delta t}{(\Delta x)^2} \right] C_{i+1}^n, \quad (43)$$

Thus $C_i^{n+1} = (E-(3-c)F)C_{i-1}^n + (1+2cF-2E-H)C_i^n + (2(1-c)F+E)C_{i+1}^n. \quad (44)$

Using Eq. (19), this scheme can be shown to be stable for;

$$0 \le s \le \frac{1-c^2}{2}. \quad (45)$$

where

$$c = -F - H, \quad (46)$$

$$s = E. \quad (47)$$

Note that Eq. (41) shows that numerical diffusion will be eliminated by the $\theta = c$ option. In the following form, the modified equivalent partial differential equation that corresponds to the finite difference formula Eq. (44) consistent with the equation of advection-diffusion can be written; $0 = O((\Delta x)^4) + \frac{\partial C(x,t)}{\partial t} + \beta\frac{\partial C(x,t)}{\partial x} - \alpha\frac{\partial^2 C(x,t)}{\partial x^2}$

$$+\beta\frac{(\Delta x)^2}{6}(1-6s-c^2)\frac{\partial^3 C(x,t)}{\partial x^3} - \beta\frac{(\Delta x)^3}{24c}(2s-12s^2+12sc^2-3c^2+4c^2)\frac{\partial^4 C(x,t)}{\partial x^4}. \quad (48)$$

Note that Eq.(36) shows that this finite difference scheme is free of numerical diffusion. This technique was first developed by Lax and Wendroff [36]. The Lax-Wendroff method is stable for a larger range of values of the Courant number than the upwind method.

IV. Numerical Experiments

Consider the measurement of groundwater pollutant concentration C under a landfill and its vicinity. The considered area is aligned with longitudinal distance, 1.0 km total length. There is a landfill which discharges leachate as a pollutant source into the underground. The pollutant parameters at the considered landfill are C_0 kg/l, $D_0 = 0.71$ km^{-2}/year, $u_0 = 0.60$ km/year, and $a = 1$ km^{-1}. In the numerical experiment, space and time are discretized by $\Delta x = 0.1$ km and $\Delta t = 0.0001$ year, respectively. The groundwater concentration is approximated by using the upwind explicit method and the Lax-Wendroff method. We obtain an analytical solution of an ideal advection-diffusion equation, proposed in [29];

$$\tilde{C}(x,t) = \frac{C_0}{2}\left(\begin{array}{l} (1+ax)^{-1} erfc\left(\frac{\ln(1+ax)}{2a\sqrt{D_0 t}} - \beta_0\sqrt{t} \right) \\ +(1+ax)^{\delta} erfc\left(\frac{\ln(1+ax)}{2a\sqrt{D_0 t}} + \beta_0\sqrt{t} \right) \end{array} \right). \quad (49)$$

where

$$\omega_0 = \left(au_0 - a^2 D_0\right), \quad (50)$$

$$\beta_0 = \sqrt{\frac{\omega_0^2}{4a^2 D_0} + au_0} = \frac{u_0 + aD_0}{2\sqrt{D_0}}, \quad (51)$$

$$\delta = \frac{u_0}{aD_0}. \quad (52)$$

If we employ the upwind explicit method, in Eqs. (34)-(37), we get the approximated groundwater pollutant along the considered area until 1.3 years, as shown in Figs. 1 and 2 and Table I. If we use the Lax-Wendroff method, in Eqs (42)-(44), we obtain the approximated groundwater pollutant concentration along the longitudinal considered area in Figs. 3 and 4 and Table II. The accuracy of the upwind explicit method and the Lax-Wendroff method are shown in Fig. 2 and Fig. 4. The accuracy of both approximations are tested by using analytical solution and the absolute error, as shown in Table III and IV.

