A New Mathematical Model of Slope Stability Analysis

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Abstract**—At present there are many existing methods to obtain the safety factor of slope, although the most used, due to its simplicity, are the Bishop's simplified method, the Spencer's method and the Morgenstern and Price procedure among others. However, the search for a method that facilitates even more the calculations has given rise to an endless number of new procedures. In this paper, a new method based on Mohr-Coulomb failure criterion is presented. A brief review is presented on analysis procedures for this class of problem. This new method is valid for both homogeneous and heterogeneous dams, and it produces accurate values factor of safety as has been proven in real projects by the authors.**

Index Terms**—Terzaghi principle, limit equilibrium, Mohr-Coulomb, factor of safety**

I. Introduction

According to [1], over several decades, the limit equilibrium method has almost dominated the profession for examining the stability of slopes, embankments, and other soil and rock structures. In this method, a slope is usually assumed to fail along a distinct failure surface that could be of circular or noncircular form [2]. When dealing with nonhomogeneous and irregular slopes, the method-of-slices is most often adopted, wherein a potential failure surface is defined and the slope is subdivided into slopes [3-7].

In 1977, [8] compared several methods-of-slices and showed that the manner in which inter-slice forces are handled has an impact on the global factor of safety.

In the one hand, [8] specify that research into the development of the method-of-slices has been along two fronts: the development of algorithms for identifying the critical failure surface corresponding to the minimum factor of safety [9-16]; and refining the simplifications with regards to satisfying equilibrium conditions on the local and global levels. A difficulty that arises is the presence of multiple local minima points, although there is no guarantee that the failure surface that has been identified corresponds to the global minimum factor of safety.

In another vein, [17] developed a method where the factor of safety is formulated as a multivariable function F(x) with the independent variables x describing the geometry of the failure surface. [15] proposed a more efficient one-dimensional optimization technique to replace the quadratic interpolation method used by Celestina and Duncan, as [9] specify. Moreover, [16] defined the failure surface by a number of nodal points connected by straight lines. The vertical coordinates of the nodal points are the variables in his method and the dynamic programming technique is employed as the optimization method. [18] defined a global optimization algorithm for finding the critical failure surface by nodal points connected by straight lines for any shape of failure. A large number of computations are needed to find the critical failure surface, as an arbitrary nodal coordinate could be irrelevant among the rest of created nodal coordinates [9].

Although the slope stability analysis has focused on the limit equilibrium method, only few studies have looked at new methods to simplify the calculation of the factor of safety. For this reason, in this paper a new mathematical method, that simplifies even more the process of calculating the slope stability analysis, is developed.

II. Presentation

A. Literature Review

As [1] point out, the calculation of slope stability consists in determining a safety factor taking into account all aspects that affect to the slope stability before being carried out.

As is known, all slice-based methods (Fellenius, Bishop, Janbu, Spencer, Lowe and Karafiath, and Morgenstern-Price) are statically indeterminate, so they require of several assumptions to solve the problem. However, the methods of analysis differ from each other in relation to the equilibrium equations used both globally and at the slice level. In this sense, and according to [19], the simplified Bishop method takes into account inter-slices normal forces, although it does not use inter-slices shear forces. On the other hand,

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Morgenstern-Price method consider both normal and tangential equilibrium as well as the moment equilibrium for each slice in non-circular and circular surfaces [20].

[18] establish that Janbu's method assumes that the inter-slice force has a horizontal direction whereas Spencer's method assumes an inter-slice force constant for all slices. Both methods include the effect of pore water by taking the water pressure to be hydrostatic below the phreatic surface. For this reason, seepage forces are ignored.

In reference to the Lowe and Karafiath method, it is known that the direction of the resultant of each inter-slice force is assumed to be equal to the average of the surface and slip surface slopes [8].

In reference to Fellenius method, it is generally considered that this method assumes that the increment of inter-slice forces through a slice is parallel to the base of the slice [21], although several authors think the method assumes no inter-slice forces [3, 10-11, 18-19].

On the other hand, Monte-Carlo is a simulation method based on the knowledge of the joint probability density function of the random variables, $f_{x1,x2,\ldots,xn}(x_1, x_2,\ldots, x_n)$, in addition to knowing precisely the integration domain defined by $g^+(x_1, x_2,\ldots, x_n) \leq 0$ [11, 13].

B. Theoretical base

According to [22], the behavioral model of the slopes of these hydraulic structures, in relation to limit equilibrium, it is based on the study of the global equilibrium of mass slope in unstable conditions. For this reason, it is necessary establish the following assumptions: a) to define the allegedly unstable surface; b) to define the acting external forces; and c) to calculate the water pressure in the failure surface.

