A Combined Laplace Transform and Boundary Element Method for a Class of Unsteady Laplace Type Problems of Anisotropic Exponentially Graded Materials

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***Abstract*—In this paper a BEM is used to solve a class of variable coefficient parabolic equations numerically. Some examples are considered to show the accuracy of the numerical solutions.**

***Index Terms*—Anisotropic functionally graded materials, unsteady Laplace equation, Laplace transform, boundary element method.**

I. Introduction

We will consider initial boundary value problems governed by a Laplace type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}\left(\mathbf{x}\right)\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial x_j}\right]=\alpha\left(\mathbf{x}\right)\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial t} \tag{1}$$

The coefficients $[\kappa_{ij}]$ $(i,j=1,2)$ is a real symmetric positive definite matrix. Also, in (1) the summation convention for repeated indices holds. Therefore equation (1) may be written explicitly as

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left(\kappa_{11}\frac{\partial\mu}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\kappa_{12}\frac{\partial\mu}{\partial x_2}\right)\\&+\frac{\partial}{\partial x_2}\left(\kappa_{12}\frac{\partial\mu}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{22}\frac{\partial\mu}{\partial x_2}\right)=\alpha\frac{\partial c}{\partial t}\end{aligned}$$

Equation (1) is usually used to model antiplane strain in elastostatics and plane thermostatic problems (see for examples [1]–[5]).

Recently a number of authors had been working on the Laplace equation to find its solutions. However the works mainly focus on problems of isotropic homogeneous materials. See for example, Guo et. al [6] and Chen and Du [7]. For such kind of materials, the boundary element method (BEM) and other methods had been successfully used to find the numerical solutions of problems associated to them. But this is not the case for inhomogeneous materials, due to the unavailability of fundamental solutions for equations of variable coefficients which govern problems of inhomogeneous media. Some progress of solving problems for inhomogeneous media of a certain class of inhomogeneities has been done. Timpitak and Pochai [8], for example, investigated finite difference solutions of unsteady diffusion-convection problems for heterogeneous media.

Azis and co-workers had been working on steady state problems of *anisotropic inhomogeneous* media of some other classes of inhomogeneities for several types of governing equations, for examples [9]–[11] for the modified Helmholtz equation, [12] for elasticity problems, [13]–[17] for the diffusion convection equation, [18]–[24] for the diffusion convection reaction equation, [25]–[29] for the Helmholtz equation.

This paper is intended to extend the recently published works in [1]–[5] for steady anisotropic Laplace type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}\left(\mathbf{x}\right)\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial x_j}\right]=0$$

to unsteady anisotropic Laplace type equation with spatially variable coefficients of the form (1).

During the last decade functionally graded materials (FGMs) have become an important topic, and numerous studies on FGMs for a variety of applications have been reported. Authors commonly define an FGM as an inhomogeneous material having a specific property such as thermal conductivity, hardness, toughness, ductility, corrosion resistance, etc. that changes spatially in a continuous fashion. Therefore equation (1) is relevant for FGMs.

Equation (1) will be transformed to a constant coefficient equation from which a boundary integral equation will derived. It is necessary to place some constraint on the class of coefficients κ_{ij} and β for which the solution obtained is valid. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (1).

II. The initial-boundary value problem

Referred to a Cartesian frame Ox_1x_2 solutions $\mu\left(\mathbf{x},t\right)$ and its derivatives to (1) are sought which are valid for time interval $t\geq 0$ and in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\mu\left(\mathbf{x},t\right)$ $(\mathbf{x}=(x_1,x_2))$ is specified and on $\partial\Omega_2$

$$P\left(\mathbf{x},t\right)=\kappa_{ij}\left(\mathbf{x}\right)\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial x_i}n_j \tag{2}$$

is specified where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\mathbf{n}=(n_1,n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$\mu\left(\mathbf{x},0\right)=0 \tag{3}$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation, and

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then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable s. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution c and its derivatives for all $(\mathbf{x}, t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula.

