Almost Periodicity in Shifts Delta(+/-) on Time Scales and Its Application to Hopfield Neural Networks

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***Abstract*—In this paper, we first give three definitions with the aid of the shift operators $\delta_\pm$, that is, almost periodicity in shifts $\delta_\pm$, almost periodic function in shifts $\delta_\pm$ and Δ-almost periodic function in shifts $\delta_\pm$, then by using the theory of calculus on time scales and some mathematical methods, the existence and uniqueness theorem of solution of linear dynamic system on almost periodic time scale in shifts $\delta_\pm$ is obtained. Finally, we applying the obtained results to study the existence and exponential stability of the almost periodic solution in shifts $\delta_\pm$ of a class of Hopfield neural networks with delays, several examples and numerical simulations are given to illustrate and reinforce the main results.**

***Index Terms*—Almost periodic time scale in shifts $\delta_\pm$; Exponential dichotomy; Hopfield neural networks; Almost periodic solution in shifts $\delta_\pm$; Global exponential stability.**

I. Introduction

It is well known that almost periodicity plays an important role in dynamic systems both theoretical study and practical applications, see [1-6]. In recent years, with the development of the theory of time scales (see [7,8]), the existence and stability of almost periodic solutions of dynamic equations on time scales received many researchers' special attention; see, for example, [9-12]. In these works, the almost periodic time scale $\mathbb{T}$ satisfies the condition "$t \pm s \in \mathbb{T}, \forall t \in \mathbb{T}$, s is the translation constant". Under this condition all almost periodic time scales are unbounded below and above. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : q > 1$ is constant and $n \in \mathbb{Z}\} \cup \{0\}$ which is neither closed under the operation $t \pm s$ nor unbounded below. Therefore, almost periodicity on time scales need to be explored further.

In [13-16], Adıvar et al. defined two shift operators, i.e. δ_+ (forward shift operator) and δ_- (backward shift operator), which does not oblige the time scale to be closed under the operations $t \pm s$. The applications of the shift operators $\delta_\pm$ on time scales; see, for example, [17-19]. Motivated by the above works, the main purpose of this paper is to define a new almost periodicity concept with the aid of the shift operators $\delta_\pm$. We first give three definitions, that is, almost periodicity in shifts $\delta_\pm$, almost periodic function in shifts $\delta_\pm$ and Δ-almost periodic function in shifts $\delta_\pm$. Under the definitions and the properties of shifts $\delta_\pm$, some basic results on almost periodic differential equations in shifts $\delta_\pm$ on time scales are established. Moreover, we applying the obtained results to study a class of Hopfield neural networks on time scales as follows

$$\begin{cases} y_i^{\Delta}(t) &= -a_i(t)y_i(t) + \sum\limits_{j=1}^{n} c_{ij}(t)g_j(y_j(t)) \\ & \quad + \sum\limits_{j=1}^{n} d_{ij}(t)f_j(y_j(\delta_-(\tau_{ij},t))) + I_i(t), \\ y_i(s) &= \phi_i(s), s \in [\delta_-(\hat{\tau},t_0),t_0]_{\mathbb{T}}, i \in \mathbb{N}, \end{cases} \tag{1}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an almost periodic time scale in shifts $\delta_\pm$, $\mathbb{N} = \{1,2,\ldots,n\}$, the integer n corresponds to the number of units in (1); $y_i(t)$ corresponds to the state of the ith unit at time t; $a_i(t) > 0$ represents the passive decay rate; c_{ij} and d_{ij} weight the strength of jth unit on the ith unit at time t; $I_i(t)$ is the input to the ith unit at time t from outside the networks; g_i and f_i denote activation functions of transmission; the function $\delta_-(\cdot,t)$ is a delay function generated by the backward shift operator δ_- on time scale $\mathbb{T}$, τ_{ij} corresponds to the signal transmission delay along the axon of the jth unit which is nonnegative and bounded, and $\hat{\tau} = \max\limits_{i,j\in\mathbb{N}}\{\tau_{ij}\}$.

II. Almost periodicity in shifts $\delta_\pm$

Let $\mathbb{T}$ is a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T},\mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T},\mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

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If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s)=\exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s,t\in\mathbb{T}$, with the cylinder transformation

$$\xi_h(z)=\begin{cases}\frac{Log(1+hz)}{h}, & \text{if } h\neq 0,\\ z, & \text{if } h=0.\end{cases}$$

Let $p,q:\mathbb{T}\to\mathbb{R}$ are two regressive functions, define

$$p\oplus q=p+q+\mu p\,q, \ominus p=-\frac{p}{1+\mu p}, p\ominus q=p\oplus(\ominus q).$$

Lemma 1. ([7]) *Assume that $p,q:\mathbb{T}\to\mathbb{R}$ are two regressive functions, then*
(i) $e_0(t,s)\equiv 1$ and $e_p(t,t)\equiv 1$;
(ii) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$;
(iii) $e_p(t,s)=\frac{1}{e_p(s,t)}=e_{\ominus p}(s,t)$;
(iv) $e_p(t,s)e_p(s,r)=e_p(t,r)$;
(v) $(e_{\ominus p}(t,s))^{\Delta}=(\ominus p)(t)e_{\ominus p}(t,s)$;
(vi) $\int_a^b p(t)e_p(c,\sigma(t))\Delta t=e_p(c,a)-e_p(c,b)$.

A comprehensive review on the shift operators $\delta_\pm$ on time scales can be found in [13-16].

Let $\mathbb{T}^*$ is a non-empty subset of the time scale $\mathbb{T}$ and $t_0\in\mathbb{T}^*$ is a fixed constant, define operators $\delta_\pm:[t_0,+\infty)\times\mathbb{T}^*\to\mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0\in\mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s\in[t_0,+\infty)_{\mathbb{T}}$ in $\delta_\pm(s,t)$ is called the shift size. The values $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t\in\mathbb{T}^*$ to the right and the left, respectively. The sets

$$\mathbb{D}_\pm:=\{(s,t)\in[t_0,+\infty)_{\mathbb{T}}\times\mathbb{T}^*:\delta_\pm(s,t)\in\mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm:[t_0,+\infty)\times\mathbb{T}^*\to\mathbb{T}^*$ exist.

Next, we give three definitions with the aid of the shift operators $\delta_\pm$, that is, almost periodicity in shifts $\delta_\pm$, almost periodic function in shifts $\delta_\pm$ and Δ-almost periodic function in shifts $\delta_\pm$.

Definition 1. *(Almost periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ is a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0\in\mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be almost periodic in shifts $\delta_\pm$ if there exists $p\in(t_0,+\infty)_{\mathbb{T}^*}$ such that $(p,t)\in\mathbb{D}_\pm$ for all $t\in\mathbb{T}^*$, that is,*

$$\{p\in(t_0,+\infty)_{\mathbb{T}^*}:(p,t)\in\mathbb{D}_\pm,\forall t\in\mathbb{T}^*\}\neq\emptyset.$$

Remark 1. *In Definition 1, if*

$$\omega_{\mathbb{T}}:=\inf\{p\in(t_0,+\infty)_{\mathbb{T}^*}:(p,t)\in\mathbb{D}_\pm,\forall t\in\mathbb{T}^*\}\neq t_0,$$

then $\omega_{\mathbb{T}}$ is called the period of the time scale $\mathbb{T}$.