With these conditions, the safety factor of slope (F) can be defined as follows (1):

$$F = \frac{R}{T} \tag{1}$$

where (R) is the force resulting from the maximum shear strength actually available along the failure surface, and (T) is the resulting force for shear strength which must be mobilized along the potentially unstable surface at the time that movement starts.

If we consider a point (A) contained in a plane (π) (Fig. 1), the breakage in said plane (π) occurs when shearing stress reaches the value given by (2):

$$\tau = c + [\sigma \cdot tg(\phi)] \tag{2}$$

where (τ) is the shear stress, (c) is the cohesion intercept, (σ) is the normal stress, and (ϕ) is the internal angle of friction.

However, the shear strength of a saturated soil (τ') is defined by a linear Mohr-Coulomb failure criterion, thus, in terms of effective stress, (2) can be written as follows (3):

$$\tau' = c' + [\sigma' \cdot tg(\phi')] \tag{3}$$

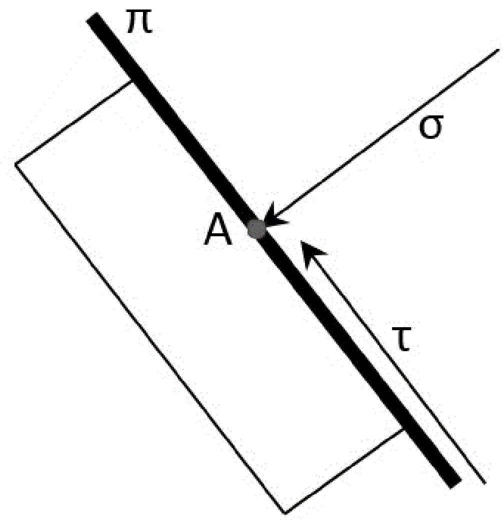

Fig. 1. Scheme of stresses produced in an inner floor element.

where (c') is the effective cohesion intercept, (σ') is the effective normal stress, and (ϕ') is the effective angle of internal friction.

(1) is obtained as a consequence of Terzaghi principle. For this reason, as it is known, Mohr circles are displaced a distance equal to interstitial pressure (u).

III. MATHEMATICAL FORMULATION

Consider a dam with a maximum flood level (h) and a leeway of full height (r), such that the total height (H) of the dam is given by (4):

$$H = h + r \tag{4}$$

Suppose that the slope angle upstream of the dam is (β).

According to [22-23], in the deformed section of the dam is produced a filtration network that exerts its effect on the part of the slope in contact with the water surface. We also know that the filtration lines have an incidence angle of 90º with the slope, to later end up having a parabolic trajectory that reaches the superstructure filter.

As is known, there are two areas to consider for calculating the safety factor of slope. The first one is located in contact with water at a height (h), where effective stress will be used to carry out the calculation. The second one is the leeway (r) which although is not in all times in contact with water, has to be considered the opposite to make the calculation from the point of view most unfavorable.

During the construction of a dam, and in the process of compaction of the upper layers, the lower layers are setting increasingly, resulting in a circle of Mohr smaller the closer we get to the last layer.

As shown in Fig. 2, on the failure plane (AB), located at a depth (z), there will be a slice bounded by two vertical planes, the slip plane and the plane of the ground surface.

In these vertical planes take places a forces (E), equal and in opposite direction, which are parallel to the failure plane defined by (AB). If we consider (W) the slice weight, its

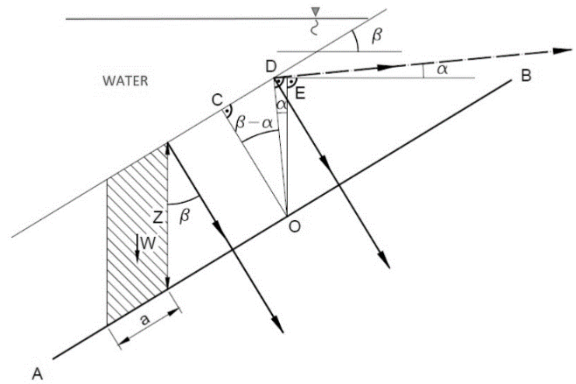

Fig. 2. Diagram of the water filtration network in a dam.

balance requires that the reaction at the base is equal, and in opposite direction, to the weight. This reaction will have a normal component (N) and another (T) parallel to the failure plane that is mobilized by shear strength.

The total normal component (N) will be the sum of the resulting effective pressures at the base (N') and the resulting interstitial pressures (U), also, at the base.