The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\kappa_{11} = \kappa_{22}$ and $\kappa_{12} = 0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

III. The Boundary Integral Equation

The coefficients κ_{ij}, α are required to take the form

$$\kappa_{ij}(\mathbf{x}) = \overline{\kappa}_{ij} g(\mathbf{x}) \quad (4)$$

$$\alpha(\mathbf{x}) = \overline{\alpha} g(\mathbf{x}) \quad (5)$$

where the $\overline{\kappa}_{ij}, \overline{\alpha}$ are constants and g is a differentiable function of $\mathbf{x}$. Further we assume that the coefficients $\kappa_{ij}(\mathbf{x})$ and $\alpha(\mathbf{x})$ are exponentially graded by taking $g(\mathbf{x})$ as an exponential function

$$g(\mathbf{x}) = \left[\exp\left(c_0 + c_i x_i\right)\right]^2 \quad (6)$$

where c_0 and c_i are constants. Therefore if

$$\overline{\kappa}_{ij} c_i c_j - \lambda = 0 \quad (7)$$

then (6) satisfies

$$\overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \lambda g^{1/2} = 0 \quad (8)$$

Use of (4)-(5) in (1) yields

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g \frac{\partial \mu}{\partial x_j}\right) = \overline{\alpha} g \frac{\partial \mu}{\partial t} \quad (9)$$

Let

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, \psi(\mathbf{x}, t) \quad (10)$$

therefore substitution of (4) and (10) into (2) gives

$$P(\mathbf{x}, t) = -P_g(\mathbf{x})\, \psi(\mathbf{x}, t) + g^{1/2}(\mathbf{x})\, P_\psi(\mathbf{x}, t) \quad (11)$$

where

$$P_g(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial g^{1/2}}{\partial x_j} n_i \qquad P_\psi(\mathbf{x}) = \overline{\kappa}_{ij} \frac{\partial \psi}{\partial x_j} n_i$$

Also, (9) may be written in the form

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left[g \frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right] = \overline{\alpha} g \frac{\partial\left(g^{-1/2}\psi\right)}{\partial t}$$

which can be simplified

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g^{1/2} \frac{\partial \psi}{\partial x_j} + g\psi \frac{\partial g^{-1/2}}{\partial x_j}\right) = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i} = -g^{-1} \frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij} \frac{\partial}{\partial x_i}\left(g^{1/2} \frac{\partial \psi}{\partial x_j} - \psi \frac{\partial g^{1/2}}{\partial x_j}\right) = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Rearrranging and neglecting the zero terms gives

$$g^{1/2} \overline{\kappa}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \overline{\kappa}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = \overline{\alpha} g^{1/2} \frac{\partial \psi}{\partial t}$$

Equation (8) then implies

$$\overline{\kappa}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \lambda \psi = \overline{\alpha} \frac{\partial \psi}{\partial t} \quad (12)$$

Taking the Laplace transform of (10), (11), (12) and applying the initial condition (3) we obtain

$$\psi^*(\mathbf{x}, s) = g^{1/2}(\mathbf{x})\, \mu^*(\mathbf{x}, s) \quad (13)$$

$$P_{\psi^*}(\mathbf{x}, s) = \left[P^*(\mathbf{x}, s) + P_g(\mathbf{x})\, \psi^*(\mathbf{x}, s)\right] g^{-1/2}(\mathbf{x}) \quad (14)$$

$$\overline{\kappa}_{ij} \frac{\partial^2 \psi^*}{\partial x_i \partial x_j} - (\lambda + s\overline{\alpha})\, \psi^* = 0 \quad (15)$$

where s is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (15) is given in the form

$$\eta(\mathbf{x}_0)\, \psi^*(\mathbf{x}_0, s) = \int_{\partial\Omega} \left[\Gamma(\mathbf{x}, \mathbf{x}_0)\, \psi^*(\mathbf{x}, s) - \Phi(\mathbf{x}, \mathbf{x}_0)\, P_{\psi^*}(\mathbf{x}, s)\right] dS(\mathbf{x}) \quad (16)$$

where $\mathbf{x}_0 = (a, b)$, $\eta = 0$ if $(a, b) \notin \Omega \cup \partial\Omega$, $\eta = 1$ if $(a, b) \in \Omega$, $\eta = \frac{1}{2}$ if $(a, b) \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a, b). The fundamental solution Φ in (16) is any solution of the equation

$$\overline{\kappa}_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} - (\lambda + s\overline{\alpha})\, \Phi = \delta(\mathbf{x} - \mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \overline{\kappa}_{ij} \frac{\partial \Phi(\mathbf{x}, \mathbf{x}_0)}{\partial x_j} n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\Phi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \ln R & \text{if } \lambda + s\overline{\alpha} = 0 \\ \frac{\imath K}{4} H_0^{(2)}(\omega R) & \text{if } \lambda + s\overline{\alpha} < 0 \\ \frac{-K}{2\pi} K_0(\omega R) & \text{if } \lambda + s\overline{\alpha} > 0 \end{cases} \quad (17)$$

