Dynamics of an Impulsive Delay SI Model in Almost Periodic Environment

Yongzhi Liao and Qilin Tang

Abstract—**In this paper, we mainly study almost periodic solution of an impulsive delay SI model with variable coefficients. First we discuss the uniform persistence and globally asymptotically stable properties of the model. After that, by constructing a suitable Lyapunov functional and obtaining some sufficient conditions, we accomplish the research of the existence of a unique almost periodic solution of the system. The practicability of the main results is verified by a living example.**

Index Terms—**Almost periodic solution, impulse, time delay, permanence.**

I. Introduction

According to reports of the food and agriculture organization, the warfare between men and pests has sustained for thousands of years. With the development of society and progress of science and technology, humans have adopted some advanced and modern weapons, such as chemical pesticides, biological pesticides, remote sensing and measure, etc., where some brilliant achievements have been obtained. However, the warfare is not the end, and will still go on. In a nutshell, pesticides are useful because they can significantly reduce the population of pest and sometimes, they will also provide the only feasible method for preventing huge economic loss. However, pesticide pollution is also considered to be a major health hazard to humans and beneficial insects. Overuse of pesticides may make pest management more difficult [1].

There are two disjoint categories: susceptible individuals and infected individuals in the SI model, whose numbers are denoted by $S(t)$ and $I(t)$ at time t, respectively. Bacterial infections tend to be of SIS type, while viral infections are correspond to SIR diseases. Logistic model is crucial to researching ecological problems and has been widely explored in mathematics and biology, and lots of excellent results are obtained. In Gao and hethcote [2], the SIS and SIR models of population logistic growth and disease-related death were studied.

Up to now, there are many literatures on the applications of microbial disease to suppress pests [3-5], and many good articles [6-13] devote to disease transmission, but few articles focused on the dynamical behaviours of disease in pest control [1, 14-21]. Specially, in SI model, Chen studied many SI models with his co-workers, these SI models mainly were investigated the pest management; the permanence and global asymptotically of system, uniformly ultimately bounded and the pest-extinction periodic solution, etc.. they have obtained abundant excellent achievements. However, there has hardly any article about the existence, uniqueness and stability of almost periodic solution [22-24] for a SI model with variable coefficients.

In [17], Jiao, Chen and Luo considered the following SI model for pest management, concerned about releasing infective pests and spraying pesticides at different fixed moments, and proved that all solutions of the following SI model were uniformly ultimately bounded.

$$\begin{cases} \dot{S}(t) &= rS(t)\left(1-\frac{S(t)+\theta I(t)}{K}\right) \\ &\quad -\beta S(t)I(t),\ \ t\neq (n+l-1)\tau,\ t\neq n\tau, \\ \dot{I}(t) &= \beta S(t)I(t) \\ &\quad -\omega I(t),\ \ t\neq (n+l-1)\tau,\ t\neq n\tau, \\ \Delta S(t) &= -\mu_1 S(t),\ \ t=(n+l-1)\tau, \\ \Delta I(t) &= -\mu_2 I(t),\ \ t=(n+l-1)\tau, \\ \Delta S(t) &= 0,\ \ t=n\tau, \\ \Delta I(t) &= -\mu,\ \ t=n\tau, 0<l<1, n=1,2,\ldots, \end{cases} \tag{1.1}$$

where $S(t)$ denotes the number of susceptible insects and $I(t)$ denotes the number of infective insects at time t, respectively, $\beta > 0$ is called the transmission coefficient, $\omega > 0$ is called the death coefficient of $I(t)$, $r > 0$ is called the intrinsic growth rate of pests, $K > 0$ is the pests capacity of environment, $\Delta I(t) = I(t^+) - I(t)$, $0 < \theta < 1$, $0 < l < 1$. $0 \leqslant \mu_1 < 1$, $0 \leqslant \mu_2 < 1$ respectively represents the portion of susceptible and infective pests which due to spraying pesticides at $t = n\tau$, $n \in Z_+$ and $Z_+ = \{1, 2, \ldots\}$. The authors obtained that system (1.1) has a susceptible pests extinction boundary periodic solution which is globally asymptotically stable. That is, according to the viewpoint, investigators gained a method to use a combination of biological and chemical tactics to eradicates pests. So this was cold susceptible pests extinction as pest-extinction.

In [18], Jiao, Chen and Cai considered the following impulsive differential equation with jumps at fixed times:

$$\begin{cases} \dot{S}(t) = rS(t)\left(1-\frac{S(t)+\theta I(t)}{K}\right) - \frac{\beta S(t)I(t)}{1+aI(t)},\ \ t\neq n\tau, \\ \dot{I}(t) = \frac{\beta S(t)I(t)}{1+aI(t)} - \omega I(t),\ \ t\neq n\tau, \\ \Delta S(t) = -\mu_1 S(t),\ \ t=n\tau, \\ \Delta I(t) = -\mu_2 I(t)+\mu,\ \ t=n\tau,\ \ n=1,2,\ldots, \end{cases} \tag{1.2}$$

where $\frac{\beta S(t)I(t)}{1+aI(t)}$ is the saturation effect of incidence rate, $r > 0$ is the intrinsic growth rate of pests, $K > 0$ is the pests carrying capacity, $\Delta I(t) = I(t^+) - I(t)$, $0 < \theta < 1$ is the infective pests' competing ability with the susceptible pests, $0 \leqslant \mu_1 < 1$, $0 \leqslant \mu_2 < 1$ represents the fraction of susceptible and infective pests removed by spraying pesticides at

Manuscript received Sep 19, 2020; revised Jan 25, 2021. This work is supported by the Funding for Applied Technology Research and Development of Panzhihua City under Grant 2017CY-S-6-2 and supported by the funding of Panzhihua College Affiliated Hospital under Grant 201501021.