Knowing the specific weight (γ) of the ground, and as shown in Fig. 2, the slice weight (W) is given by (5):

$$W = \gamma \cdot z \cdot a \cdot \cos(\beta) \tag{5}$$

And the value of its components (6) and (7):

$$N = W \cdot \cos(\beta) = \gamma \cdot z \cdot a \cdot cos^2(\beta) \tag{6}$$

$$T = W \cdot \sin(\beta) = \gamma \cdot z \cdot a \cdot co\,s(\beta) \cdot \sin(\beta) \tag{7}$$

To obtain both shear (τ) (8) and normal (σ) (9) stress, it will suffice to divide equations (6) and (7) by (a):

$$\sigma = \gamma \cdot z \cdot cos^2(\beta) \tag{8}$$

$$\tau = \gamma \cdot z \cdot co\,s(\beta) \cdot \sin(\beta) \tag{9}$$

If (u) is the value of the interstitial pressure in the failure plane, so that (U) is constant, and based on the Terzaghi principle, the maximum shear stress (10) that can occur at the base of the slice will be:

$$\tau_{max} = c' + [(\sigma - u) \cdot tg(\phi')] \tag{10}$$

Which provides a value of (R) (11) equal to:

$$R = a \cdot \{c' + [(\sigma - u) \cdot tg(\phi')]\} \tag{11}$$

Substituting this value into (1) yields (12):

$$F = \frac{a \cdot \{c' + [(\sigma - u) \cdot tg(\phi')]\}}{T} \tag{12}$$

If we now substitute (7) and (8) in (12) and simplify, we obtain (13):

$$F = \frac{c' + [(\gamma \cdot z \cdot cos^2(\beta) - u) \cdot tg(\phi')]}{\gamma \cdot z \cdot co\,s(\beta) \cdot \sin(\beta)} \tag{13}$$

As is known, the existence of a water filtration network in a dam requires determining the value of the interstitial pressure (u) in the sliding plane. If water filtration network is rectilinear, which does not happen in reality (in Fig. 2, the filtration network appears as discontinuous line) it would cut at the point (D) of the dam (see Fig. 2).

As both (D) and (O) points are located on the same equipotential line (Fig. 2), we can write (14):

$$h_D = z_D + \left(\frac{u_D}{\gamma_w}\right) = h_O = z_O + \left(\frac{u_O}{\gamma_w}\right) \tag{14}$$

As point (D) is on the surface of the dam slope, we can consider that the interstitial pressure in (D) is zero, which allows us to express the interstitial pressure at the point (O) as (15):

$$h_O = (z_D - z_O) \cdot \gamma_w \tag{15}$$

Where the difference (z_D - z_O) is the segment (EO) of Fig. 2, which can be written as follows (16, 17 and 18):

$$EO = DO \cdot \cos(\alpha) \tag{16}$$

$$DO = \frac{CO}{\cos(\beta - \alpha)} \tag{17}$$

$$CO = z \cdot co\,s(\beta) \tag{18}$$

And substituting (17) and (18) into (16) yield (19):

$$EO = \frac{z \cdot \cos(\beta) \cdot \cos(\alpha)}{\cos(\beta - \alpha)} \tag{19}$$

And by introducing (19) into (15) we have the following interstitial pressure value (u) (20):

$$u = \frac{\gamma_W \cdot z \cdot \cos(\beta) \cdot \cos(\alpha)}{\cos(\beta - \alpha)} \tag{20}$$

Finally, substituting (20) into (13), and taking into account that the water filtration network is perpendicular to the slope upstream of the dam, which means that α = (90 –β), and after simplifying and introducing an average shape factor called (JER) obtained as a result of the OTR2010-PC36 research project, in order to homogenize the different typologies of existing dams, we obtain the new safety factor of slope (F) defined as follows (21):

$$F = JER \cdot \sum_{i=1}^{n} [F_1 + F_2] \tag{21}$$

where (n) is the number of slices in which the slope of the dam is divided.

It is very important to know that when water table is not taken into account, we have to use (22) to obtain (21), while (23) will be used in opposite case. The values of coefficient F_1 (24) and F_2 (25) are shown below:

$$JER_{WFL} = 0.182 \tag{22}$$

$$JER_{FL} = 0.1613 \tag{23}$$

$$F_1 = \frac{c\prime}{\gamma \cdot z \cdot \cos(\beta) \cdot \sin(\beta)} \tag{24}$$

$$F_2 = tg(\phi') \cdot \left(\frac{\cos(2\beta+90)}{tg(\beta)} - \frac{\cos(90-\beta)}{\cos(2\beta+90)}\right) \tag{25}$$

IV. Examples of Application

In order to check this new method called Ra-Ma (**Ra**mírez-**Ma**dueño), it was applied to four examples to find, and comparing it with the classic methods (Fellenius, Bishop and Janbu), the most critical failure surface. Soil parameters used in each example are shown in table I. All examples consisted of a slope with height 10 m, slope angle of 1.75:1 and different types of soil.