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \frac{1}{R} \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \\ \frac{-\imath K\omega}{4} H_1^{(2)}(\omega R)\, \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \\ \frac{K\omega}{2\pi} K_1(\omega R)\, \overline{\kappa}_{ij} \frac{\partial R}{\partial x_j} n_i \end{cases}$$

$$\begin{cases} \text{if } \lambda + s\overline{\alpha} = 0 \\ \text{if } \lambda + s\overline{\alpha} < 0 \\ \text{if } \lambda + s\overline{\alpha} > 0 \end{cases}$$

where

$$K = \ddot{\tau}/D$$

$$\omega = \sqrt{|\lambda + s\overline{\alpha}|/D}$$

$$D = \left[\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\dot{\tau} + \overline{\kappa}_{22}\left(\dot{\tau}^2 + \ddot{\tau}^2\right)\right]/2$$

$$R = \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2}$$

$$\dot{x}_1 = x_1 + \dot{\tau} x_2$$

$$\dot{a} = a + \dot{\tau} b$$

$$\dot{x}_2 = \ddot{\tau} x_2$$

$$\dot{b} = \ddot{\tau} b$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\kappa}_{11} + 2\overline{\kappa}_{12}\tau + \overline{\kappa}_{22}\tau^2 = 0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (17) are given by

$$\begin{aligned} \frac{\partial R}{\partial x_1} &= \frac{1}{R}\left(\dot{x}_1 - \dot{a}\right) \\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}\left(\dot{x}_1 - \dot{a}\right)\right] + \ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2 - \dot{b}\right)\right] \end{aligned}$$

Use of (13) and (14) in (16) yields

$$\eta g^{1/2}\mu^* = \int_{\partial\Omega}\left[\left(g^{1/2}\Gamma - P_g\Phi\right)\mu^* - \left(g^{-1/2}\Phi\right)P^*\right]dS \tag{18}$$

This equation provides a boundary integral equation for determining μ^* and its derivatives at all points of Ω.

Knowing the solutions $\mu^*(\mathbf{x}, s)$ and its derivatives $\partial\mu^*/\partial x_1$ and $\partial\mu^*/\partial x_2$ which are obtained from (18), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $\mu(\mathbf{x}, t)$ and its derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The Stehfest formula is

$$\begin{aligned} \mu(\mathbf{x}, t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \mu^*(\mathbf{x}, s_m) \\ \frac{\partial\mu(\mathbf{x}, t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial\mu^*(\mathbf{x}, s_m)}{\partial x_1} \\ \frac{\partial\mu(\mathbf{x}, t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial\mu^*(\mathbf{x}, s_m)}{\partial x_2} \end{aligned} \tag{19}$$

where

$$\begin{aligned} s_m &= \frac{\ln 2}{t} m \\ V_m &= (-1)^{\frac{N}{2}+m} \times \\ &\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m, \frac{N}{2}\right)} \frac{k^{N/2}(2k)!}{\left(\frac{N}{2} - k\right)!k!(k-1)!(m-k)!(2k-m)!} \end{aligned}$$

A simple script is developed and embedded into the main FORTRAN code to calculate the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any number N. Table (I) shows the values of V_m for $N = 4, 6, 8, 10$.

TABLE I

VALUES OF V_m OF THE STEHFEST FORMULA FOR $N = 4, 6, 8, 10$

V_m	$N=4$	$N=6$	$N=8$	$N=10$
V_1	-2	1	$-1/3$	$1/12$
V_2	26	-49	145/3	$-385/12$
V_3	-48	366	-906	1279
V_4	24	-858	16394/3	$-46871/3$
V_5		810	$-43130/3$	505465/6
V_6		-270	18730	-236957.5
V_7			$-35840/3$	1127735/3
V_8			8960/3	$-1020215/3$
V_9				164062.5
V_{10}				-32812.5

IV. NUMERICAL EXAMPLES

In order to justify the analysis derived in the previous sections, we will consider several problems either as test examples of analytical solutions or problems without simple analytical solutions.