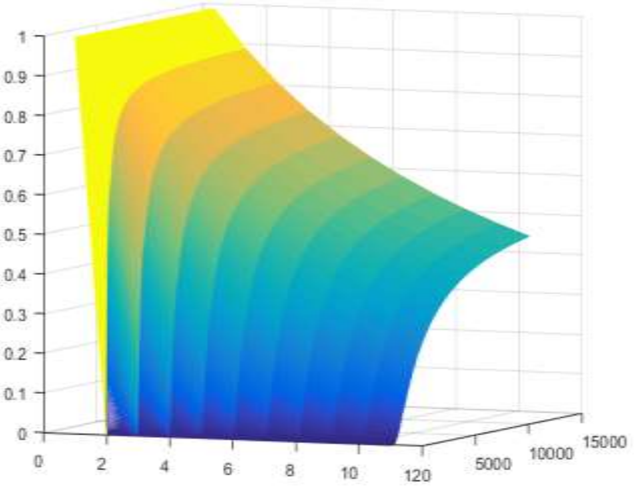

Fig 1. Approximated groundwater pollutant by using the upwind explicit method

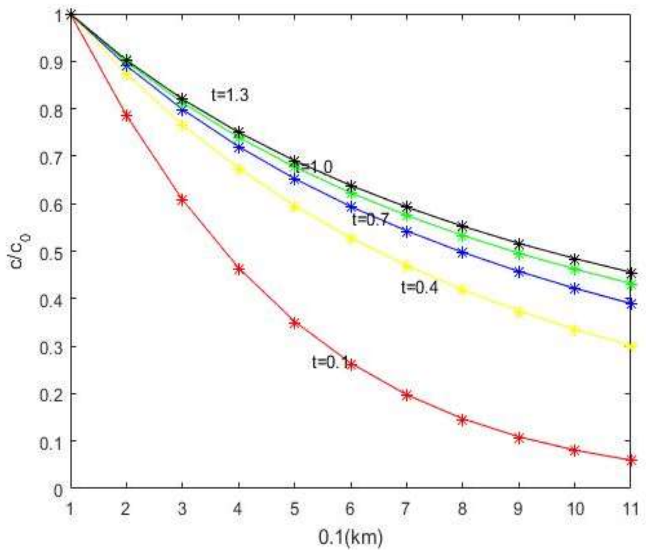

Fig 2. Groundwater pollutant using the upwind explicit method at 0.1, 0.4, 0.7, 1.0, and 1.3 years, when star represents the upwind explicit method solution and curved line represents analytical solution

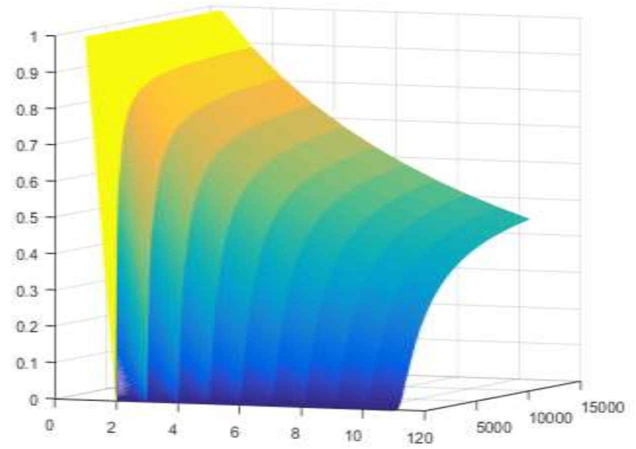

Fig 3. Approximated groundwater pollutant by using the Lax-Wendroff method

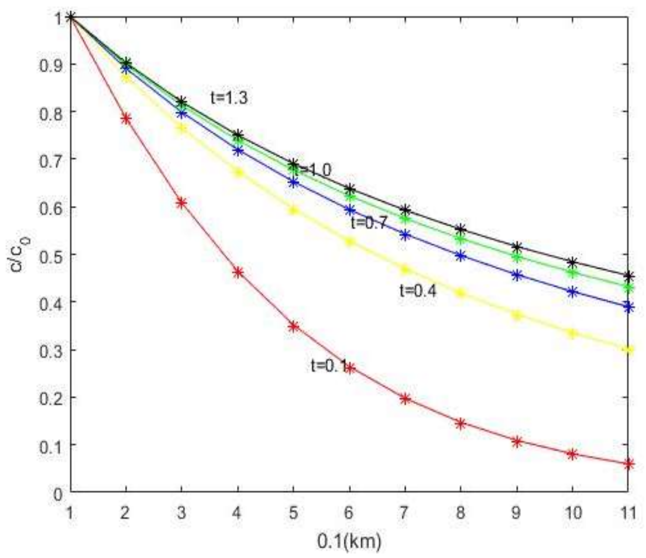

Fig 4. Groundwater pollutant concentration at 0.1, 0.4, 0.7, 1.0, and 1.3 years, when star represents the Lax-Wendroff method solution and curved line represents analytical solution