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$t = n\tau$, respectively, $n \in Z_+$, $a > 0$. According to the result of the model (1.2), investigators told that the combination of biological and chemical methods is used to eliminate pests or keep the number of pests below the damage level.

Now, based on (1.1)-(1.2), we incorporate pulse vaccination, pulse removal and a saturated contact rate into the following SI model and investigate the existence, uniqueness and stability of almost periodic solution.

$$\begin{cases} \dot{S}(t) &= S(t)[b(t) - d(t)S(t)] - \frac{\beta(t)I(t-\tau_1)S(t)}{1+\alpha(t)S(t-\tau)}, \\ \dot{I}(t) &= I(t)[b_1(t) - d_1(t)I(t)] \\ &\quad + \frac{\beta(t)I(t)S(t-\tau')}{1+\alpha(t)S(t-\tau')}, \ t \neq t_k, \\ S(t_k^+) &= (1-\theta_k)S(t_k), \\ I(t_k^+) &= (1-\mu_k)I(t_k), \ t = t_k, \ k = 0, 1, 2, \ldots. \end{cases} \quad (1.3)$$

In model (1.3), θ_k is the proportion of those pulse vaccinated successfully and μ_k is the proportion of infected individuals pulse removed at each fixed time t_k. $b(t)$ denotes Intrinsic growth rate of susceptible population. The constants τ and τ_1 are the gestation periods, while τ' is the length of the infectious period. $\beta(t)S(t)/(1+\alpha(t)S(t))$ denotes the saturation exposure rate of the disease. For this system, we mainly discuss its existence of a unique almost periodic solution.

By the basic theories of impulsive differential equations in [25-26], system (1.3) has a unique solution $X(t) = X(t, X_0) \in PC([0,+\infty), R^2)$ and $PC([0,+\infty), R^2) = \{\phi : [0,+\infty) \to R^2, \phi \text{ is continuous for } t \neq \tau_k$. Also $\phi(\tau_k^-)$ and $\phi(\tau_k^+)$ exist, and $\phi(\tau_k^-) = \phi(\tau_k)$, $k = 1, 2, \cdots\}$ for each initial value $x_0 = x(0) \in R^{2+}$.

For a given continuous function $f(t)$, we will use the following notations and assumptions:

$$f^l = \inf_{t\in[0,+\infty)} f(t), \qquad f^L = \sup_{t\in[0,+\infty)} f(t).$$

In this paper, we suppose system (1.3) satisfy the following conditions:

(H_1) The impulse times t_k satisfy $0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$ and $\lim_{k\to\infty} t_k = +\infty$.

(H_2) The parameters θ_k and μ_k are real sequences satisfying $0 < \theta_k < 1$, $0 < \mu_k < 1$, $k = 1, 2, \ldots$.

(H_3) $\prod_{0<t_k<t}(1-\theta_k)$ and $\prod_{0<t_k<t}(1-\mu_k)$ are almost periodic, and there exist positive constants m_1, m_2, M_1 and M_2, such that $m_1 \le \prod_{0<t_k<t}(1-\theta_k) \le M_1$, $m_2 \le \prod_{0<t_k<t}(1-\mu_k) \le M_2$, for all $t \ge 0$.

(H_4) The functions $b(t)$, $b_1(t)$, $d(t)$, $d_1(t)$, $\beta(t)$ and $\alpha(t)$ are positive continuous almost periodic functions. These functions satisfy: $b^l \le b(t) \le b^L$, $b_1^l \le b_1(t) \le b_1^L$, $d^l \le d(t) \le d^L$, $d_1^l \le d_1(t) \le d_1^L$, $\beta^l \le \beta(t) \le \beta^L$, and $\alpha^l \le \alpha(t) \le \alpha^L$.

(H_5) Time delays τ, τ' and τ_1 are nonnegative constants.

We will only consider the solution of system (1.3) with initial condition

$$\big(S(s), I(s)\big) = \big(\varphi(s), \psi(s)\big) \quad \text{for} \quad -\tau^* \le s \le 0, \ (1.4)$$

where $\varphi(0) > 0, \psi(0) > 0;\ \varphi, \psi \in C^1\big([-\tau^*, 0], [0,+\infty)\big)$, $\tau^* = \max\{\tau, \tau_1, \tau'\}$.

In Section II, we present some notations and lemmas. In Sections III-IV, sufficient conditions are obtained for the permanence and globally asymptotically stable properties of system (1.3). In Section V, we present some new sufficient conditions and by constructing a Lyapunov functional to study the existence of a unique almost periodic solution of system (1.3). In Section VI, we shall give an example to illustrate our result.

II. Preliminaries

In this section, we shall recall some definitions and state some lemmas for discussing the permanence and global attractivity of system (1.3).

Definition 1. ([20]) Functions $S, I \in ([-\tau^*, +\infty), [0, +\infty))$ are said to be a solution of system (1.1) on $[-\tau^*, +\infty)$ provided

(1) $S(t)$ and $I(t)$ are absolutely continuous on each interval $(0, t_1] and (t_k, t_{k+1}]$, $k = 1, 2, \ldots$.

(2) For any $t_k, k = 1, 2, \ldots, S(t_k^+), I(t_k^+), S(t_k^-)$ and $I(t_k^-)$ exist and $S(t_k^-) = S(t_k), I(t_k^-) = I(t_k)$.

(3) $S(t)$ and $I(t)$ satisfy (1.3) for almost everywhere $(a.e.)$ in $[0,+\infty)/t_k$ and satisfy $S(t_k^+) = (1-\theta_k)S(t_k), I(t_k^+) = (1-\mu_k)I(t_k)$ for every $t = t_k, k = 1, 2, \ldots$.