Definition 2. *(Almost periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ is an almost periodic time scale in shifts $\delta_\pm$. A real-valued function f defined on $\mathbb{T}^*$ is almost periodic in shifts $\delta_\pm$ if the ε-translation set of f*

$$E\{\varepsilon,f\}=\{(p,t)\in\mathbb{D}_\pm:|f(\delta_\pm^p(t))-f(t)|<\varepsilon,\forall t\in\mathbb{T}^*\}$$

is a relatively dense set in $\mathbb{T}^*$ for all $\varepsilon>0$; that is, for any given $\varepsilon>0$, there exists a constant $l(\varepsilon)>t_0$, such that in any interval of length $\delta_\pm^{l(\varepsilon)}(\cdot)$, there exists at least a $p\in E\{\varepsilon,f\}$ such that

$$|f(\delta_\pm^p(t))-f(t)|<\varepsilon,\forall t\in\mathbb{T}^*,$$

where $\delta_\pm^p:=\delta_\pm(p,t)$, p is called the ε-translation constant of f, $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon,f\}$.

Remark 2. *In Definition 2, if there exists a $\omega\in[\omega_{\mathbb{T}},+\infty)_{\mathbb{T}^*}$ and $\omega\in E\{\varepsilon,f\}$ such that $f(\delta_\pm^\omega(t))=f(t)$, $\forall t\in\mathbb{T}^*$, the smallest constant ω is called the period of f.*

Remark 3. *If $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$, then the operator $\delta_\pm^p(t))=t\pm p$ associated with the initial point $t_0=0$, then Definition 2 is converted to the traditional definition of almost periodic function on $\mathbb{R}$ or $\mathbb{Z}$.*

Now, we give two examples to illustrate the Definition 2 is more generality.

Example 1. Let $\mathbb{T}=\mathbb{R}$ and $t_0=1$. The operators

$$\delta_-(p,t)=\begin{cases}\frac{t}{p}, & \text{if } t\geq 0,\\ pt, & \text{if } t<0,\end{cases}\quad \text{for } p\in[1,+\infty)$$

and

$$\delta_+(p,t)=\begin{cases}pt, & \text{if } t\geq 0,\\ \frac{t}{p}, & \text{if } t<0,\end{cases}\quad \text{for } p\in[1,+\infty)$$

are backward and forward shift operators (on the set $\mathbb{R}^*=\mathbb{R}-\{0\}$) associated with the initial point $t_0=1$.

Consider the function

$$f(t)=\cos\left(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi\right), t\in\mathbb{R}^*.$$

By direct computation,

$$\begin{aligned}f(\delta_\pm^p(t)) &= \begin{cases}f(tp^{\pm1}), & \text{if } t\geq 0,\\ f(\frac{t}{p^{\pm1}}), & \text{if } t<0,\end{cases}\\ &= \cos\left(\frac{\ln|t|\pm\ln(\frac{1}{p})}{\ln(\frac{1}{2})}\pi\right)\\ &= \cos\left(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi\pm\frac{\ln(\frac{1}{p})}{\ln(\frac{1}{2})}\pi\right).\end{aligned}$$

Let $l=p=2^{2n}, n\in\mathbb{Z}$, for any given $\varepsilon>0$, then

$$|f(\delta_\pm^p(t))-f(t)|=0<\varepsilon,\forall t\in\mathbb{R}^*.$$

Therefore, $\cos\left(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi\right)$ is an almost periodic function in shifts $\delta_\pm$. Moreover, $\cos\left(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi\right)$ is a 4-periodic function in shifts $\delta_\pm$.

Remark 4. *The function $f(t)=\cos\left(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi\right)$ is not almost periodic in the sense of the definition of almost periodic function which has been defined in [9], but is almost periodic in shifts $\delta_\pm$. See, Figure1,*

Example 2. Let $\mathbb{T}=\overline{q^{\mathbb{Z}}}$, $\overline{q^{\mathbb{Z}}}=\{q^n:n\in\mathbb{Z},q>1\}\cup\{0\}$, and $t_0=1$. The operators

$$\delta_-(p,t)=\frac{t}{p}, \text{for } p\in[1,+\infty)$$

and

$$\delta_+(p,t)=pt, \text{for } p\in[1,+\infty)$$

are backward and forward shift operators (on the set $\overline{q^{\mathbb{Z}}}^*=\overline{q^{\mathbb{Z}}}$) associated with the initial point $t_0=1$.

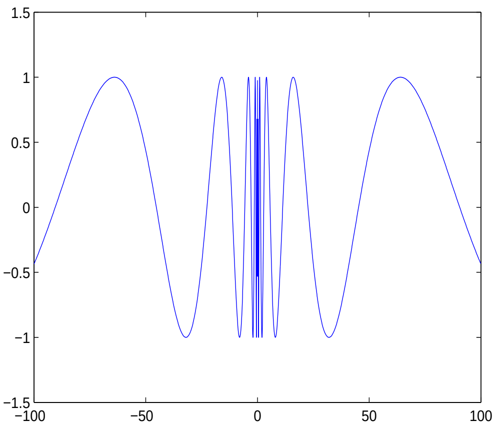

Fig. 1. Graph of $f(t) = \cos\left(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi\right)$.

Consider the function

$$f(t) = (-1)^{\frac{\ln t}{\ln q}}, t \in \overline{q^{\mathbb{Z}}}.$$

Let $l = p = q^{2n}, n \in \mathbb{Z}$, then

$$f(\delta_{\pm}^{p}(t)) = (-1)^{\frac{\ln t}{\ln q} \pm 2n} = (-1)^{\frac{\ln t}{\ln q}} = f(t),$$

and for any given $\varepsilon > 0$,

$$|f(\delta_{\pm}^{p}(t)) - f(t)| = 0 < \varepsilon, \forall t \in \overline{q^{\mathbb{Z}}}.$$

Therefore, $(-1)^{\frac{\ln t}{\ln q}}$ is an almost periodic function in shifts $\delta_{\pm}$. Moreover, $(-1)^{\frac{\ln t}{\ln q}}$ is a q^2-periodic function in shifts $\delta_{\pm}$.

Remark 5. *The function $f(t) = (-1)^{\frac{\ln t}{\ln q}}$ is not almost periodic in the sense of the definition of almost periodic function which has been defined in [9], since there is not any positive constant p such that $f(t+p) = f(t)$ holds. But $f(t)$ is almost periodic in shifts $\delta_{\pm}$. See, Figure2,*

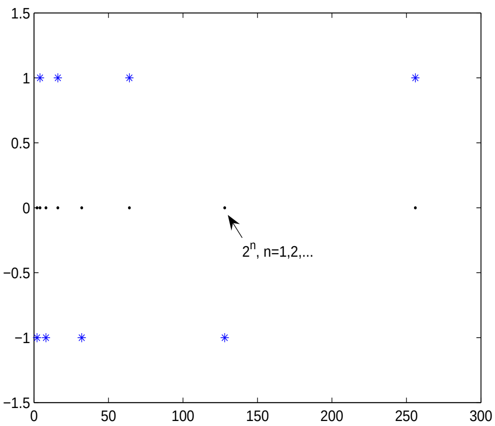

Fig. 2. Graph of $f(t) = (-1)^{\frac{\ln t}{\ln q}}$, $q = 2$.

Definition 3. *(Δ-almost periodic function in shifts $\delta_{\pm}$) Let $\mathbb{T}$ is an almost periodic time scale in shifts $\delta_{\pm}$. A real-valued function f defined on $\mathbb{T}^*$ is Δ-almost periodic in shifts $\delta_{\pm}$ if the ε-translation set of f*

$$E^{\Delta}\{\varepsilon, f\} = \{(p,t) \in \mathbb{D}_{\pm} : |f(\delta_{\pm}^{p}(t))\delta_{\pm}^{\Delta p}(t) - f(t)| < \varepsilon, \forall t \in \mathbb{T}^*\}$$

is a relatively dense set in $\mathbb{T}^*$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > t_0$, such that in any interval of length $\delta_{\pm}^{l(\varepsilon)}(\cdot)$, there exists at least a $p \in E^{\Delta}\{\varepsilon, f\}$ such that

$$|f(\delta_{\pm}^{p}(t))\delta_{\pm}^{\Delta p}(t) - f(t)| < \varepsilon, \forall t \in \mathbb{T}^*,$$

where $\delta_{\pm}^{p} := \delta_{\pm}(p,t)$, p is called the ε-translation constant of f, $l(\varepsilon)$ is called the inclusion length of $E^{\Delta}\{\varepsilon, f\}$.