TABLE I APPROXIMATED GROUNDWATER POLLUTANT CONCENTRATION USING THE UPWIND EXPLICIT METHOD ALONG A CONSIDERED AREA BETWEEN 0.1-1.3 YEARS

$C(x,t)$						
x / t	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.00	0.784135	0.605103	0.460967	0.347623	0.260112
0.4	1.00	0.872311	0.764294	0.672245	0.593332	0.525337
0.7	1.00	0.890264	0.797834	0.719028	0.651159	0.592208
1.0	1.00	0.897854	0.812104	0.739122	0.676290	0.621665
1.3	1.00	0.901852	0.819641	0.749780	0.689690	0.637468
x / t	0.6	0.7	0.8	0.9	1.0	
0.1	0.193494	0.143325	0.105852	0.078029	0.057461	
0.4	0.466487	0.415354	0.370764	0.331754	0.297521	
0.7	0.540626	0.495203	0.454980	0.419182	0.387179	
1.0	0.573775	0.531482	0.493890	0.460286	0.430094	
1.3	0.591678	0.551214	0.515214	0.482989	0.453988	

TABLE II APPROXIMATED GROUNDWATER POLLUTANT CONCENTRATION USING THE LAX-WENDROFF METHOD ALONG A CONSIDERED AREA BETWEEN 0.1-1.3 YEARS

$C(x,t)$						
x / t	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.00	0.784700	0.606089	0.462209	0.348974	0.261460
0.4	1.00	0.872553	0.764744	0.672869	0.594096	0.526211
0.7	1.00	0.890418	0.798121	0.719432	0.651662	0.592795
1.0	1.00	0.897961	0.812305	0.739407	0.676647	0.622086
1.3	1.00	0.901930	0.819789	0.749990	0.689954	0.637780
x / t	0.6	0.7	0.8	0.9	1.0	
0.1	0.194764	0.144474	0.106860	0.078893	0.058189	
0.4	0.467447	0.416374	0.371827	0.332843	0.298623	
0.7	0.541284	0.495520	0.455745	0.419986	0.388015	
1.0	0.574251	0.532005	0.494453	0.460884	0.430722	
1.3	0.592032	0.551606	0.515638	0.483442	0.454466	

TABLE III THE ABSOLUTE ERROR OF THE UPWIND EXPLICIT METHOD APPROXIMATION WHERE $e(x,t)=\left|C(x,t)-\tilde{C}(x,t)\right|$

$e(x,t)$						
x / t	0.0	0.1	0.2	0.3	0.4	0.5
0.1	0.00	0.001853	0.003230	0.004071	0.004434	0.004431
0.4	0.00	0.000792	0.001473	0.002041	0.002501	0.002865
0.7	0.00	0.000510	0.000941	0.001320	0.001647	0.001923
1.0	0.00	0.000350	0.000660	0.000932	0.001170	0.001377
1.3	0.00	0.000256	0.000484	0.000686	0.000864	0.001021
x / t	0.6	0.7	0.8	0.9	1.0	
0.1	0.004181	0.003787	0.003328	0.002857	0.002410	
0.4	0.003142	0.003346	0.003486	0.003573	0.003616	
0.7	0.002156	0.002349	0.002508	0.002636	0.002739	
1.0	0.001556	0.001711	0.001845	0.001959	0.002057	
1.3	0.001159	0.001281	0.001389	0.001483	0.001565	

TABLE IV THE ABSOLUTE ERROR OF THE LAX-WENDROFF METHOD APPROXIMATION WHERE $e(x,t)=\left|C(x,t)-\tilde{C}(x,t)\right|$

x \ t	$e(x,t)$ 0.0	0.1	0.2	0.3	0.4	0.5
0.1	0.00	0.001287	0.002244	0.002829	0.003083	0.003082
0.4	0.00	0.000550	0.001023	0.001417	0.001737	0.001990
0.7	0.00	0.000348	0.000653	0.000917	0.001144	0.001336
1.0	0.00	0.000243	0.000458	0.000647	0.000812	0.000956
1.3	0.00	0.000178	0.000336	0.000476	0.000600	0.000709
x \ t	0.6	0.7	0.8	0.9	1.0	
0.1	0.002910	0.002638	0.002319	0.001993	0.001682	
0.4	0.002183	0.002325	0.002423	0.002484	0.002514	
0.7	0.001497	0.001632	0.001742	0.001832	0.001903	
1.0	0.001081	0.001189	0.001281	0.001361	0.001429	
1.3	0.000805	0.000890	0.000964	0.001030	0.001087	

V. DISCUSSION

The upwind explicit method and the Lax-Wendroff method give good agreement for approximated groundwater pollutant concentration in an ideal case, as shown in Fig. 2 and Fig. 4, respectively. In both simulations, the ground water pollutant measurement is simulated for a long period of time, around 0-1.3 years, as shown in Tables I-II and Fig. 1 and Fig. 3, respectively. In the numerical aspect, the Lax-Wendroff method gives better approximated solutions than the upwind explicit method, as shown by the absolute error between Table III and Table IV. The proposed numerical techniques provide accurate approximated solutions.