Definition 2. ([21]) The set of sequences $\{\tau_k^j = \tau_{k+j} - \tau_k\}, k, j \in Z$ is uniformly almost periodic when for any $\varepsilon > 0$, there exists a relatively dense set of ε-almost periods common for any sequences.

Definition 3. ([21]) The function $\varphi \in PC(R, R)$ is said to be almost periodic if

(a) $\{\tau_k^j\}, k, j \in Z$ is uniformly almost periodic;

(b) for any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and $t' - t'' < \delta$, then $\phi(t') - \psi(t'') < \varepsilon$;

(c) for any $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $\phi(t+\tau) - \psi(t) < \varepsilon$ for all $t \in R$, satisfying the condition $t - \tau_k > \varepsilon, k \in Z$.

Consider the following nonimpulsive system

$$\begin{cases} \dot{x}(t) = x(t)\big[b(t) - D(t)x(t) - \frac{B(t)y(t-\tau_1)}{1+A(t)x(t-\tau)}\big] \\ \dot{y}(t) = y(t)\big[b_1(t) - D_1(t)y(t) + \frac{B_1(t)x(t-\tau')}{1+A_1(t)x(t-\tau')}\big], \end{cases} \quad (2.1)$$

where

$$D(t) = d(t)\prod_{0<t_k<t}(1-\theta_k)$$

,

$$D_1(t) = d_1(t)\prod_{0<t_k<t}(1-\mu_k),$$

$$B_1(t) = \beta(t)\prod_{0<t_k<t}(1-\theta_k),$$

$$A(t) = A_1(t) = \alpha(t)\prod_{0<t_k<t}(1-\theta_k)$$

with initial condition

$$\big(x(s), y(s)\big) = \big(\varphi(s), \psi(s)\big) \quad \text{for} \quad -\tau^* \le s \le 0, \ (2.2)$$

where $\varphi(0) > 0, \psi(0) > 0;\ \varphi, \psi \in C^1\big([-\tau^*, 0], [0,+\infty)\big)$.

From H_3-H_4, We have

$$d^l m_1 \leqslant D(t) \leqslant d^L M_1, \quad d^l m_2 \leqslant D_1(t) \leqslant d^L M_2,$$

$$\beta^l m_2 \leqslant B(t) \leqslant \beta^L M_2, \quad \beta^l m_1 \leqslant B_1(t) \leqslant \beta^L M_1,$$

$$\alpha^l m_1 \leqslant A(t) \leqslant \alpha^L M_1.$$

By the solution $(x(t), y(t))$ of (2.1) and (2.2), two absolutely continuous functions $x(t), y(t)$ defined on $[-\tau^*, 0]$, which satisfies (2.1) a.e. for $t \geqslant 0$ and $x(t) = \varphi(t), y(t) = \psi(t)$ on $[-\tau^*, 0]$.

Lemma 1. *Let $(x(t), y(t))^{\mathrm{T}}$ be any solution of system (2.1) such that $x(0) > 0, y(0) > 0$, then $x(t) > 0, y(t) > 0$ for all $t \geq 0$.*

Proof: From (2.1), we have $x'(t) = x(t)P(t)$, where $P(t) = b(t) - D(t)x(t) - \frac{B(t)y(t-\tau_1)}{1+A(t)x(t-\tau)}$, thus when $x(0) > 0$, we can obtain

$$x(t) = x(0)\exp\{\int_0^t P(t)ds\} > 0.$$

Similarly, $y(t) > 0$. This completes the proof. ■

Lemma 2. *For systems (1.3) and (2.1), the following results hold:*

(1) if $(x(t), y(t))^{\mathrm{T}}$ is a solution of (2.1) on $[-\tau^*, +\infty)$, then

$$(S(t), I(t))^{\mathrm{T}} = \Big(\prod_{0<t_k<t}(1-\theta_k)x(t), \prod_{0<t_k<t}(1-\mu_k)y(t)\Big)^{\mathrm{T}}$$

is a solution of (1.3);

(2) if $(S(t), I(t))^{\mathrm{T}}$ is a solution of (1.3) on $[-\tau^*, +\infty)$, then

$$(x(t), y(t))^{\mathrm{T}} = \Big(\prod_{0<t_k<t}(1-\theta_k)^{-1}S(t), \prod_{0<t_k<t}(1-\mu_k)^{-1}I(t)\Big)^{\mathrm{T}}$$

is a solution of (2.1).

Proof: (1) suppose that $(x(t), y(t))^{\mathrm{T}}$ is a solution of (2.1), Let $S(t) = \prod_{0<t_k<t}(1-\theta_k)x(t)$, $I(t) = \prod_{0<t_k<t}(1-\mu_k)y(t)$, then for any $t \neq t_k, k = 1, 2, \ldots$, by substituting $x(t) = \prod_{0<t_k<t}(1-\theta_k)^{-1}S(t)$, $y(t) = \prod_{0<t_k<t}(1-\mu_k)^{-1}I(t)$ into system (2.1), we can easily verify that the first two equations of (1.1) hold. For $t = t_k, k = 1, 2, \ldots$, we have

$$\begin{aligned}
S(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_k<t}(1-\theta_k)x(t)\\
&= \prod_{0<t_j\leqslant t_k}(1-\theta_j)x(t)\\
&= (1-\theta_k)\prod_{0<t_j<t_k}(1-\theta_j)x(t)\\
&= (1-\theta_k)S(t_k).\\
I(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_k<t}(1-\mu_k)y(t)\\
&= \prod_{0<t_j\leqslant t_k}(1-\mu_j)y(t)\\
&= (1-\mu_k)\prod_{0<t_j<t_k}(1-\mu_j)y(t)\\
&= (1-\mu_k)I(t_k).
\end{aligned}$$

So the last two equations of (1.3) also hold. Thus $(S(t), I(t))^{\mathrm{T}}$ is a solution of (1.3). This proves the conclusion of (1).

(2) Since $x(t), y(t)$ is continuous on each interval $(t_k, t_{k+1}]$, it is sufficient to check the continuity of $x(t), y(t)$ at the impulse points $t_k, k = 1, 2, \ldots$. Since $x(t) = \prod_{0<t_k<t}(1-\theta_k)^{-1}S(t)$, $y(t) = \prod_{0<t_k<t}(1-\mu_k)^{-1}I(t)$, we have

$$\begin{aligned}
x(t_k^+) &= \prod_{0<t_j\leqslant t_k^+}(1-\theta_j)^{-1}S(t_k^+)\\
&= \prod_{0<t_j<t_k}(1-\theta_j)^{-1}S(t_k) = x(t_k),\\
x(t_k^-) &= \prod_{0<t_j<t_k^-}(1-\theta_j)^{-1}S(t_k^+)\\
&= \prod_{0<t_j<t_k}(1-\theta_j)^{-1}S(t_k) = x(t_k),
\end{aligned}$$

hence $x(t)$ is continuous on $[0, +\infty)$, similar to proof $y(t)$ is continuous on $[0, +\infty)$. Come back (2.2), we obtain $x(t), y(t)$ are continuous on $[-\tau^*, +\infty)$. It is easy to check that $(x(t), y(t))^{\mathrm{T}}$ satisfies (2.1), that is, it is a solution of (2.1). Therefore, this implies that the proof of Lemma 2 is now completed. ■

As a direct corollary of Lemma 2.2 of Chen [27], we have

Lemma 3. ([27]) *If $a > 0, b > 0$ and $\dot{x} \geqslant x(b - ax)$, when $t \geqslant 0$ and $x(0) > 0$, we have*

$$\lim_{t\to\infty}\inf x(t) \geqslant \frac{b}{a}.$$

If $a > 0, b > 0$ and $\dot{x} \leqslant x(b - ax)$, when $t \geqslant 0$ and $x(0) > 0$, we have

$$\lim_{t\to\infty}\sup x(t) \leqslant \frac{b}{a}.$$

III. Uniform permanence

Following, we will proof the permanence of system (2.1).

Theorem 1. *Let (H_2)-(H_4) hold. Furthermore, assume that*

(H_6) $b^l - \beta^L M_2 N_2 > 0.$

holds, then system (2.1) with initial condition (2.2) is uniformly permanent, i.e., there exists a $T > 0$, such that for all $t > T$, any solution $(x(t), y(t))^{\mathrm{T}}$ of the system (2.1) satisfies

$$n_1 \leqslant \lim_{t\to\infty}\inf x(t) \leqslant \lim_{t\to\infty}\sup x(t) \leqslant N_1,$$
$$n_2 \leqslant \lim_{t\to\infty}\inf y(t) \leqslant \lim_{t\to\infty}\sup y(t) \leqslant N_2,$$

where

$$n_1 = \frac{b^l - \beta^L M_2 N_2}{d^L M_1},$$

$$n_2 = \frac{b_1^l + \frac{\beta^l m_1 n_1}{1+\alpha^L M_1 N_1}}{d_1^L M_2},$$

$$N_1 = \frac{b^L}{d^l m_1}, N_2 = \frac{b_1^L + \beta^L M_1 N_1}{d_1^l m_2}.$$

Proof: Let $(x(t), y(t))^{\mathrm{T}}$ be any solution of system (2.1). By (H_2)-(H_4) and Lemma 1, when $t - \tau^* = T_0 > 0$, for system (2.1), we have,

$$\dot{x}(t) \leqslant x(t)[b^L - d^l m_1 x(t)],$$
$$\dot{y}(t) \leqslant y(t)[b_1^L - d_1^l m_2 y(t) + \beta_1 x(t-\tau')].$$

From Lemma 3, we obtain

$$\lim_{t\to\infty}\sup x(t) \leqslant \frac{b^L}{d^l m_1} \triangleq N_1.$$

Thus, for an arbitrary positive constant ε, there exists a $T_1 > T_0 > 0$, such that for all $t-\tau' > T_1$

$$x(t-\tau') \leqslant N_1+\varepsilon.$$

Then

$$\dot{y}(t) \leqslant y(t)[b_1^L+\beta^L M_1(N_1+\varepsilon)-d_1^l m_2 y(t)].$$

From Lemma 3, and setting $\varepsilon \to 0$, we obtain

$$\lim_{t\to\infty} \sup y(t) \leqslant \frac{b_1^L+\beta^L M_1 N_1}{d_1^l m_2} \triangleq N_2.$$

Therefore, for an arbitrary positive constant ε, there exists a $T_2 > T_1 > 0$, such that for all $t-\tau_1 > T_2$

$$y(t-\tau_1) \leqslant N_2+\varepsilon.$$

From system (2.1), we have

$$\dot{x}(t) \geqslant x(t)[b^l-\beta^L M_2(N_2+\varepsilon)-d^L M_1 x(t)],$$

Thus from H_6 and Lemma 3, and setting $\varepsilon\to 0$, it follows that

$$\lim_{t\to\infty} \inf x(t) \geq \frac{b^l-\beta^L M_2 N_2}{d^L M_1} \triangleq n_1.$$

Thus, for an arbitrary positive constant ε, there exists a sufficient big $T_3 \geqslant T_2 > 0$, such that for all $t-\tau' > T_3$

$$n_1-\varepsilon \leqslant x(t-\tau') \leqslant N_1+\varepsilon.$$

Again from system (2.1), we have

$$\dot{y}(t) \geqslant y(t)\left[b_1^l+\frac{\beta^l m_1(n_1-\varepsilon)}{1+\alpha^L M_1(N_1+\varepsilon)}-d_1^L M_2 y(t)\right].$$

Similarly, it follows that

$$\lim_{t\to\infty} \inf y(t) \geq \frac{b_1^l+\frac{\beta^l m_1 n_1}{1+\alpha^L M_1 N_1}}{d_1^L M_2} \triangleq n_2.$$

This completes the proof of Lemma 2.4. ■

Remark 1. From the proof of Theorem 1, we know that under the conditions of Theorem 1, the set $S=\{(x(t),y(t))\in R^2: n_1\leqslant x(t)\leqslant N_1, n_2\leqslant y(t)\leqslant N_2, n_1,n_2>0)\}$ is an invariant set of system (2.1). That is, system (2.1) with above-mentioned condition is uniformly permanent.