Lemma 2. *Let $f \in C(\mathbb{T}, \mathbb{R})$ is an almost periodic function in shifts $\delta_{\pm}$, then f(t) is bounded on $\mathbb{T}$.*

Proof: We only consider the forward shift operator δ_{+}, the other case is similar. For given $\varepsilon \leq 1$, there exists a constant l, such that in any interval of length $\delta_{+}^{l}(\cdot)$, there exists at least a $p \in E\{\varepsilon, f\}$, such that $|f(\delta_{+}^{p}(t)) - f(t)| < \varepsilon, \forall t \in \mathbb{T}$. In addition, $f \in C(\mathbb{T}, \mathbb{R})$, then in the limited interval $[t_0, \delta_{+}^{l}(t_0)]_{\mathbb{T}}$, there exists a constant $M > 0$, such that $|f(t)| < M$. For any given $t \in \mathbb{T}$, we can take $p \in E\{\varepsilon, f\} \bigcap [\delta_{-}(t,t_0), \delta_{-}(t, t_0 + l)]_{\mathbb{T}}$, then we have $\delta_{+}^{p}(t) \in [t_0, \delta_{+}^{l}(t_0)]_{\mathbb{T}}$. Hence, we can obtain $|f(\delta_{+}^{p}(t))| < M$ and $|f(\delta_{+}^{p}(t)) - f(t)| < 1$. So for all $t \in \mathbb{T}$, we have $|f(t)| < M + 1$. This completes the proof.

Definition 4. ([9]) *Let $x \in \mathbb{R}^n$, and $A(t)$ is an $n \times n$ rd-continuous matrix function on $\mathbb{T}$, the linear system*

$$x^{\Delta}(t) = A(t)x(t), \ t \in \mathbb{T}, \tag{2}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant k, α, projection P and the fundamental solution matrix $X(t)$ of (2),satisfying

$$|X(t)PX^{-1}(\sigma(s))|_0 \leq ke_{\ominus\alpha}(t, \sigma(s)),$$
$$s,t \in \mathbb{T}, t \geq \sigma(s),$$
$$|X(t)(I-P)X^{-1}(\sigma(s))|_0 \leq ke_{\ominus\alpha}(\sigma(s), t),$$
$$s,t \in \mathbb{T}, t \leq \sigma(s),$$

where $|\cdot|_0$ is the Euclidean norm.

Definition 5. *The solution $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ of a system is said to be exponentially stable, if there exists a positive α such that for any $\xi \in [\delta_{-}(\tau_0, t_0), t_0]_{\mathbb{T}}$, $\tau_0 > t_0$, there exists $N = N(\xi) \geq 1$ such that for any solution $x = (x_1, x_2, \cdots, x_n)^T$ satisfying*

$$\|x - x^*\| \leq N\|\varphi - x^*\|e_{\ominus\alpha}(t, \xi), \ t \in [t_0, +\infty)_{\mathbb{T}},$$

where φ is the initial condition, and

$$\|\varphi - x^*\| = \max_{1 \leq i \leq n} \sup_{\xi \in [\delta_{-}(\tau_0, t_0), t_0]_{\mathbb{T}}} |\varphi_i(\xi) - x_i^*(\xi)|.$$

Lemma 3. ([20]) *If the following conditions*

(1) $D^{+}x_i^{\Delta}(t) \leq \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}\bar{x}_j(t)$, $t \in [t_0, +\infty)_{\mathbb{T}}$, $i, j = 1, 2, \cdots, n$,
where $a_{ij} \geq 0 (i \neq j)$, $b_{ij} \geq 0$, $\sum_{i=1}^{n} \bar{x}_i(t_0) > 0$, $\bar{x}_i(t) = \sup_{s \in [\delta_{-}(\tau_0, t), t]_{\mathbb{T}}} x_i(s)$, and $\tau_0 > 0$ is a constant;

(2) $\widetilde{M} := -(a_{ij} + b_{ij})_{n \times n}$ *is an M-matrix;*

hold, then there exists a constant $\gamma_i > 0$, $a > 0$, such that the solutions of inequality (1) satisfies

$$x_i(t) \leq \gamma_i \left(\sum_{j=1}^{n} \bar{x}_j(t_0) \right) e_{\ominus a}(t, t_0),$$
$$\forall \ t \in (t_0, +\infty)_{\mathbb{T}}, \ i = 1, 2, \cdots, n.$$

Consider the following almost periodic system in shifts $\delta_\pm$

$$x^\Delta(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}, \tag{3}$$

where $A(t)$ is an almost periodic matrix function in shifts $\delta_\pm$, $f(t)$ is a Δ-almost periodic vector function in shifts $\delta_\pm$, and $\delta_\pm^\Delta(\cdot,t)$ are bounded.

Let $A(t) = \big(a_{ij}(t)\big)_{n\times n}$, $A^u = \big(\sup(a_{ij}(t))\big)_{n\times n}$, $1 \le i,j \le n$, $t \in \mathbb{T}$.

Theorem 1. *If the linear system (2) admits exponential dichotomy, and $-A^u$ is an M-matrix, then system (3) has a unique almost periodic solution in shifts $\delta_\pm$, which is globally exponentially stable. And the solution*

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} e_A(t)Pe_A^{-1}(\sigma(s))f(s)\Delta s \\ &\quad -\int_{t}^{+\infty} e_A(t)(I-P)e_A^{-1}(\sigma(s))f(s)\Delta s, \end{aligned} \tag{4}$$

where $e_A(t)$ is the fundamental solution matrix of (2).

Proof: First, we prove that $x(t)$ is a bounded almost periodic solution in shifts $\delta_\pm$ of system (3). In fact,

$$\begin{aligned} & x^\Delta(t) - A(t)x(t) \\ =\ & e_A^\Delta(t)\int_{-\infty}^{t} Pe_A^{-1}(\sigma(s))f(s)\Delta s \\ & +e_A(\sigma(t))Pe_A^{-1}(\sigma(t))f(t) \\ & -e_A^\Delta(t)\int_{t}^{+\infty}(I-P)e_A^{-1}(\sigma(s))f(s)\Delta s \\ & +e_A(\sigma(t))(I-P)e_A^{-1}(\sigma(t))f(t) \\ & -A(t)e_A(t)\int_{-\infty}^{t} Pe_A^{-1}(\sigma(s))f(s)\Delta s \\ & +A(t)e_A(t)\int_{t}^{+\infty}(I-P)e_A^{-1}(\sigma(s))f(s)\Delta s \\ =\ & e_A(\sigma(t))(P+I-P)e_A^{-1}(\sigma(t))f(t) \\ =\ & f(t). \end{aligned}$$

By Lemma 1, Lemma 2 and Definition 4,

$$\begin{aligned} |x|_0 &= \bigg|\int_{-\infty}^{t} e_A(t)Pe_A^{-1}(\sigma(s))f(s)\Delta s \\ &\quad -\int_{t}^{+\infty} e_A(t)(I-P)e_A^{-1}(\sigma(s))f(s)\Delta s\bigg|_0 \\ &\le \bigg(\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s \\ &\quad +\int_{t}^{+\infty} e_{\ominus\alpha}(\sigma(s),t)\Delta s\bigg)k|f|_0 \\ &\le \left(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}\right)k|f|_0, \end{aligned}$$

that is, $x(t)$ is a bounded solution of system (3).

Furthermore,

$$\begin{aligned} & \int_{-\infty}^{\delta_\pm^p(t)} X(\delta_\pm^p(t))PX^{-1}(\sigma(s))f(s)\Delta s \\ =\ & \int_{-\infty}^{t} X(\delta_\pm^p(t))PX^{-1}(\sigma(\delta_\pm^p(s)))f(\delta_\pm^p(s))\delta_\pm^{\Delta p}(s)\Delta s; \\ & \int_{\delta_\pm^p(t)}^{+\infty} X(\delta_\pm^p(t))(I-P)X^{-1}(\sigma(s))f(s)\Delta s \\ =\ & \int_{t}^{+\infty} X(\delta_\pm^p(t))(I-p)X^{-1}(\sigma(\delta_\pm^p(s))) \\ & \times f(\delta_\pm^p(s))\delta_\pm^{\Delta p}(s)\Delta s, \end{aligned}$$

and $f(t)$ is a Δ-almost periodic function in shifts $\delta_\pm$, then we can check that $x(t)$ is an almost periodic solution in shifts $\delta_\pm$ of system (3).

Next, we show that the solution $x(t)$ is globally exponentially stable and uniqueness.

Let $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ is a solution of (3), and $x = (x_1, x_2, \cdots, x_n)^T$ is an arbitrary solution of (3). From system (3), we have

$$(x(t) - x^*(t))^\Delta = A(t)x(t) - A(t)x^*(t). \tag{5}$$

Assume that the initial condition of (3) is

$$\phi(s) = (\phi_1(s), \cdots, \phi_n(s))^T, s \in [\delta_-(\tau_0,t_0), t_0]_\mathbb{T}, \tau_0 > 0,$$

then the initial condition of (5) is

$$\hat\phi(s) = \phi(s) - x^*, s \in [\delta_-(\tau_0,t_0), t_0]_\mathbb{T}.$$

Let $V(t) = |x(t) - x^*(t)|$, the upper right derivative $D^+V^\Delta(t)$ along the solutions of system (5) is as follows

$$\begin{aligned} D^+V^\Delta(t) &= \operatorname{sign}(x(t)-x^*(t))(x(t)-x^*(t))^\Delta \\ &\le A^uV(t) + O\overline{V}(t), \end{aligned}$$

where O is an $n\times n$-matrix with all its elements are zeros, "$\le$" denotes the relationship between the components of vectors of the two sides, respectively.

Since $-(A+O) = -A$ is an M-matrix, according to Lemma 3, then there exist constants $\alpha > 0$, $\gamma_0 > 0$, for any $\xi \in [\delta_-(\tau_0,t_0), t_0]_\mathbb{T}$,

$$\begin{aligned} & |x_i(t) - x_i^*(t)| \\ \le\ & \gamma_0\Big[\sup_{\xi\in[\delta_-(\tau_0,t_0),t_0]_\mathbb{T}} |\phi_i(\xi) - x_i^*(\xi)|\Big]e_{\ominus\alpha}(t,t_0) \\ \le\ & \frac{\gamma_0}{e_{\ominus\alpha}(t_0,\xi)}\Big[\sup_{\xi\in[\delta_-(\tau_0,t_0),t_0]_\mathbb{T}} |\phi_i(\xi) - x_i^*(\xi)|\Big]e_{\ominus\alpha}(t,\xi), \\ & t \in [t_0, +\infty)_\mathbb{T}. \end{aligned}$$

Then, there exists a positive constant $\eta > \frac{e_{\ominus\alpha}(t_0,\xi)}{\gamma_0}$, such that

$$\|x - x^*\| \le N\|\phi - x^*\|e_{\ominus\alpha}(t,\xi),\ t \in [t_0, +\infty)_\mathbb{T}, \tag{6}$$

where $N = N(\xi) = \frac{\eta\gamma_0}{e_{\ominus\alpha}(t_0,\xi)} > 1$, and $\|x\| = \max\limits_{1\le i\le n}\sup\limits_{t\in[t_0,+\infty)_\mathbb{T}} |x_i(t)|$.

By Definition 5, the solution $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ is globally exponentially stable, that is, the solution of system (3) is globally exponentially stable.

In inequality (6), let $t \to +\infty$, then $e_{\ominus\alpha}(t,\xi) \to 0$, so we can get $x = x^*$. Hence, system (3) has a unique solution. This completes the proof.

Lemma 4. ([9]) *Let $c_i(t)$ is an almost periodic function in shifts $\delta_\pm$ on $\mathbb{T}$, where $c_i(t) > 0$, $-c_i(t) \in \mathcal{R}^+$, $\forall\ t \in \mathbb{T}$ and*

$$\min_{1\le i\le n}\big\{\inf_{t\in\mathbb{T}} c_i(t)\big\} = \widetilde{m} > 0,$$

then the linear system

$$x^\Delta(t) = \operatorname{diag}\big(-c_1(t), -c_2(t), \cdots, -c_n(t)\big)x(t) \tag{7}$$

admits an exponential dichotomy on $\mathbb{T}$.

By Lemma 4 and Theorem 1, we can obtain the following result.

Corollary 1. *In system (3), if*

$$A(t)=\mathrm{diag}(-a_1(t),-a_2(t),\cdots,-a_n(t)),$$

and $\min\limits_{1\le i\le n}\big\{\inf\limits_{t\in\mathbb{T}} a_i(t)\big\}=\hat{a}>0$*, then the system (3) has a unique almost periodic solution in shifts* $\delta_\pm$*, which is defined in (4).*

III. An application

In this section, by using the definitions and the results developed in Section 2, we shall study the existence and exponential stability of almost periodic solution in shifts $\delta_\pm$ of system (1).

Firstly, we make the following assumptions

(H_1) $a_i(t), c_{ij}(t), d_{ij}(t), I_i(t)$ are almost periodic functions in shifts $\delta_\pm$ defined on $\mathbb{T}$, $i,j\in\mathbb{N}$.

(H_2) $f_j, g_j\in C(\mathbb{R},\mathbb{R})$ and satisfy $f_j(0)=0,\ g_j(0)=0$, respectively. Moreover, there exists positive constants L_j^f, L_j^g such that $|f_j(x)-f_j(y)|\le L_j^f|x-y|,\ |g_j(x)-g_j(y)|\le L_j^g|x-y|$, $j\in\mathbb{N}$.

(H_3) $\lambda=\min\limits_{i\in\mathbb{N}}\big\{\inf\limits_{t\in\mathbb{T}} a_i(t)\big\}>0$, and $1-\mu(t)a_i(t)>0,\ \forall\ t\in\mathbb{T},\ i\in\mathbb{N}$.

(H_4) $\delta_\pm^\Delta(\cdot,t)$ are bounded functions on $\mathbb{T}$.

By Lemma 2, we know that all almost periodic functions in shifts $\delta_\pm$ are bounded. For convenience, we denote $\overline{a}=\sup\limits_{t\in\mathbb{T}}|a(t)|,\ \underline{a}=\inf\limits_{t\in\mathbb{T}}|a(t)|$ for any $a(t)\in APS(\mathbb{T})$, where $APS(\mathbb{T})$ is a set of all almost periodic functions in shifts $\delta_\pm$ on the time scale $\mathbb{T}$.

Theorem 2. *Assume that* $(H_1)-(H_4)$ *hold, and*

$$w_1=\max_{i\in\mathbb{N}}\left\{\frac{1}{\underline{a}_i}\left[\sum_{j=1}^n \bar{d}_{ij}L_j^f+\sum_{j=1}^n \bar{c}_{ij}L_j^g\right]\right\}<1,$$

then system (1) has a unique almost periodic solution in shifts $\delta_\pm$ *in* $\|z-z_0\|\le\frac{w_1w_2}{1-w_1}$*, where*

$$\begin{aligned}z_0 &= \Big\{\int_{-\infty}^t e_{-a_1}(t,\sigma(s))I_1(s)\Delta s,\cdots,\\ &\quad \int_{-\infty}^t e_{-a_n}(t,\sigma(s))I_n(s)\Delta s\Big\},\end{aligned}$$

and $w_2=\max\limits_{i\in\mathbb{N}}\left\{\frac{\overline{I}_i}{\underline{a}_i}\right\}$.

Proof: Let $\mathbb{B}=\big\{z|z=\big(\psi_1,\psi_2,\cdots,\psi_n\big)^T\big\}$, where z is a continuous almost periodic function in shifts $\delta_\pm$ on time scale $\mathbb{T}$ with the norm

$$\|z\|=\max_{i\in\mathbb{N}}\big\{\sup_{t\in\mathbb{T}}|\psi_i(t)|\big\},$$

then $\mathbb{B}$ is a Banach space.

For any $z\in\mathbb{B}$, consider the solution $y_z(t)$ of the nonlinear almost periodic dynamic system in shifts $\delta_\pm$

$$\begin{aligned}y_i^\Delta(t) &= -a_i(t)y_i(t)+\sum_{j=1}^n d_{ij}(t)f_j(\psi_j(\delta_-(\tau_{ij},t)))\\ &\quad +\sum_{j=1}^n c_{ij}(t)g_j(\psi_j(t))+I_i(t), i\in\mathbb{N}.\end{aligned}\tag{8}$$

Since $\min\limits_{i\in\mathbb{N}}\big\{\inf a_i(t)\big\}>0$, according to Corollary 1, the uniqueness solution of system (8) can be expressed as the following form

$$\begin{aligned}&y_z(t)=\\ &\Big\{\int_{-\infty}^t e_{-a_1}(t,\sigma(s))\Big[\sum_{j=1}^n d_{1j}(s)f_j(\psi_j(\delta_-(\tau_{1j},s)))\\ &+\sum_{j=1}^n c_{1j}(s)g_j(\psi_j(s))+I_1(s)\Big]\Delta s,\cdots,\\ &\int_{-\infty}^t e_{-a_n}(t,\sigma(s))\Big[\sum_{j=1}^n d_{nj}(s)f_j(\psi_j(\delta_-(\tau_{nj},s)))\\ &+\sum_{j=1}^n c_{nj}(s)g_j(\psi_j(s))+I_n(s)\Big]\Delta s\Big\}.\end{aligned}\tag{9}$$

Define a mapping $\Phi:\mathbb{B}\to\mathbb{B}$, and

$$\Phi(z)(t)=y_z(t),\ \forall\ z\in\mathbb{B}.$$

Set

$$\mathbb{B}^*=\Big\{z|z\in\mathbb{B},\|z-z_0\|\le\frac{w_1w_2}{1-w_1}\Big\},$$

then $\mathbb{B}^*$ is a closed convex subset of $\mathbb{B}$. According to the definition of the norm on $\mathbb{B}$, we have

$$\begin{aligned}\|z_0\| &= \max_{i\in\mathbb{N}}\Big\{\sup_{t\in\mathbb{T}}\Big|\int_{-\infty}^t e_{-a_i}(t,\sigma(s))I_i(s)\Delta s\Big|\Big\}\\ &\le \max_{i\in\mathbb{N}}\left\{\frac{\overline{I}_i}{\underline{a}_i}\right\}=w_2.\end{aligned}$$

Therefore,

$$\|z\|\le\|z-z_0\|+\|z_0\|=\frac{w_2}{1-w_1}.$$

First, we show that the mapping Φ is a self-mapping from $\mathbb{B}^*$ to $\mathbb{B}^*$. In fact, for any $z\in\mathbb{B}^*$, we have

$$\begin{aligned}&\|\Phi(z)-z_0\|\\ =\ &\max_{i\in\mathbb{N}}\sup_{t\in\mathbb{T}}\Big\{\Big|\int_{-\infty}^t e_{-a_i}(t,\sigma(s))\\ &\times\Big[\sum_{j=1}^n d_{ij}(s)f_j(\psi_j(\delta_-(\tau_{ij},s)))\\ &+\sum_{j=1}^n c_{ij}(s)g_j(\psi_j(s))\Big]\Delta s\Big|\Big\}\\ \le\ &\max_{i\in\mathbb{N}}\sup_{t\in\mathbb{T}}\Big\{\int_{-\infty}^t e_{-a_i}(t,\sigma(s))\\ &\times\Big[\sum_{j=1}^n \bar{d}_{ij}L_j^f|\psi_j(\delta_-(\tau_{ij},s))|\\ &+\sum_{j=1}^n \bar{c}_{ij}L_j^g|\psi_j(s)|\Big]\Delta s\Big\}\\ \le\ &\max_{i\in\mathbb{N}}\sup_{t\in\mathbb{T}}\Big\{\int_{-\infty}^t e_{-a_i}(t,\sigma(s))\\ &\times\Big[\sum_{j=1}^n \bar{d}_{ij}L_j^f+\sum_{j=1}^n \bar{c}_{ij}L_j^g\Big]\Delta s\Big\}\|z\|\\ \le\ &\max_{i\in\mathbb{N}}\left\{\frac{1}{\underline{a}_i}\left[\sum_{j=1}^n \bar{d}_{ij}L_j^f+\sum_{j=1}^n \bar{c}_{ij}L_j^g\right]\right\}\|z\|\\ =\ &w_1\|z\|\le\frac{w_1w_2}{1-w_1},\end{aligned}$$

which implies that $\Phi(z)(t) \in \mathbb{B}^*$. Therefore, the mapping Φ is a self-mapping from $\mathbb{B}^*$ to $\mathbb{B}^*$.

Next, we show that the mapping Φ is a contraction mapping of $\mathbb{B}^*$. In fact, in view of $(H_1)-(H_4)$, for any $z, \tilde{z} \in \mathbb{B}$,

$$z = \big(\psi_1, \psi_2, \cdots, \psi_n\big)^T, \quad \tilde{z} = \big(\tilde{\psi}_1, \tilde{\psi}_2, \cdots, \tilde{\psi}_n\big)^T,$$

we have

$$\begin{aligned}
& \|\Phi(z) - \Phi(\tilde{z})\| \\
= & \max_{i\in\mathbb{N}} \sup_{t\in\mathbb{T}} \Bigg\{ \Bigg| \int_{-\infty}^{t} e_{-a_i}(t, \sigma(s)) \\
& \times \Bigg[\sum_{j=1}^{n} d_{ij}(s)[f_j(\psi_j(\delta_-(\tau_{ij}, s))) \\
& -f_j(\tilde{\psi}_j(\delta_-(\tau_{ij}, s)))] \\
& + \sum_{j=1}^{n} c_{ij}(s)[g_j(\psi_j(s)) - g_j(\tilde{\psi}_j(s))] \Bigg] \Delta s \Bigg| \Bigg\} \\
\leq & \max_{i\in\mathbb{N}} \sup_{t\in\mathbb{T}} \Bigg\{ \int_{-\infty}^{t} e_{-a_i}(t, \sigma(s)) \\
& \times \Bigg[\sum_{j=1}^{n} |d_{ij}(s)||f_j(\psi_j(\delta_-(\tau_{ij}, s))) \\
& -f_j(\tilde{\psi}_j(\delta_-(\tau_{ij}, s)))| \\
& + \sum_{j=1}^{n} |c_{ij}(s)||g_j(\psi_j(s)) - g_j(\tilde{\psi}_j(s))| \Bigg] \Delta s \Bigg\} \\
\leq & \max_{i\in\mathbb{N}} \sup_{t\in\mathbb{T}} \Bigg\{ \int_{-\infty}^{t} e_{-a_i}(t, \sigma(s)) \\
& \times \Bigg[\sum_{j=1}^{n} \bar{d}_{ij} L_j^f |\psi_j(\delta_-(\tau_{ij}, s)) - \tilde{\psi}_j(\delta_-(\tau_{ij}, s))| \\
& + \sum_{j=1}^{n} \bar{c}_{ij} L_j^g |\psi_j(s) - \tilde{\psi}_j(s)| \Bigg] \Delta s \Bigg\} \\
\leq & \max_{i\in\mathbb{N}} \sup_{t\in\mathbb{T}} \Bigg\{ \int_{-\infty}^{t} e_{-a_i}(t, \sigma(s)) \\
& \times \Bigg[\sum_{j=1}^{n} \bar{d}_{ij} L_j^f + \sum_{j=1}^{n} \bar{c}_{ij} L_j^g \Bigg] \Delta s \Bigg\} \|z - \tilde{z}\| \\
\leq & \max_{i\in\mathbb{N}} \Bigg\{ \frac{1}{\underline{a}_i} \Bigg[\sum_{j=1}^{n} \bar{d}_{ij} L_j^f + \sum_{j=1}^{n} \bar{c}_{ij} L_j^g \Bigg] \Bigg\} \|z - \tilde{z}\| \\
= & w_1 \|z - \tilde{z}\|.
\end{aligned}$$

This implies that the mapping Φ is a contraction mapping. Hence, Φ has exactly one fixed point z^* in $\mathbb{B}^*$ such that $\Phi(z^*) = z^*$, that is, system (1) has a unique almost periodic solution in shifts $\delta_\pm$ in $\mathbb{B}^*$. This completes the proof.

Theorem 3. *Assume that $(H_1)-(H_4)$ and the conditions of Theorem 2 hold. Furthermore, if $A-(CL^g+DL^f)$ is an M-matrix, where $A = \operatorname{diag}(\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n)$, $C = (\bar{c}_{ij})_{n\times n}, D = (\bar{d}_{ij})_{n\times n}, L^g = \operatorname{diag}(L_1^g, L_2^g, \cdots, L_n^g), L^f = \operatorname{diag}(L_1^f, L_2^f, \cdots, L_n^f)$, then the almost periodic solution in shifts $\delta_\pm$ of system (1) is globally exponentially stable.*

Proof: According to Theorem 2, system (1) has an almost periodic solution in shifts $\delta_\pm$, denote it as $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$. Assume that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is an arbitrary solution of (1).

Let $u(t) = x(t) - x^*(t)$, then for $i \in \mathbb{N}$, the system (1) can be written as

$$\begin{aligned}
u_i^\Delta(t) &= -a_i(t)u_i(t) + \sum_{j=1}^{n} c_{ij}(t)p_j(u_j(t)) \\
&\quad + \sum_{j=1}^{n} d_{ij}(t)q_j(u_j(\delta_-(\tau_{ij}, t))), \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
& p_j(u_j(t)) = g_j(x_j(t)) - g_j(x_j^*(t)), q_j(u_j(\delta_-(\tau_{ij}, t))) \\
& = f_j(x_j(\delta_-(\tau_{ij}, t))) - f_j(x_j^*(\delta_-(\tau_{ij}, t))).
\end{aligned}$$

The initial condition of system (10) is $\Psi(s) = \psi(s) - x^*, \ s \in [\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}$.

From (H_2), we have

$$|p_j(u_j)| \leq L_j^g |u_j|, |q_j(u_j)| \leq L_j^f |u_j|, \ j \in \mathbb{N}.$$

Let $V_i(t) = |u_i(t)|$, calculating the upper right derivative $D^+V^\Delta(t)$ along the solutions of system (10),

$$\begin{aligned}
& D^+V_i^\Delta(t) \\
= & \operatorname{sign}(u_i(t))u_i^\Delta(t) \\
\leq & -\underline{a}_i|u_i(t)| + \sum_{j=1}^{n} \bar{c}_{ij}L_j^g|u_j(t)| + \sum_{j=1}^{n} \bar{d}_{ij}L_j^f|\bar{u}_j(t)| \\
\leq & -\underline{a}_iV_i(t) + \sum_{j=1}^{n} \bar{c}_{ij}L_j^gV_j(t) + \sum_{j=1}^{n} \bar{d}_{ij}L_j^f\overline{V}_j(t),
\end{aligned}$$

that is

$$D^+V^\Delta(t) \leq (-A + CL^g)V(t) + DL^f\overline{V}(t), \ t \in \mathbb{T},$$

where "$\leq$" denotes the relationship between the components of vectors of the two sides, respectively.

Since $A - (CL^g + DL^f)$ is an M-matrix, according to Lemma 3, there exist constants $\mu > 0, \ r > 0$, such that

$$\begin{aligned}
V_i(t) &= |u_i(t)| \\
&\leq r \sup_{\zeta\in[\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}} |\psi_i(\zeta) - x^*(\zeta)|e_{\ominus\mu}(t, t_0), \ i \in \mathbb{N},
\end{aligned}$$

that is

$$\begin{aligned}
& |x_i(t) - x_i^*(t)| \\
\leq & r \sup_{\zeta\in[\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}} |\psi_i(\zeta) - x^*(\zeta)|e_{\ominus\mu}(t, t_0) \\
\leq & \frac{r}{e_{\ominus\mu}(t_0, \zeta)}\|\psi - x^*\|e_{\ominus\mu}(t, \zeta), \ i \in \mathbb{N}.
\end{aligned}$$

Let $N = N(\zeta) = \frac{r}{e_{\ominus\mu}(t_0, \zeta)}$, then

$$\|x - x^*\| \leq N\|\psi - x^*\|e_{\ominus\mu}(t, \zeta), \ t \in \mathbb{T}.$$

From Definition 5, the solution $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ of system (1) is globally exponentially stable. This completes the proof.

IV. Numerical simulations

Consider the following system

$$\begin{cases}
y_i^\Delta(t) = -a_i(t)y_i(t) + \sum\limits_{j=1}^{2} c_{ij}(t)g_j(y_j(t)) \\
\qquad + \sum\limits_{j=1}^{2} d_{ij}(t)f_j(y_j(\delta_-(\tau_{ij}, t))) + I_i(t), \\
y_i(s) = \phi_i(s), s \in [\delta_-(\hat{\tau}, t_0), t_0]_{\mathbb{T}}, i = 1, 2.
\end{cases} \qquad (11)$$

The first two cases. Let $\mathbb{T} = \mathbb{R}$ and $t_0 = 1$, the operators $\delta_{\pm}(p,t) = \delta_{\pm}^{p}(t)$, which have been defined in Example 1.

Case I. Almost periodic solution in shifts $\delta_{\pm}$. Let

$$\begin{aligned}
&\mathrm{diag}(a_1(t), a_2(t)) \\
&= \begin{bmatrix} 2+\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) & 0 \\ 0 & 2-\cos(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \end{bmatrix}, \\
&(c_{ij}(t))_{2\times 2} \\
&= \begin{bmatrix} 0 & 0.5\sin\sqrt{2}(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \\ 0.3\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) & 0 \end{bmatrix}, \\
&(d_{ij}(t))_{2\times 2} \\
&= \begin{bmatrix} 0.4\sin 2(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) & 0 \\ 0 & 0.5\sin\sqrt{2}(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \end{bmatrix}, \\
&(I_i(t))_{2\times 1} = \begin{bmatrix} 3\sin\sqrt{5}(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \\ 3\cos 2(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \end{bmatrix}, \\
&f_j(y_j(\delta_-(\tau_{ij},t))) = \tanh(y_i(t)), \\
&g_j(y_j(t)) = \frac{1}{2}(|y_j(t)+1| - |y_j(t)-1|).
\end{aligned}$$

By direct computation, we can obtain

$$L_j^f = L_j^g = 1, i,j \in \mathbb{N}, \lambda = 1 > 0, Q = 0.9 < 1,$$

and

$$A - (CL^g + DL^f) = \begin{bmatrix} 0.6 & -0.5 \\ -0.3 & 0.5 \end{bmatrix}$$

is an M-matrix.