VI. CONCLUSION

The long-term conditions for calculating groundwater contamination was tested in heterogeneous soil. An updated model of groundwater quality was applied over a long time. The polluting concentration positions were approximated using numerical technique, and the concentration of groundwater pollutants at their monitoring stations was assumed to be the model's initial and boundary condition. To approximate the model solution, explicit two-level methods, the upwind method and the Lax-Wendroff method, were used. The Lax-Wendroff method provided a better approximation than the traditional upwind method. The proposed simulation can be used to warn of what happens in the future with groundwater contamination. The proposed numerical techniques provide an accurate approximate solution and do not result in excessive numerical diffusion.

REFERENCES

[1] Ikem A., Osibanjo O., Sridhar M.K.C., Sobande A. Evaluation of groundwater quality characteristics near two waste sites in Ibadan and Lagos, Nigeria. Water, Air, and Soil Pollution, 140: 2002, 307–333.

[2] Secretariat of the Basel Convention. Environment. ; 2013.

[3] Anne, J.L. and Fred, G.L. 1993. “Groundwater pollution by municipal landfill: Leachate composition, detection and water quality significant”. Proceedings of 4th international landfill symposium, Sardinia, Italy. October, 1993, 1093-1103.

[4] Gourav Suthar, Rishabh Singh, R. K. Malik, Sakshi Gupta, groundwater contamination from landfill leachates an analysis for developing countries (OAIJSE) Volume 2, Issue 9, ISSN(Online):2017, 2456-3293.

[5] Minghua, Municipal solid waste management in Pudong New Area, China, Waste Manage 29: 2009, 1227–1233.

[6] Daabes, assessment of heavy metals and organics in municipal solid waste leachates from landfills with different ages in Journal of Environmental Protection, 4, 2013, 344-352.

[7] Barde, S.S Jain, S.T.Mali, Groundwater Contamination due to Landfill Leachate at Urali Devachi and Validation with Argus One, International Journal of Research in Advent Technology, Vol.2, No.5, E-ISSN: 2321-9637, 2014.

[8] Kjeldsen P, Barlaz MA, Rooker AP, Baun A, Ledin A, Christensen TH. Present and Long-Term Composition of MSW Landfill Leachate: A Review. Crit Rev Environ Sci Technol. 2002; 32(4): 297–336.

[9] Saarela J. Pilot investigations of surface parts of three closed landfills and factors affecting them. Environ Monitor Assess. 2003; 84(1/2): 183– 92.

[10] Abu-Rukah Y, Al-Kofahi O. The assessment of the effect of landfill leachate on ground-water quality—a case study. El-Akader landfill site—north Jordan. J Arid Environ. 49(3): 615–30, 2001.

[11] Christensen TH, Kjeldsen P. Basic biochemical processes in landfills. NewYork: AcademicPress; 1989.

[12] Lukmon Salami, Olaosebikan A. Olafadehan, Gutti Babagana, & Alfred A. Susu, Prediction of concentration Profiles of contaminants in groundwater polluted by leachates from a landfill site, IJRRAS 15(3): 365-377, 2013.

[13] Rajkumar M, Ae N, Prasad MN, Freitas H. Potential of sider ophore producing bacteria for improving heavy metal phytoextr action. Trends Bio technol. 28(3): 2010, 142–9.

[14] Daabes, assessment of heavy metals and organics in municipal solid waste leachates from landfills with different ages in Journal of Environmental Protection. 4: 2013, 344-352.

[15] Deepanjan Majumdar, The blue baby syndrome, the Indian Academy of Sciences, ISSN: 0971-8044 Print 0973-712X (Online), 2003.

[16] Comly H. Cyanosis in infants caused by nitrates in well water. JAMA129: 1945, 112-116.

[17] Walton G. Survey of literature relating to infant methemoglobinemia due to nitrate-contaminated water. Am J Public Health 41:1951, 987-995.