We assume each problem belongs to a system which is valid in given spatial and time domains and governed by equation (1) and satisfying the initial condition (3) and some boundary conditions as mentioned in Section II. The characteristics of the system which are represented by the coefficients $\kappa_{ij}(\mathbf{x}), \alpha(\mathbf{x})$ in equation (1) are assumed to be of the form (4) and (5) in which $g(\mathbf{x})$ is an exponential function of the form (6). The coefficients $\kappa_{ij}(\mathbf{x}), \alpha(\mathbf{x})$ may represent respectively the diffusivity or conductivity and the change rate of the unknown or dependent variable $\mu(\mathbf{x}, t)$.

Standard BEM with constant elements is employed to obtain numerical results. The value of N in (18) for the Stehfest formula is chosen to be $N = 10$. For a simplicity, a unit square (depicted in Figure 1) will be taken as the geometrical domain for all problems. A number of 320 elements of equal length, namely 80 elements on each side of the unit square, are used. And the time interval is chosen to be $0 \leq t \leq 5$. A FORTRAN script is developed to compute the solutions and a specific FORTRAN command is imposed to calculate the elapsed CPU time for obtaining the results as to measure the efficiency of the numerical procedure.

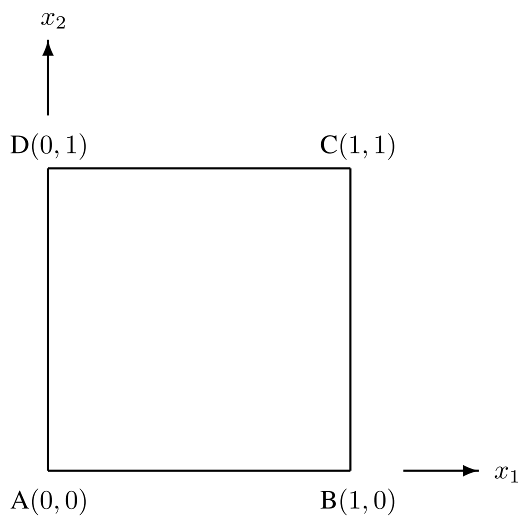

Fig. 1. The domain Ω

For all problems the inhomogeneity function is taken to be

$$g^{1/2}(\mathbf{x}) = \exp\left[-0.75 + 0.45x_1 + 0.3x_2\right]$$

and the constant anisotropy coefficient $\overline{\kappa}_{ij}$

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix}$$

so that 7 implies

$$\lambda = 0.3645$$

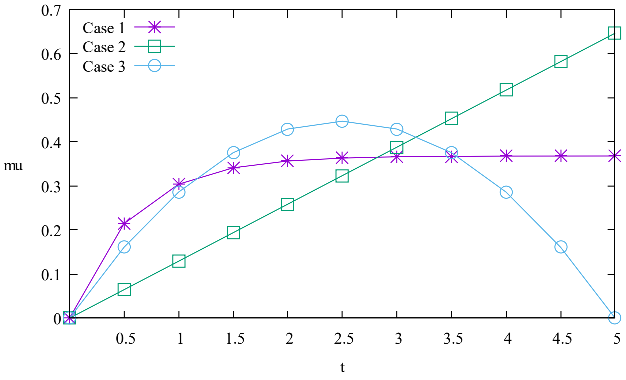

Fig. 2. Solutions μ at $(x_1, x_2) = (0.5, 0.5)$ for Problem 1

A. Examples with analytical solutions

1) Problem 1:: Another aspect that will be justified is the accuracy of the numerical solutions. The analytical solutions are assumed to take a separable variables form

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where $h(\mathbf{x}), f(t)$ are continuous functions. The boundary conditions are assumed to be (see Figure 1)

P is given on side AB
P is given on side BC
μ is given on side CD
P is given on side AD

Case 1:: We take

$$\begin{aligned} h(\mathbf{x}) &= 1 - 0.25x_1 - 0.75x_2 \\ f(t) &= 1 - \exp(-1.75t) \end{aligned}$$

Thus for $h(\mathbf{x})$ to satisfy (15)

$$\overline{\alpha} = -0.3645/s$$

Case 2:: For the analytical solution we take

$$\begin{aligned} h(\mathbf{x}) &= \cos(1 - 0.25x_1 - 0.75x_2) \\ f(t) &= t/5 \end{aligned}$$

So that in order for $h(\mathbf{x})$ to satisfy (15)

$$\overline{\alpha} = -1.04575/s$$

Case 3:: We take

$$\begin{aligned} h(\mathbf{x}) &= \exp(-1 + 0.25x_1 + 0.75x_2) \\ f(t) &= 0.16t(5 - t) \end{aligned}$$

Therefore (15) gives

$$\overline{\alpha} = 0.31675/s$$

Table II shows the accuracy of the numerical solutions μ and the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ solutions in the domain. For the Case 1, 2 and 3 the errors mainly occur in the fourth decimal place for the $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ solutions.

TABLE II
COMPARISON OF THE NUMERICAL (NUM) AND THE ANALYTICAL (ANAL) SOLUTIONS AT $(x_1, x_2) = (0.5, 0.5)$ FOR PROBLEM 1

t	μ		$\frac{\partial\mu}{\partial x_1}$		$\frac{\partial\mu}{\partial x_2}$	
	Num	Anal	Num	Anal	Num	Anal
Case 1						
0.0005	0.0003	0.0003	-0.0001	-0.0001	-0.0004	-0.0004
0.5	0.2147	0.2149	-0.0885	-0.0885	-0.2781	-0.2782
1.0	0.3041	0.3044	-0.1254	-0.1254	-0.3939	-0.3941
1.5	0.3414	0.3418	-0.1407	-0.1408	-0.4420	-0.4425
2.0	0.3568	0.3573	-0.1472	-0.1472	-0.4623	-0.4626
2.5	0.3635	0.3638	-0.1498	-0.1499	-0.4711	-0.4710
3.0	0.3664	0.3665	-0.1510	-0.1510	-0.4745	-0.4745
3.5	0.3674	0.3677	-0.1515	-0.1515	-0.4760	-0.4760
4.0	0.3679	0.3681	-0.1518	-0.1517	-0.4766	-0.4766
4.5	0.3681	0.3683	-0.1517	-0.1517	-0.4766	-0.4768
5.0	0.3682	0.3684	-0.1517	-0.1518	-0.4772	-0.4769
Case 2						
0.0005	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.5	0.0647	0.0647	0.0145	0.0145	0.0398	0.0398
1.0	0.1294	0.1293	0.0291	0.0290	0.0795	0.0796
1.5	0.1941	0.1940	0.0436	0.0436	0.1193	0.1193
2.0	0.2588	0.2587	0.0581	0.0581	0.1590	0.1591
2.5	0.3235	0.3234	0.0726	0.0726	0.1988	0.1989
3.0	0.3882	0.3880	0.0872	0.0871	0.2385	0.2387
3.5	0.4528	0.4527	0.1017	0.1017	0.2783	0.2785
4.0	0.5175	0.5174	0.1162	0.1162	0.3180	0.3182
4.5	0.5822	0.5820	0.1307	0.1307	0.3578	0.3580
5.0	0.6469	0.6467	0.1453	0.1452	0.3976	0.3978
Case 3						
0.0005	0.0002	0.0002	0.0001	0.0001	0.0002	0.0002
0.5	0.1610	0.1609	0.0544	0.0544	0.1537	0.1537
1.0	0.2862	0.2861	0.0967	0.0967	0.2733	0.2733
1.5	0.3756	0.3754	0.1269	0.1269	0.3587	0.3587
2.0	0.4293	0.4291	0.1451	0.1450	0.4100	0.4099
2.5	0.4472	0.4470	0.1511	0.1511	0.4271	0.4270
3.0	0.4293	0.4291	0.1451	0.1450	0.4100	0.4099
3.5	0.3757	0.3754	0.1269	0.1269	0.3588	0.3587
4.0	0.2863	0.2861	0.0967	0.0967	0.2734	0.2733
4.5	0.1611	0.1609	0.0544	0.0544	0.1539	0.1537
5.0	0.0002	0.0000	0.0001	0.0000	0.0002	0.0000

Figure 2 shows a variation of the μ solution values at point $(x_1, x_2) = (0.5, 0.5)$ as the time increases from $t = 0.0005$ to $t = 5$. As expected, the variation follows the way the associated function $f(t)$ changes. Specifically for the Case 1 of associated function $f(t) = 1 - \exp(-1.75t)$ the μ solution will tend to approach a steady state solution. This is also expected, as the function $f(t) = 1 - \exp(-1.75t)$ will converge to 1 as t gets bigger.