Apparently, system (2.1) is equivalent to system (1.3), which implies that system (1.3) and (1.4) is uniformly permanent under the conditions of Theorem 1.

IV. Globally asymptotic stability

Definition 4. A bounded positive solution $(x_1(t),y_1(t))^T$ of system (2.1) is said to be globally asymptotically stable if for any other positive bounded solution $(x_2(t),y_2(t))^T$ of system (2.1), the following equality holds:

$$\lim_{t\to\infty}[|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|]=0.$$

The following lemma is from [28], and will be employed in establishing the globally asymptotic stability of system (2.1).

Lemma 4. ([28]) *Let h be a real number, $f(t)$ be a nonnegative integrable uniformly continuous function for $t\in[h,+\infty)$, then $\lim_{t\to+\infty} f(t)=0$.*

Followingly, we will adopt the idea of Chen [28] to prove the global attractivity of the positive solution of system (2.1).

Theorem 2. *If (H_1)-(H_6) hold. Assume further that*

(H_7) *there exist positive constants ρ_1,ρ_2,γ such that*

$$-\rho_1 d^L M_1 n_1+C_1+C_2<-\gamma,$$
$$-\rho_2 d_1^L M_2 n_2+C_3<-\gamma.$$

Then system (1.3) with initial condition (1.4) is globally asymptotically stable, where

$$C_1=\frac{\rho_1\beta^L M_2 N_2\alpha^L M_1 N_1}{(1+\alpha^l m_1 n_1)^2},$$

$$C_2=\frac{\rho_2\beta^L M_1 N_1}{(1+\alpha^l m_1 n_1)^2},$$

$$C_3=\rho_1\beta^L M_2 N_2\frac{1+\alpha^L M_1 N_1}{(1+\alpha^l m_1 n_1)^2}.$$

Proof: Since system (2.1) is equivalent to system (1.3), to finish the proof of Theorem 2, we only need to prove that system (2.1) is globally asymptotically stable. Let $x(t)=e^{u(t)}, y(t)=e^{v(t)}$, then system (2.1) is transformed into

$$\begin{cases}\dot{u}(t)=b(t)-D(t)e^{u(t)}-\frac{B(t)e^{v(t-\tau_1)}}{1+A(t)e^{u(t-\tau)}}\\ \dot{v}(t)=b_1(t)-D_1(t)e^{v(t)}+\frac{B_1(t)e^{u(t-\tau')}}{1+A_1(t)e^{u(t-\tau')}}.\end{cases}\tag{4.1}$$

It is obvious that the globally asymptotically of system (2.1) is equivalent to that of system (4.1).

For $Z(t)\in R_+^2$, we define $\|Z(t)\|=|u(t)|+|v(t)|$. Let $Z_1(t)=\big(u_1(t),v_1(t)\big)^T$ and $Z_2(t)=\big(u_2(t),v_2(t)\big)^T$ be two solutions of system (4.1). we can obtain the associated product $\dot{u}_i(t)$, $\dot{v}_i(t)$, $(i=1,2)$ of system (4.1)

Construct a Lyapunov functional $V(t)=V(t,Z_1(t),Z_2(t))$ as follows

$$V(t)=F_1(t)+F_2(t),\quad t\geqslant T>0,$$

$$F_1(t)=\rho_1|u_1(t)-u_2(t)|+\rho_2|v_1(t)-v_2(t)|,$$

$$F_2(t)=C_1\int_{t-\tau}^{t}|u_2(s)-u_1(s)|\mathrm{d}s$$

$$+C_2\int_{t-\tau'}^{t}|u_2(s)-u_1(s)|\mathrm{d}s+C_3\int_{t-\tau_1}^{t}|v_2(s)-v_1(s)|\mathrm{d}s.$$

For calculate the right derivative $D^+V(t)$ of $V(t)$ along the solutions of system (4.1), we first calculate

$$\begin{aligned}D^+F_1(t) &= \rho_1 sign(u_1(t)-u_2(t))(\dot{u_1}(t)-\dot{u_2}(t))\\ &\quad+\rho_2 sign(v_1(t)-v_2(t))(\dot{v_1}(t)-\dot{v_2}(t))\\ &= \rho_1 sign(u_1(t)-u_2(t))\Big[D(t)\big(e^{u_2(t)}-e^{u_1(t)}\big)\\ &\quad+\frac{B(t)e^{v_2(t-\tau_1)}}{1+A(t)e^{u_2(t-\tau)}}-\frac{B(t)e^{v_1(t-\tau_1)}}{1+A(t)e^{u_1(t-\tau)}}\Big]\\ &\quad+\rho_2 sign(v_1(t)-v_2(t))\Big[D_1(t)\big(e^{v_2(t)}-e^{v_1(t)}\big)\\ &\quad-\frac{B_1(t)e^{u_2(t-\tau')}}{1+A_1(t)e^{u_2(t-\tau')}}+\frac{B_1(t)e^{u_1(t-\tau')}}{1+A_1(t)e^{u_1(t-\tau')}}\Big].\end{aligned}$$