[18] Lynda Knobeloch, Barbara Salna, Adam Hogan, Jeffrey Postle, and Henry Anderson, Blue Babies and Nitrate-Contaminated Well Water, Grand Rounds in Environmental Medicine, Volume 108, 7:2000, 675-678.

[19] A. Kumar, D.K, Jaiswal, N. Kumar, Analytical solutions to one-dimensional advection-diffusion equation whit variable coefficients in semi-infinite media, J. Hydrol. 380: 2010, 330-337.

[20] Li, G, Jackson, CR:Simple, accurate and efficient revisions to MacCormack and Saulyev schemes: high Peclet numbers. Appl. Math. Comput. 186:2007, 610-622.

[21] Lingyu Li, Ziwen Jiang, and Zhe Yin,Fourth-order compact finite difference method for solving two-dimensional convection–diffusion equation,Lietal. Advances in Difference Equations. 2018; 234.

[22] Lingyu Li, Ziwen Jiang, and Li, L.Y. and Yin, Z. Numerical Simulation of Groundwater Pollution Problems Based on Convection Diffusion Equation. American Journal of Computational Mathematics, 7: 2017, 350-370.

[23] Pochai, N, Tangmanee, s, Crane, LJ,Miller, JJH: A water quality computation in the uniform channel. J. Interdiscip. Math. 11:2008, pp.803-814.

[24] Pochai, N: A numerical computation of non-dimensional form of stream water quality model with hydrodynamic advection-dispersion-reaction equation. Nonlinear Anal. Hybrid Syst. 13:2009, 666-673.

[25] Yule, D. F. and W. R. Gardner, “Longitudinal and transverse dispersion coefficients in unsaturated Plainfield sand”, Water Resource. Res., Vol.14, 582-588, 1978.

[26] Pongnu, N. and Pochai, N. Numerical simulation of groundwater measurement using alternating direction methods, Journal of Interdisciplinary Mathematics 20:2017, 513–541.

[27] M. Dehghan: Weighted finite difference techniques for the one-dimensional advection-diffusion Equation. Journal of Applied Mathematics and Computational. 147:2004, 307-319.

[28] Klankaew, P. and Pochai, N. “Numerical Simulation of a Nonlinear Thin Fluid Film Flow Velocity Model of a Third Grade Fluid on a Moving Belt using Finite Difference Method with Newton Iterative Scheme,” Lecture Notes in Engineering and Computer Science: Proceedings of the International MultiConference of Engineers and Computer Scientists 2019, IMECS 2019, 13-15March, 2019, Hong Kong, pp318-321.

[29] A. A. Opanuga, Member, IAENG, E.A. Owoloko, H. I. Okagbue, O.O. Agboola, “Finite Difference Method and Laplace Transform for

Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2017, WCE 2017, 5-7July, 2017, London, U.K., pp65-69.

[30] A. Cheniguel, "Numerical Method for the Heat Equation with Dirichlet and Neumann Conditions," Lecture Notes in Engineering and Computer Science: Proceedings of the International MultiConference of Engineers and Computer Scientists 2014, IMECS 2014, 12–14March, 2014, Hong Kong, pp535-539.

[31] A. Cheniguel and M. Reghioua, "On the Numerical Solution of Three-Dimensional Diffusion Equation with an Integral Condition," Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering and Computer Science 2013, WCECS 2013, 23-25October, 2013, San Francisco, USA, pp1017-1021.

[32] Svetislav SavovĆ, Alexander Djordjevich, Finite difference solution of the one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media. International Journal of Heat and Mass Transfer 55: 2012, 91-94.

[33] J.D. Anderson, Computational Fluid Dynamics, McGraw-hill, New York, 1995.

[34] A.C Hindmarsh, P.M.Gresho, D.F. Griffiths, The stability of explicit Euler time-integration for certain finite difference approximation of the advection-diffusion equation, Int. J. Numerical Methods Fluids 4 ,1984, 853-897.

[35] R.F. Warming, B.J. Hyett, The modified equation approach to the stability and accuracy analysis of finite-difference method, J.comput. Phys. 14 (2): 1974, 159-179.

[36] P.D. Lax, B. Wendroff, Difference schemes with high order of accuracy for solving hyperbolic equation, comun. Pure Appl. Math. 17:1964, 381-398.

N. Pochai is a researcher of Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand.

S. Khatbanjong is an assistant researcher of Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand.