New Fifth Order Iterative Method for Finding Multiple Root of Nonlinear Function

Waikhom Henarita Chanu, Sunil Panday, and Mona Dwivedi

***Abstract*—In this paper, we have established a new fifth order iterative method for finding multiple roots of nonlinear univariate function with known multiplicity. Many researchers have generated several order techniques, whenever the second and the higher-order derivatives of the function exist in a neighborhood of the root. But the cost of evaluating the second derivative of the function is itself a cumbersome problem. The proposed iteration technique does not require the evaluation of second and higher-order derivatives. We used the weight function approach to derive the proposed technique. The convergence analysis of the proposed method is described exhaustively to reveal the fifth order convergence. Programs are developed in Mathematica 12.2 software to demonstrate the efficacy of the proposed method over the existing method on several numerical test functions. In addition, the presented CPU-time also confirms the improved performance of the proposed methods as compared to some standard iterative methods in the literature.**

***Index Terms*—Nonlinear equations, Iterative methods, Order of convergence, Error.**

I. Introduction

FINDING the roots of nonlinear equations are most challenging problems in Numerical Analysis. But it has many applications in engineering and scientific computations [1]–[3]. Analytical methods to solve such equations are rarely available and therefore, it is indispensable to obtain approximate solutions based on iterative methods [1], [2], [4]. The iterative method for finding roots of the following nonlinear equation

$$\Psi(x) = 0 \tag{1}$$

where $\Psi : D \subset R \to R$ is a nonlinear differentiable univariate function defined on an open interval D subset of the set of real numbers R, is to begin with any initial approximation of the root and generate a sequence of approximations of the solution. The Newton-Raphson iterative methods for finding simple roots of univariate function [5], [6] is defined as:

$$x_{n+1} = x_n - \frac{\Psi(x_n)}{\Psi'(x_n)} \tag{2}$$

It is a widely known quadratically convergent scheme to obtain the simple roots of the non-linear equation but linearly convergent for equation having multiple roots. Let r be a multiple roots of $\Psi(x) = 0$ with multiplicity m, i.e., $\Psi^{(j)}(r) = 0,\ j = 0, 1, 2, ..., m-1$ and $\Psi^{(m)}(r) \neq 0$. The construction of new iterative methods for multiple roots is also one of the challenging problems in Numerical Analysis. The modified Newton method for finding multiple roots is written as [7]:

$$x_{n+1} = x_n - m\frac{\Psi(x_n)}{\Psi'(x_n)} \tag{3}$$

The iterative method given in equation (3) is quadratically convergent for multiple roots [7]. Many researchers worldwide have developed multipoint iterative methods based on modifications of the Newton method for finding the root of nonlinear equations [8]–[11]. In 2011, Zhou et. al. [3] had also developed a new fourth-order method. Kim and Geum [12] suggested a new method for finding multiple roots. Soleymani and Babajee [13] proposed a fourth-order method for finding multiple roots. In 2020, a family of Chebyshev's method was developed by M. Barrada et. al. [14]. Barrada and Benkjouya had developed a third-order family of Halley's methods [15].

The third-order Chebyshev's method (CM) for finding multiple roots is written as [6], [16]:

$$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{\Psi(x_n)}{\Psi'(x_n)} - \frac{m^2}{2}\frac{\Psi(x_n)^2\Psi''(x_n)}{\Psi'(x_n)^3} \tag{4}$$

In 2012, Thukral [17] developed a fifth-order method (TM1), which is written as follows:

$$\begin{aligned} y_n &= x_n - m\frac{\Psi(x_n)}{\Psi'(x_n)} \\ x_{n+1} &= y_n - m\left(1 + \left(\frac{\Psi(y_n)}{\Psi(x_n)}\right)^{\frac{2}{m}}\right)\frac{\Psi(y_n)}{\Psi'(y_n)} \end{aligned} \tag{5}$$