By applying Lemma 4, it follows that there exists a large enough $T'\geqslant T_3>0$, for any $t>T'$, such that

$$D^+F_1(t)$$

$$\begin{aligned}
&\leqslant -\rho_1 d^L M_1 n_1 |u_1(t)-u_2(t)| \\
&\quad -\rho_2 d_1^L M_2 n_2 |v_1(t)-v_2(t)| \\
&\quad +\rho_1 \beta^L M_2 N_2 \frac{1+\alpha^L M_1 N_1}{(1+\alpha^l m_1 n_1)^2} |v_1(t-\tau_1)-v_2(t-\tau_1)| \\
&\quad +\frac{\rho_1 \beta^L M_2 N_2 \alpha^L M_1 N_1}{(1+\alpha^l m_1 n_1)^2} |u_1(t-\tau)-u_2(t-\tau)| \\
&\quad +\frac{\rho_2 \beta^L M_1 N_1}{(1+\alpha^l m_1 n_1)^2} |u_1(t-\tau')-u_2(t-\tau')| \\
&= -\rho_1 d^L M_1 n_1 |u_1(t)-u_2(t)| \\
&\quad -\rho_2 d_1^L M_2 n_2 |v_1(t)-v_2(t)| \\
&\quad +C_1 |u_1(t-\tau)-u_2(t-\tau)| \\
&\quad +C_2 |u_1(t-\tau')-u_2(t-\tau')| \\
&\quad +C_3 |v_1(t-\tau_1)-v_2(t-\tau_1)|.
\end{aligned}$$

It follows from the mean value theorem and Remark 1 that

$$\begin{aligned}
n_1|u_2(t)-u_1(t)| &\leqslant |e^{u_2(t)}-e^{u_1(t)}| \\
&= e^{\theta(t)}|u_2(t)-u_1(t)| \\
&\leqslant N_1|u_2(t)-u_1(t)|,
\end{aligned}$$

$$\begin{aligned}
n_2|v_2(t)-v_1(t)| &\leqslant |e^{v_2(t)}-e^{v_1(t)}| \\
&= e^{\eta(t)}|v_2(t)-v_1(t)| \\
&\leqslant N_2|v_2(t)-v_1(t)|,
\end{aligned}$$

$$\begin{aligned}
&e^{u_2(t-\tau)}e^{v_1(t-\tau_1)} - e^{u_1(t-\tau)}e^{v_2(t-\tau_1)} \\
&= e^{u_2(t-\tau)}\big(e^{v_1(t-\tau_1-e^{v_2(t-\tau_1)})}\big) \\
&\quad +e^{v_2(t-\tau_1)}\big(e^{u_2(t-\tau)}-e^{u_1(t-\tau)}\big).
\end{aligned}$$

where $\theta(t)$ lies between $u_2(t)$ and $u_1(t)$, $\eta(t)$ lies between $v_2(t)$ and $v_1(t)$.

$$\begin{aligned}
&D^+F_2(t) \\
&= C_1|u_2(t)-u_1(t)| - C_1|u_2(t-\tau)-u_1(t-\tau)| \\
&\quad +C_2|u_2(t)-u_1(t)| - C_2|u_2(t-\tau')-u_1(t-\tau')| \\
&\quad +C_3|v_2(t)-v_1(t)| - C_3|v_2(t-\tau_1)-v_1(t-\tau_1)|.
\end{aligned}$$

Thus, there exists a positive constant $\gamma > 0$ and large enough $T > T'$ such that, for all $t > T$, we have

$$\begin{aligned}
D^+V(t) &\leqslant (-\rho_1 d^L M_1 n_1 + C_1 + C_2)|u_1(t)-u_2(t)| \\
&\quad +(-\rho_2 d_1^L M_2 n_2 + C_3)|v_1(t)-v_2(t)| \\
&< -\gamma(|u_1(t)-u_2(t)| + |v_1(t)-v_2(t)|).
\end{aligned}$$

Integrating both sides of the above inequality from T to t produces

$$\begin{aligned}
&V(t) + \gamma\int_T^t (|u_1(s)-u_2(s)| + |v_1(t)-v_2(s)|)\mathrm{d}s \\
&\leqslant V(T) < +\infty, \quad t \geqslant T.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_T^t (|u_1(s)-u_2(s)| + |v_s(t)-v_2(1)|)\mathrm{d}s \\
&\leqslant \gamma^{-1}V(T) < +\infty, \quad t \geqslant T.
\end{aligned}$$

Hence

$$|u_1(t)-u_2(t)| + |v_1(t)-v_2(t)| \in L^1(T,+\infty).$$

That is, $|u_1(t)-u_2(t)|$, $|v_1(t)-v_2(t)|$ are bounded on $[T,+\infty)$. On the other hand, it is easy to see that $\dot{u}_1(t)$, $\dot{u}_2(t)$, $\dot{v}_1(t)$ and $\dot{v}_2(t)$ are bounded for $t \geqslant T$, Therefore, $|u_1(t)-u_2(t)|$, $|v_t(t)-v_2(t)|$ are uniformly continuous on $[T,+\infty)$. By Lemma 4, one can conclude that

$$\lim_{t\to\infty}[|u_1(t)-u_2(t)| + |v_1(t)-v_2(t)|] = 0$$

And so

$$\lim_{t\to\infty}|u_1(t)-u_2(t)| = 0, \quad \lim_{t\to\infty}|v_1(t)-v_2(t)| = 0.$$

This ends the proof of Theorem 2. ■

V. Almost periodic dynamics

In the section, we can investigate the existence and uniqueness of almost periodic solutions for functional differential equations by using Lyapunov functional as follows and referring the ideas from to [29, p. 388].

Let $C = C([-r,0],\mathbb{R}^n), H \in \mathbb{R}_+$ or $H = +\infty$. Denote $C_H = \{\varphi : \varphi \in C, |\varphi| < H\}, |\varphi| = \sup_{\theta\in[-r,0]} |\varphi(\theta)|$.

Consider the system

$$\dot{x}(t) = f(t, x(t)), \tag{5.1}$$

where $f(t,\phi)$ is continuous in $(t,\phi) \in \mathbb{R} \times C_H$ and almost periodic in the uniformly for $\phi \in C_H, C_H \subseteq C, \forall\alpha > 0, \exists L(\alpha) > 0$, such that $|f(t,\phi)| \leqslant L(\alpha)$ as $t \in \mathbb{R}, \varphi \in C_\alpha$.

To investigate the almost periodic solution of system (5.1), we introduce the associate product system of system (5.1):

$$\dot{x}(t) = f(t, x(t)), \quad \dot{y}(t) = f(t, y(t)). \tag{5.2}$$

Lemma 5. ([29]) *Suppose that for $t \geqslant 0, \phi, \psi \in C_H$, there exists a continuous Lyapunov functional $V(t,\phi,\psi)$ which has the following properties:*

(1) *$u(|\phi-\psi|) \leqslant V(t,\phi,\psi) \leqslant v(|\phi-\psi|)$ where $u(s)$ and $v(s)$ are continuous nondecreasing functions, and $u(s) \to 0$ as $s \to 0$.*
(2) *$|V(t,\phi_1,\psi_1) - V(t,\phi_2,\psi_2)| \leqslant L(|(\phi_1-\phi_2)-(\psi_1-\psi_2))|)$, where L is a positive constant.*
(3) *$\dot{V}_{(5.2)}(t,\phi,\psi) \leqslant -\lambda V(t,\phi,\psi)$, where λ is a positive constant.*

Moreover, one assumes that system (5.1) has a solution $x(t,\sigma,\varphi)$ such that $|x_t(\sigma,\varphi)| \leqslant H_1$ for $t \geqslant \sigma \geqslant 0, H > H_1 > 0$. Then system (5.1) has a unique almost periodic solution which is uniformly asymptotically stable.

According to Lemma 5 and Remark 1, we first obtain a sufficient condition which guarantees the existence of a bounded solution of system (1.3), and then construct an adaptive Lyapunov functional for system (1.3).

Our main result of this paper is as follows:

Theorem 3. *If (H_1)-(H_4) and (H_6)-(H_7) hold. Assume further that*

(H_8) *The set of sequences $\{t_k^j = t_{k+j} - t_k\}, k, j \in Z$ is uniformly almost periodic.*

Then system (1.3) has a unique positive almost periodic solution, which is globally asymptotically stable.

Proof: Since system (4.1) is equivalent to system (2.1), to finish the proof of Theorem 3, first, we only need to prove that system (4.1) has a unique almost periodic solution.

From Remark 1, the invariant set of system (2.1) is

$$S=\{(x(t),y(t))\in R^2|n_1\leqslant x(t)\leqslant N_1,$$
$$n_2\leqslant y(t)\leqslant N_2,n_1,n_2>0)\},$$

then the invariant set of system (4.1) is

$$S_1=\{(u(t),v(t))\in R^2|\ln n_1\leqslant u(t)\leqslant \ln N_1,$$
$$\ln n_2\leqslant v(t)\leqslant \ln N_2\}.$$

In the following, we shall prove that (4.1) has a unique almost periodic solution $Z(t)=\big(u(t),v(t)\big)$ in $S_1=\{(u(t),v(t))\in R^2|\ln n_1\leqslant u(t)\leqslant \ln N_1,\ln n_2\leqslant v(t)\leqslant \ln N_2\}$. Suppose that $Z_1(t)=\big(u_1(t),v_1(t)\big)^T$ and $Z_2(t)=\big(u_2(t),v_2(t)\big)^T$ are any two solutions of system (4.1).

For $Z(t)\in R^2_+$. we define $\|Z(t)\|=|u(t)|+|v(t)|$. Consider the associated product system of (4.1):

$$\begin{cases} \dot{u_1}(t)=b(t)-D(t)e^{u_1(t)}-\frac{B(t)e^{v_1(t-\tau_1)}}{1+A(t)e^{u_1(t-\tau)}},\\ \dot{v_1}(t)=b_1(t)-D_1(t)e^{v_1(t)}+\frac{B_1(t)e^{u_1(t-\tau')}}{1+A_1(t)e^{u_1(t-\tau')}},\\ \dot{u_2}(t)=b(t)-D(t)e^{u_2(t)}-\frac{B(t)e^{v_2(t-\tau_1)}}{1+A(t)e^{u_2(t-\tau)}},\\ \dot{v_2}(t)=b_1(t)-D_1(t)e^{v_2(t)}+\frac{B_1(t)e^{u_2(t-\tau')}}{1+A_1(t)e^{u_2(t-\tau')}}. \end{cases}\quad(5.3)$$

Construct a Lyapunov functional $V(t)=V(t,Z_1(t),Z_2(t))$ as follows

$$\begin{aligned} &V(t)=F_1(t)+F_2(t),\quad t>0,\\ &F_1(t)=\rho_1|u_1(t)-u_2(t)|+\rho_2|v_1(t)-v_2(t)|\\ &F_2(t)=C_1\int_{t-\tau}^{t}|u_2(s)-u_1(s)|\mathrm{d}s\\ &\qquad+C_2\int_{t-\tau'}^{t}|u_2(s)-u_1(s)|\mathrm{d}s\\ &\qquad+C_3\int_{t-\tau_1}^{t}|v_2(s)-v_1(s)|\mathrm{d}s. \end{aligned}$$

Similar to that of the analysis of Theorem 3.1 of [23], it is easy to know that conditions (1) and (2) of Lemma 5 are satisfied by $F_1(t)$ and $F_2(t)$.