The elapsed CPU time for the computation of the numerical solutions at 19×19 spatial positions and 11 time steps is 4534.875 seconds for the Case 1, 6466.40625 seconds for the Case 2, and 2443.078125 seconds for the Case 3. The longer computation time for the Cases 1 and 2 is produced by the iterative calculation of the polynomial approximation of the Hankel and Bessel functions in the fundamental solutions (17).

B. Examples without analytical solutions

The aim is to show the effect of inhomogeneity and anisotropy of the considered material on the solution μ.

1) Problem 2:: The material is supposed to be either inhomogeneous or homogeneous and either anisotropic or isotropic. If the material is homogeneous then

$$g(\mathbf{x}) = 1$$

and if it is isotropic then

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So that there are four cases regarding the material, namely anisotropic inhomogeneous, anisotropic homogeneous, isotropic inhomogeneous and isotropic homogeneous material. The corresponding value of λ for each case is obtained from equation (7). We set $\overline{\alpha} = 1$ and the boundary conditions are (see Figure 1)

$$\begin{aligned} P &= f(t) \text{ on side AB} \\ P &= 0 \text{ on side BC} \\ \mu &= 0 \text{ on side CD} \\ P &= 0 \text{ on side AD} \end{aligned}$$

Four cases of the function $f(t)$ will be considered, namely

$$\begin{aligned} &\text{Case 1: } f(t) = 1 \\ &\text{Case 2: } f(t) = 1 - \exp(-1.75t) \\ &\text{Case 3: } f(t) = t/5 \\ &\text{Case 4: } f(t) = 0.16t(5-t) \end{aligned}$$

In fact, for the case of isotropic and homogeneous material the system is geometrically symmetric about the axis $x_1 = 0.5$, but asymmetric about $x_2 = 0.5$ (see Figure 3). Figure 3 also indicates that the solution μ tends to follow the variation of the function $f(t)$ associated for the boundary condition on the side AB. In addition, Figure 4 also shows the effect of anisotropy and inhomogeneity on the asymmetry of the solution μ.

Figure 5 shows again the effect of anisotropy and inhomogeneity on the solution μ and the tendency of the solution μ to agree the variation of the corresponding function $f(t)$. In particular, for bigger t the boundary conditions on the side AB with $f(t) = f_1(t) = 1$ and $f(t) = f_2(t) = 1 - \exp(-1.75t)$ are identical. This is verified by the results in Figure 5, the two plots for the cases when $f(t) = f_1(t) = 1$ and $f(t) = f_2(t) = 1 - \exp(-1.75t)$ will coincide as t goes to infinity.

After all, the results suggest it is important to put the anisotropy and inhomogeneity into account in any practical application.

V. Conclusion

A combined Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic functionally graded materials which are governed by the equation (1). It is easy and accurate. It involves a time variable free fundamental solution and therefore that is why it is more accurate. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution usually has singular time points and the procedure may involve round error propagation.

It has been applied to three classes of functionally graded materials, namely quadratically, exponentially and trigonometrically graded materials. As the coefficients $\kappa_{ij}(\mathbf{x}), \alpha(\mathbf{x})$ do depend on the spatial variable $\mathbf{x}$ only and on the same inhomogeneity or gradation function $g(\mathbf{x})$, it is interesting to extend the study in the future to the case when the coefficients depend on different gradation functions varying also with the time variable t.