In 2013, Thukral [18] developed another fifth-order method (TM2), which is written as follows:

$$\begin{aligned} y_n &= x_n - m\frac{\Psi(x_n)}{\Psi'(x_n)} \\ z_n &= x_n - m\left(\sum_{i=1}^{3} i\left(\frac{\Psi(y_n)}{\Psi(x_n)}\right)^{\frac{i}{m}}\right)\left(\frac{\Psi(x_n)}{\Psi'(x_n)}\right) \\ x_{n+1} &= z_n - m\left(\frac{\Psi(z_n)}{\Psi(x_n)}\right)^{\frac{1}{m}}\left(\frac{\Psi(x_n)}{\Psi'(x_n)}\right) \end{aligned} \tag{6}$$

In 2019, Bhel and Al-Hamadan [19] had developed the following optimal fourth order method (BAM) for finding multiple roots:

$$\begin{aligned} y_n &= x_n - m\frac{\Psi(x_n)}{\Psi'(x_n)} \\ z_n &= x_n - m\frac{\Psi(x_n)}{\Psi'(x_n)}\left(\frac{1-\mu}{1-2\mu}\right)Q(\mu) \end{aligned} \tag{7}$$

where $\mu = (\Psi(y_n)/\Psi(x_n))^{\frac{1}{m}}$ and $Q(\mu)$ is weight function. Inspired by the recent remarkable activities in this direction, we propose an elegant way to achieve a fifth-order iterative method for finding multiple roots of nonlinear equations.

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In this paper, we have constructed the new fifth-order iterative method with the first step different from the traditional Newton step. This creates an entirely new approach for finding multiple roots of nonlinear equations. The analysis of convergence of newly proposed methods is studied to reveal the fifth-order convergence. We have demonstrated the performance of the newly proposed method on several numerical examples. It is observed that the newly proposed methods have better numerical results as compare to some standard methods available in the literature.
The remaining portion of this paper is organized as follows: In section II, we have developed a new fifth-order iterative method using the weight function approach and Taylor's expansion approach for finding multiple roots of a nonlinear equation. The theoretical proof of convergence is also provided in this section. In section III, we have compared existing methods with the newly proposed methods by taking several numerical examples to show the better performance of the newly proposed fifth-order method in solving the nonlinear equation (1). Some concluding remarks are presented in section IV.

II. Construction of new scheme and convergence analysis

In this section, we introduce a new fifth order iteration methods with multiplicity $m \geq 1$ as follows:

$$\begin{aligned} y_n &= x_n + m\frac{\Psi(x_n)}{\Psi'(x_n)} \\ z_n &= x_n - \frac{m}{2^m}\frac{\Psi(y_n)}{\Psi'(x_n)} \\ x_{n+1} &= z_n - mW(h)\frac{\Psi(z_n)}{\Psi'(z_n)} \end{aligned} \quad (8)$$

where $W(h)$ is real valued weight function with

$$h = 2^m\frac{\Psi(x_n)}{\Psi(y_n)}$$

Theorem 1: Let $x = \alpha$ be a multiple zero with multiplicity m of a functions Ψ: $R \to R$ in the region enclosing α. Then, the iterative methods defined by equation (8) has fifth-order convergence, if the following conditions are fulfilled:

$$W(1) = 1, W'(1) = 0, W''(1) = \frac{8}{m^2} - \frac{4}{m}$$

Then, the newly proposed method in equation (8) satisfies the following error equation:

$$\theta_{n+1} = \frac{-C}{96m^4}\theta_n^5 + O[\theta_n]^6 \quad (9)$$

where

$$\begin{aligned} C &= k_1^2(m-2)(96k_2m(m-3) \\ &\quad + k_1^2(72 + m(162 + m(mW'''(1) - 66)))) \end{aligned}$$

and $\theta_n = x_n - \alpha$ is the error at n^{th} iteration.