Calculate the right derivative $\dot V(t)$ of $V(t)$ along the solutions of system (5.3). From the proofs of Theorem 2, we have

$$\begin{aligned} \dot{F_1}(t)\ =\ &\rho_1 sign(u_1(t)-u_2(t))(\dot{u_1}(t)-\dot{u_2}(t))\\ &+\rho_2 sign(v_1(t)-v_2(t))(\dot{v_1}(t)-\dot{v_2}(t))\\ \leqslant\ &-\rho_1 d^L M_1 n_1|u_1(t)-u_2(t)|\\ &-\rho_2 d_1^L M_2 n_2|v_1(t)-v_2(t)|\\ &+C_1|u_1(t-\tau)-u_2(t-\tau)|\\ &+C_2|u_1(t-\tau')-u_2(t-\tau')|\\ &+C_3|v_1(t-\tau_1)-v_2(t-\tau_1)|. \end{aligned}$$

Thus

$$\begin{aligned} \dot V(t)\ \leqslant\ &-\rho_1 d^L M_1 n_1|u_1(t)-u_2(t)|\\ &-\rho_2 d_1^L M_2 n_2|v_1(t)-v_2(t)|\\ &+C_1|u_2(t)-u_1(t)|\\ &+C_2|u_2(t)-u_1(t)|+C_3|v_2(t)-v_1(t)|\\ <\ &-\gamma(|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|). \end{aligned}$$

It has a positive constant δ such that

$$\dot V(t)_{(5.3)}\leqslant-\delta V(t).$$

From Lemma 5, condition (3) is satisfied. Therefore, system (2.1) has a unique almost periodic solution, which is uniformly asymptotic stable.

Next, we prove that system (1.3) has a unique almost periodic solution. From Lemma 2, we know that

$$(S(t),I(t))^{\mathrm T}=\big(\prod_{0<t_k<t}(1-\theta_k)x(t),\prod_{0<t_k<t}(1-\mu_k)y(t)\big)^{\mathrm T}$$

is a solution of system (1.3). Since condition (H_8) holds, similar to the proofs of Lemma 31 and Theorem 79 in [27], we can prove that

$$(S(t),I(t))^{\mathrm T}=\big(\prod_{0<t_k<t}(1-\theta_k)x(t),\prod_{0<t_k<t}(1-\mu_k)y(t)\big)^{\mathrm T}$$

is almost periodic. Therefore, $(S(t),I(t))^{\mathrm T}$ is a unique almost periodic solution of system (1.3), because of the uniqueness of $(x(t),y(t))^T$. This completes the proof of Theorem 3. ■

From Theorem 3, we can directly obtain the following corollary.

Corollary 1. *Suppose that $\theta_k=0$, $\mu_k=0$, $k=1,2,\ldots$, in system (1.3). And further assume that the following conditions hold:*

(H_6') $b^l-\beta^L N_2'>0.$

(H_7') *there exist positive constants ρ_1', ρ', γ' such that*
$$-\rho_1' d^L n_1'+C_1'+C_2'<-\gamma',\quad -\rho_2' d_1^L n_2'+C_3'<-\gamma',$$
where n_1', n_2', N_2', C_1', C_2', C_3' correspond n_1, n_2, N_2, C_1, C_2, C_3 of Theorems 1-2, respectively, when $m_1=m_2=M_1=M_2=1$. Then system (1.3) admits a unique almost periodic solution, which is uniformly asymptotic stable.

VI. An Example

Lastly, an example is provided to illustrate our results.

Consider the following impulsive delay SI model with variable coefficients:

$$\begin{cases} \dot S(t)\ =S(t)[(3+0.5\sin\sqrt2 t)\\ \qquad -(3+2\sin 2t)S(t)]\\ \qquad -\frac{(0.03+0.02\cos\sqrt2 t)I(t-\tau_1)S(t)}{1+(0.02+0.01\sin\sqrt3 t)S(t-\tau)},\\ \dot I(t)\ =I(t)[(1+0.5\cos\sqrt t)\\ \qquad -(0.3+0.2\cos 2t)I(t)]\\ \qquad +\frac{(0.03+0.02\cos\sqrt2 t)I(t)S(t-\tau')}{1+(0.02+0.01\sin\sqrt3 t)S(t-\tau')},\ \ t\neq t_k,\\ S(t_k^+)=(1-\theta_k)S(t_k),\\ I(t_k^+)=(1-\mu_k)I(t_k),\ \ t=t_k=k,\ \ k=0,1,2,\ldots. \end{cases}\quad(6.1)$$

We can ensure

$$\frac14\le\prod_{0<t_k<t}(1-\theta_k)\le\frac12,$$
$$\frac13\le\prod_{0<t_k<t}(1-\mu_k)\le\frac12$$

are hold by properly selecting θ_k, μ_k.

Obviously, for system (6.1), the conditions (H_1)-(H_5) and (H_7) are satisfied. From Theorem 3, by calculating we can obtain

$$N_1=14,\quad N_2=55.5,\quad n_1=0.445,\quad n_2\approx2,$$

$$C_1 \approx 0.29, \quad C_2 = 0.35, \quad C_3 \approx 1.68,$$

which imply (H_7) are satisfied, i.e.,

$$-\rho_1 d^L M_1 n_1 + C_1 + C_2 \approx -0.45 \triangleq -\gamma_1,$$
$$-\rho_2 d_1^L M_2 n_2 + C_3 \approx -3.21 \triangleq -\gamma_2.$$

Let $\gamma = \delta = 0.4$. By Theorem 3. system (6.1) admits a unique almost periodic solution, which is uniformly asymptotic stable.

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