According to Theorem 2 and Theorem 3, we can conclude that (11) has an almost periodic solution in shifts $\delta_{\pm}$, and the solution is exponential stability. See Figure 3.

Case II. Periodic solution in shifts $\delta_{\pm}$. Let

$$\begin{aligned}
&\mathrm{diag}(a_1(t), a_2(t)) \\
&= \begin{bmatrix} 2+\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) & 0 \\ 0 & 2-\cos(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \end{bmatrix}, \\
&(c_{ij}(t))_{2\times 2} = \begin{bmatrix} 0 & 0.5\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \\ 0.3\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) & 0 \end{bmatrix}, \\
&(d_{ij}(t))_{2\times 2} = \begin{bmatrix} 0.4\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) & 0 \\ 0 & 0.5\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \end{bmatrix}, \\
&(I_i(t))_{2\times 1} = \begin{bmatrix} 3\sin(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \\ 3\cos(\frac{\ln|t|}{\ln(\frac{1}{2})}\pi) \end{bmatrix}, \\
&f_j(y_j(\delta_-(\tau_{ij},t))) = \tanh(y_i(t)), \\
&g_j(y_j(t)) = \frac{1}{2}(|y_j(t)+1| - |y_j(t)-1|).
\end{aligned}$$

Then system (11) is a 4-periodic system in shifts $\delta_{\pm}$. Similarly to the computation of Case I, according to Theorem 2 and Theorem 3, we can conclude that (11) has a 4-periodic solution in shifts $\delta_{\pm}$, and the solution is exponential stability. See Figure 4.

The following two cases. Let $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, the operators $\delta_{\pm}(p,t) = t - p$, then system (11) is deduced into the traditional almost periodic system.

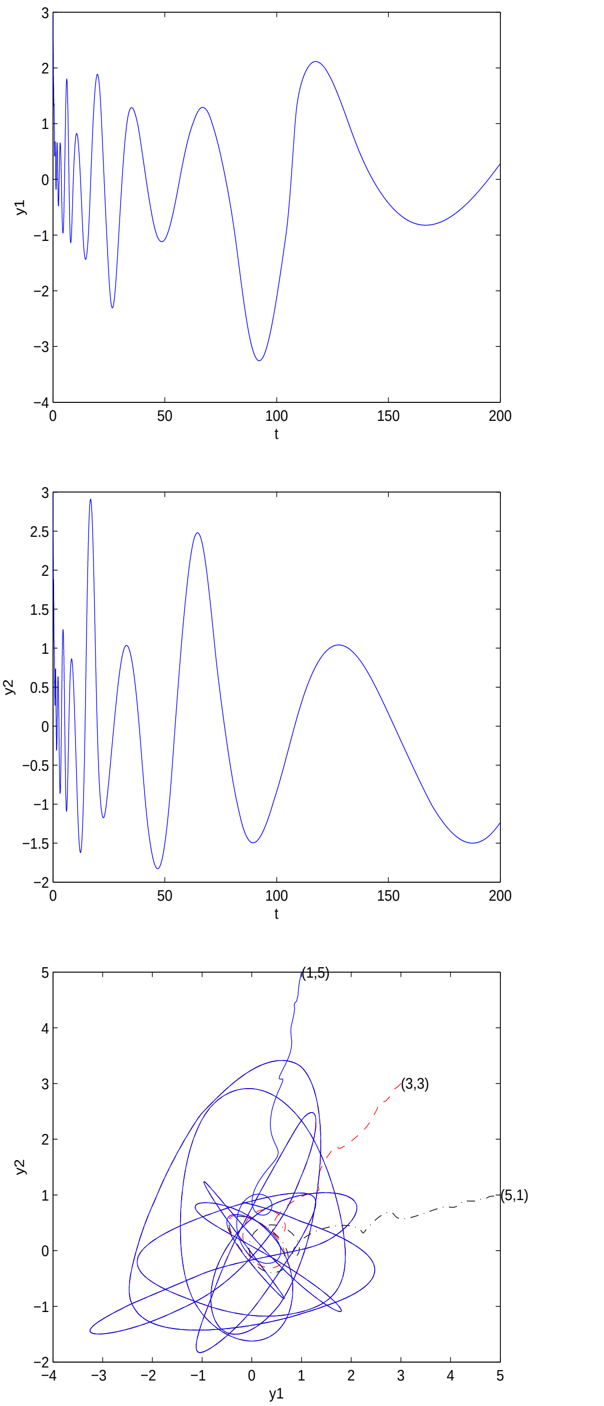

Fig. 3. Dynamic behaviors of system (11), the parameters are given in Case I.

Case III. Almost periodic solution. Let

$$\begin{aligned}
&\mathrm{diag}(a_1(t), a_2(t)) = \begin{bmatrix} 2+\sin t & 0 \\ 0 & 2-\cos t \end{bmatrix}, \\
&(c_{ij}(t))_{2\times 2} = \begin{bmatrix} 0 & 0.5\sin\sqrt{2}t \\ 0.3\sin t & 0 \end{bmatrix}, \\
&(d_{ij}(t))_{2\times 2} = \begin{bmatrix} 0.4\sin 2t & 0 \\ 0 & 0.5\sin\sqrt{2}t \end{bmatrix}, \\
&(I_i(t))_{2\times 1} = \begin{bmatrix} 3\sin\sqrt{5}t \\ 3\cos 2t \end{bmatrix}, \\
&f_j(y_j(\delta_-(\tau_{ij},t))) = \tanh(y_i(t)), \\
&g_j(y_j(t)) = \frac{1}{2}(|y_j(t)+1| - |y_j(t)-1|).
\end{aligned}$$

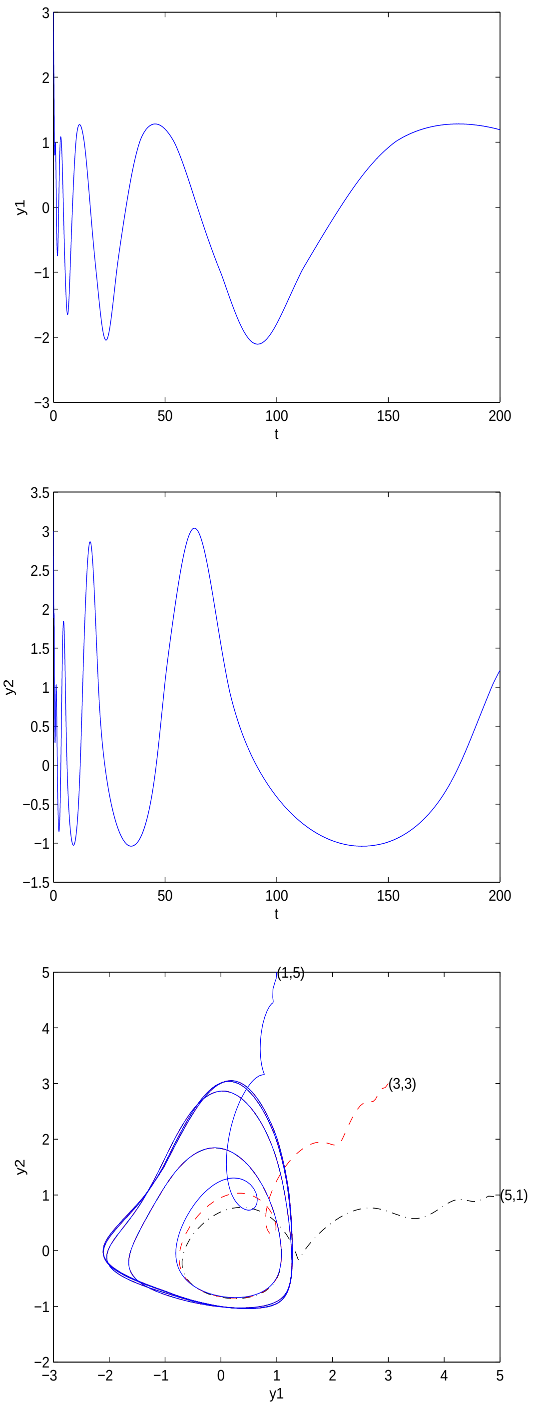

Fig. 4. Dynamic behaviors of system (11), the parameters are given in Case II.

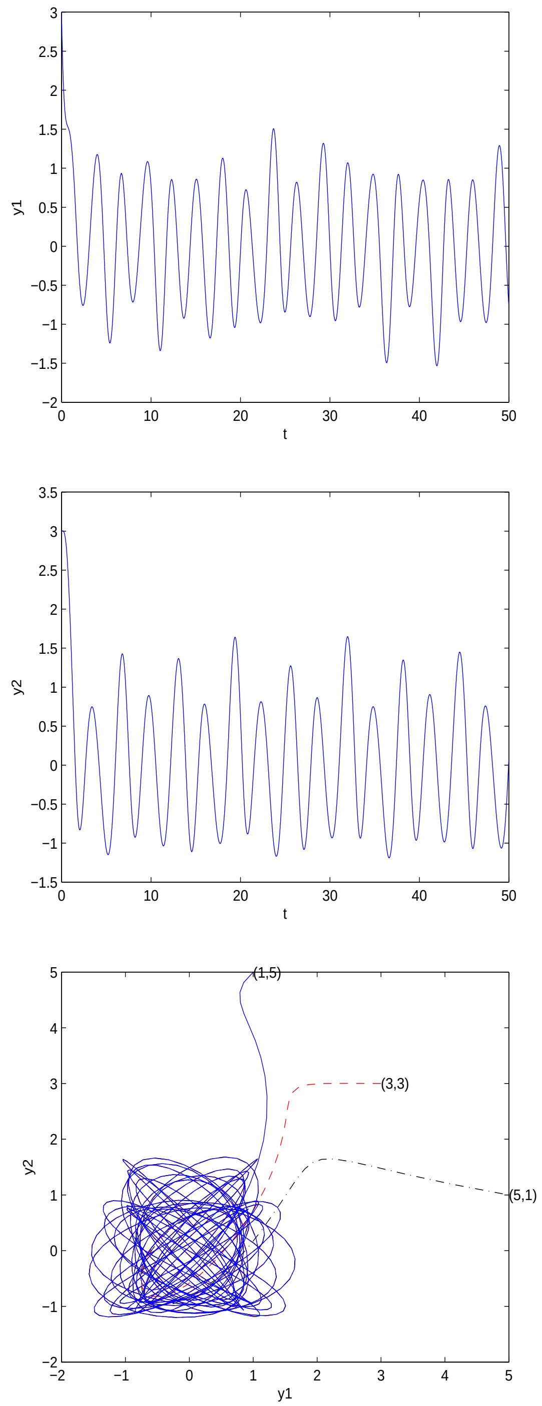

Fig. 5. Dynamic behaviors of system (11), the parameters are given in Case III.

Similarly to the computation of Case I, according to Theorem 2 and Theorem 3, we can conclude that (11) has an almost periodic solution, and the solution is exponential stability. See Figure 5.

Case IV. Periodic solution. Let

$$\begin{aligned}
&\mathrm{diag}(a_1(t), a_2(t)) = \begin{bmatrix} 2+\sin t & 0 \\ 0 & 2-\cos t \end{bmatrix}, \\
&(c_{ij}(t))_{2\times 2} = \begin{bmatrix} 0 & 0.5\sin t \\ 0.3\sin t & 0 \end{bmatrix}, \\
&(d_{ij}(t))_{2\times 2} = \begin{bmatrix} 0.4\sin t & 0 \\ 0 & 0.5\sin t \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&(I_i(t))_{2\times 1} = \begin{bmatrix} 3\sin t \\ 3\cos t \end{bmatrix}, \\
&f_j(y_j(\delta_-(\tau_{ij}, t))) = \tanh(y_i(t)), \\
&g_j(y_j(t)) = \frac{1}{2}(|y_j(t)+1| - |y_j(t)-1|).
\end{aligned}$$

Similarly to the computation of Case I, according to Theorem 2 and Theorem 3, we can conclude that (11) has a periodic solution, and the solution is exponential stability. See Figure 6.

V. Conclusion

From the examples and numerical simulations in Section IV, the coefficients of the first two cases are almost periodic

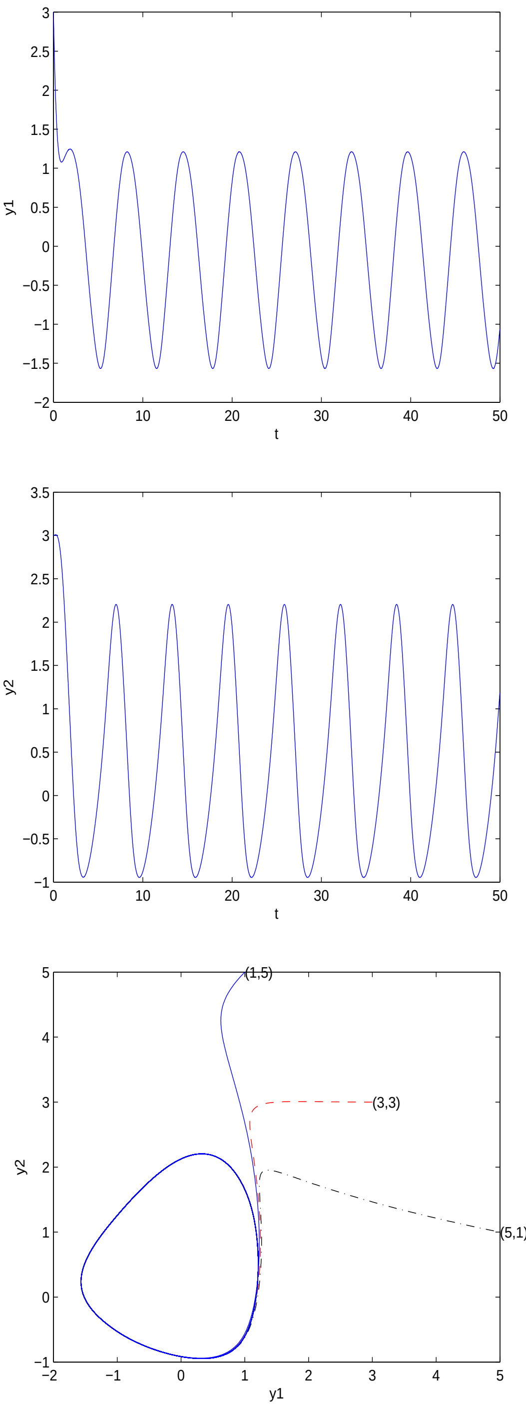

Fig. 6. Dynamic behaviors of system (11), the parameters are given in Case IV.

functions in shifts $\delta_{\pm}$, but not almost periodic functions; and if we choose suitable shift operators, almost periodic function in shifts $\delta_{\pm}$ will be deduced into almost periodic function, see cases III and IV; that is, almost periodic function which has been defined in [9] is a special case of almost periodic function in shifts $\delta_{\pm}$. Moreover, we can study the existence of almost periodic solution in shifts $\delta_{\pm}$ on more general time scales, even the time scale is not satisfy additivity or the time scale is bounded. Therefore, the obtained results in this paper improve and supplement that of the previous studies.

We would like to point out here that the obtained results in this paper can be used to study many other dynamic systems; see [21-24]. We leave this for future work.

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