TABLE I
SOIL PARAMETERS USED IN THE EXAMPLES

	Soil no.	Effective cohesion (kPa)	Effective angle of friction (º)	Unit weight (kN/m^3)
Example 1	1	10	15	20.2
	2	10	15	19.6
Example 2	1	6	24	19.6
	2	6	24	20.2
Example 3	1	28.73	20	19.6
Example 4	1	5	35	17

Safety factors for all example problems are shown in table II. As can be observed, all methods obtained a safety factor value higher than one in each example, although in the first case could be convenient a little variation of dam height or slope angle to obtain a safety factor more favorable. In Fig. 3, it is shown the critical failure surface corresponding to example 4 for all methods specified in table II.

TABLE II
SAFETY FACTOR FOR EXAMPLE PROBLEMS

Method	Ex. 1	Ex. 2	Ex. 3	Ex. 4
Fellenius	1.01	1.33	1.87	1.91
Bishop	1.113	1.512	2.002	2.187
Janbu	1.023	1.38	1.82	1.99
Ra-Ma	1.09	1.48	1.97	2.15

V. Discussion

As is known, all slice-based methods (Fellenius, Bishop, Janbu, Spencer, Lowe and Karafiath, and Morgenstern-Price) are statically indeterminate, so they require of several assumptions to solve the problem. However, the methods of analysis differ from each other in relation to the equilibrium equations used both globally and at the slice level. In this sense, and according to [19], the simplified Bishop method takes into account inter-slices normal forces, although it does not use inter-slices shear forces. Otherwise, Morgenstern-Price method consider both normal and tangential equilibrium as well as the moment equilibrium for each slice in non-circular and circular surfaces [20].

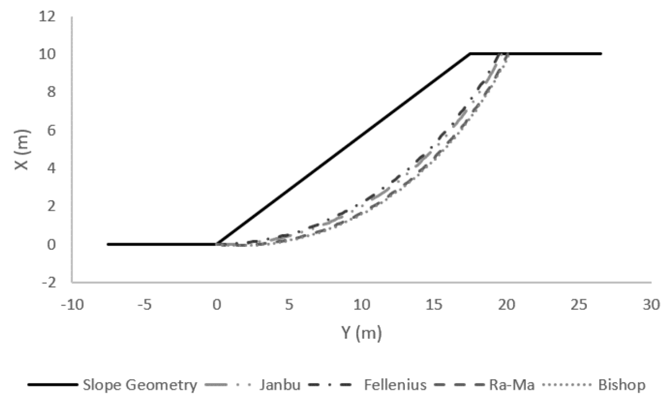

Fig. 3. Critical failure surfaces for each method of the example 4.

On the other hand, [18] establish that Janbu's method assumes that the inter-slice force has a horizontal direction whereas Spencer's method assumes an inter-slice force constant for all slices. Both methods include the effect of pore water by taking the water pressure to be hydrostatic below the phreatic surface. For this reason, seepage forces are ignored. Moreover, it is known, in reference to the Lowe and Karafiath method [8], that the direction of the resultant of each inter-slice force is assumed to be equal to the average of the surface and slip surface slopes.

In reference to Fellenius method, it is generally considered that this method assumes that the increment of inter-slice forces through a slice is parallel to the base of the slice, although several authors think the method assumes no inter-slice forces [3, 10-11, 18-19].

Unlike the previous methods, the new proposed methodology simplifies the required calculus being a method valid to obtain the factor of safety in hydraulic infrastructures using variables collected easily. In relation to the number of iterations required to carry out a satisfactory calculus of the safety factor of slope, it is important to take into account that when the slope angle is between 20º and 60º, a low number of computations are required to obtain the best solution.

As can be seen in table II, the new proposed method "Ra-Ma" obtains intermediate safety factor values between the other three methods (Fellenius, Bishop and Janbu).

VI. Conclusions

In the previous sections a new method was presented to obtain the critical failure surface during slope stability analysis. From the results shown it may be concluded that there is a clear consistency in this new method to calculate the safety factor.

(21), obtained by dimensionless calculation, is much simpler than all the others that currently exist, which facilitates calculation. Used in a total of 34 dam construction projects has revealed the great importance of shape factors in the calculation of this type of superstructures. As future studies, it could be interesting to test the new equation in environments with strong seismicity, where the dynamic acceleration reaches values much higher than the maximum allowed by the construction regulations of each country.

In these cases, with strong seismicity, can be useful the use of B-spline curve interpolation model [24] because structure movement under dynamic forces could be adjusted to this kind of curves.

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