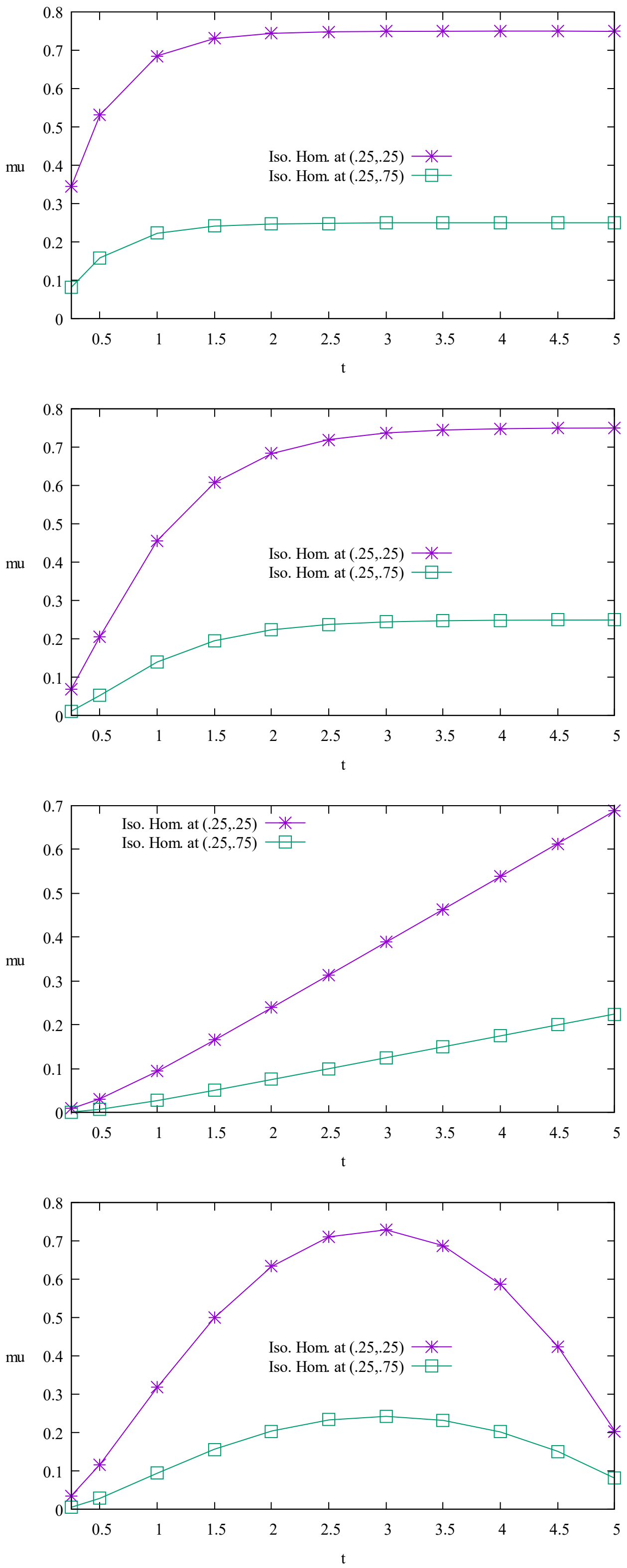

Fig. 3. Asymmetry of solution μ about $x_2 = 0.5$ for Case 1 (first row), Case 2 (second row), Case 3 (third row), Case 4 (fourth row) of Problem 2

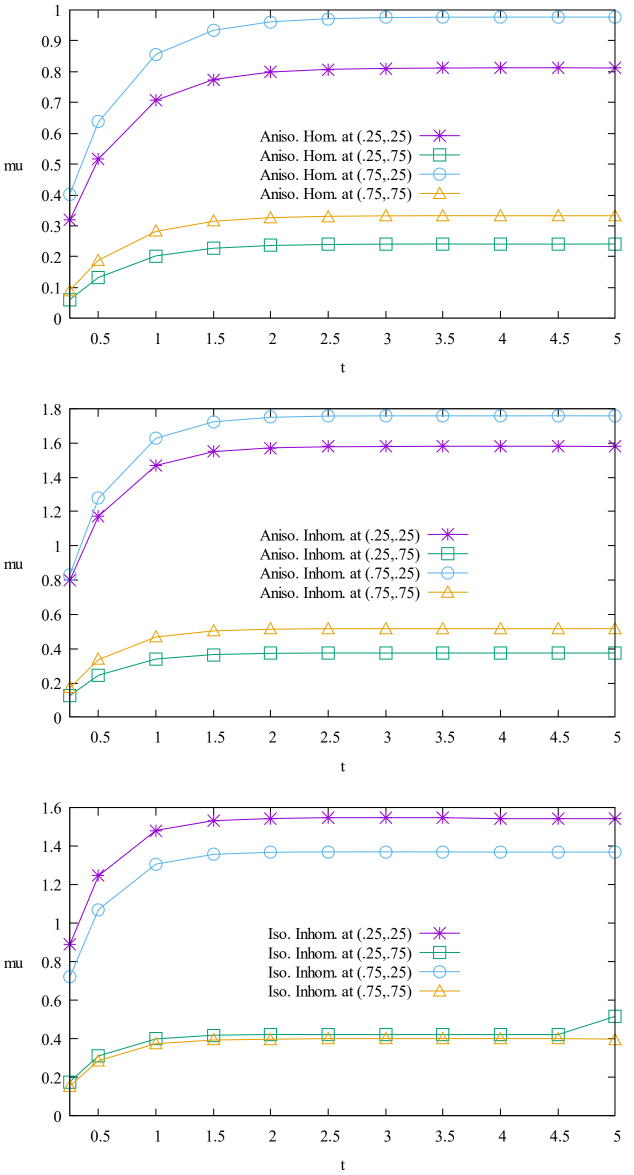

Fig. 4. The effect of anisotropy and inhomogeneity on the asymmetry of the solution μ for Case 1 of Problem 2

In order to use the boundary integral equation (18), the values $\mu(\mathbf{x}, t)$ or $P(\mathbf{x}, t)$ of the boundary conditions as stated in Section (II) of the original system in time variable t have to be Laplace transformed first. This means that from the beginning when we set up a problem, we actually put a set of approached boundary conditions. Therefore it is really important to find a very accurate technique of numerical Laplace transform inversion.

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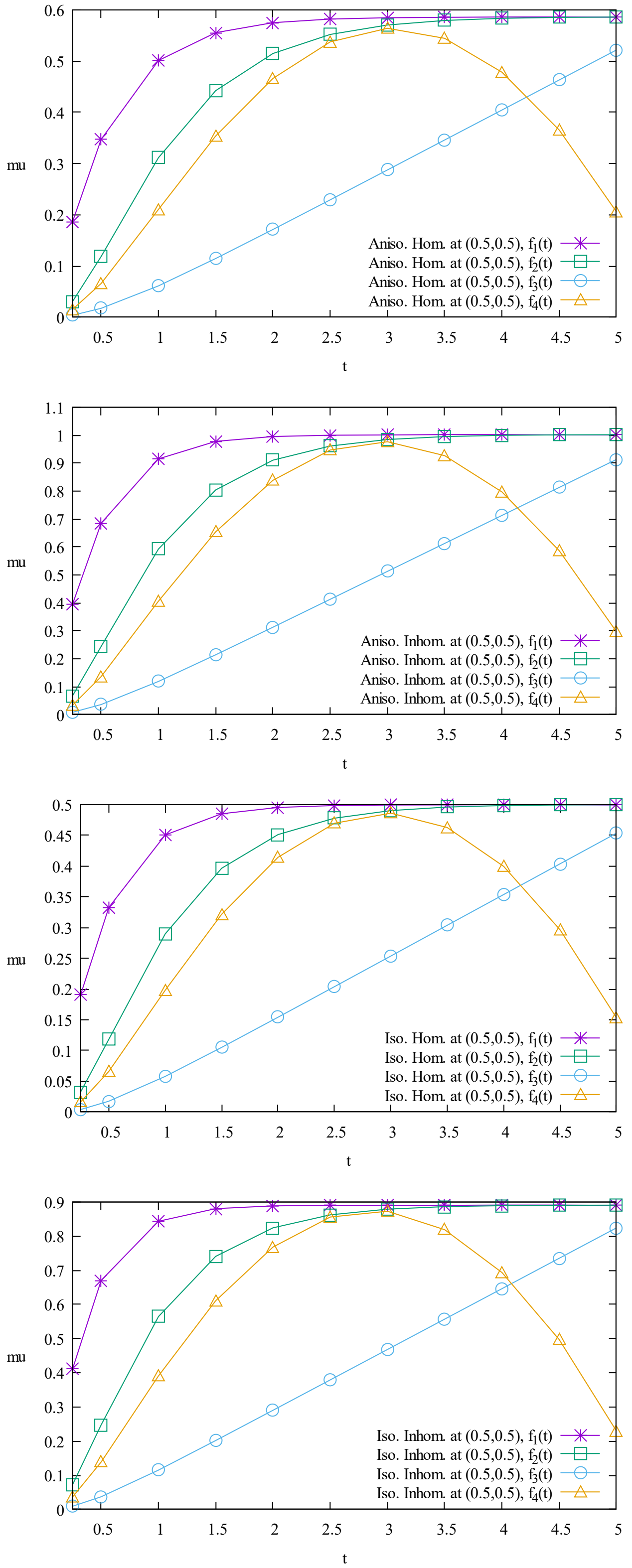

Fig. 5. Solutions μ at $(x_1, x_2) = (0.5, 0.5)$ for Problem 2

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