Proof: Since $x = \alpha$ be a multiple root of $\Psi(x)$ with multiplicity m. Let us assume that $\theta_n = x_n - \alpha$ be the error at the n^{th} iteration. We expand $\Psi(x_n)$ and $\Psi'(x_n)$ in powers of θ_n by Taylor's series expansion as follows:

$$\begin{aligned} \Psi(x_n) = \frac{\Psi^{(m)}(\alpha)}{m!}\theta_n^m\Big(&1 + \theta_n k_1 + \theta_n^2 k_2 + \theta_n^3 k_3 + \theta_n^4 k_4 \\ &+ \theta_n^5 k_5 + \theta_n^6 k_6 + O[\theta_n]^7\Big) \end{aligned} \quad (10)$$

and

$$\begin{aligned} \Psi'(x_n) = \frac{\Psi^{(m)}(\alpha)}{m!}\theta_n^{m-1}\Big(&m + (m+1)k_1\theta_n \\ &+ (m+2)k_2\theta_n^2 + (m+3)k_3\theta_n^3 + (m+4)k_4\theta_n^4 \\ &+ (m+5)k_5\theta_n^5 + (m+6)k_6\theta_n^6 + O[\theta_n]^7\Big) \end{aligned} \quad (11)$$

where $k_i = \frac{m!}{(m+i)!}\frac{\Psi^{(m+i)}(\alpha)}{\Psi^{(m)}(\alpha)},\ i = 1, 2, 3, ..., 6$ respectively.

$$\begin{aligned} y_n - \alpha = 2\theta_n &- \frac{k_1\theta_n^2}{m} + \frac{(k_1^2(1+m) - 2k_2m)\theta_n^3}{m^2} \\ &+ \frac{(k_1k_2m(4+3m) - 3k_3m^2 - k_1^3(1+m)^2)\theta_n^4}{m^3} \\ &+ \Big(\frac{k_1^4(1+m)^3 + 2k_1k_3m^2(3+2m)}{m^4} \\ &\frac{-2k_1^2k_2m(1+m)(3+2m))}{m^4} \\ &+ \frac{2m^2(-2k_4m + k_2^2(2+m))}{m^4}\Big)\theta_n^5 \\ &+ \Big(\frac{k_1^4k_2m(1+m)^2(8+5m) - k_1^5(1+m)^4}{m^5} \\ &+ \frac{k_1m^2(k_4m(8+5m) - k_2^2(2+m)(6+5m))}{m^5} \\ &\frac{-k_1^2K_3m(1+m)(9+5m)}{m^5} \\ &+ \frac{m^2(k_2k_3(12+5m) - 5k_5m)}{m^5}\Big)\theta_n^6 + O[\theta_n]^7 \end{aligned} \quad (12)$$

Now, we ought to expand $\Psi(y_n)$ by using the Taylor Series expansion in powers of θ_n as follows:

$$\begin{aligned} \Psi(y_n) = \theta_n^m\frac{\Psi^{(m)}(\alpha)}{m!}\Big(&2^m + 3\times 2^{(m-1)}k_1\theta_n \\ &+ \frac{2^{(m-3)}(24k_2m - k_1^2(5+3m))\theta_n^2}{m} \\ &+ \frac{2^{(m-4)}(312k_3m^2 - 24k_1k_2m(9+4m)}{3m^2} \\ &+ \frac{k_1^3(m+2)(23m+17)}{3m^2}\theta_n^3 \\ &+ \frac{\gamma\theta_n^4}{m^3} + O[\theta_n]^5\Big) \end{aligned} \quad (13)$$

where $\gamma = 2^{(m-7)}(-544k_1k_3m^2(3+m) - 64m^2(-28k_4m + k_2^2(13+5m))$ $+16k_1^2k_2m(58 + 5m(17+5m)) - k_1^4(98 + m(303 + m(290 + 77m))))$.
By utilizing the equation (11) and (13) in the second step of equation (8), we get

$$\begin{aligned} z_n - \alpha = k_1(-\frac{1}{2} + \frac{1}{m})\theta_n^2 &+ \frac{(m-1)}{8m^2}\Big(k_1^2(8+7m) \\ &- 16k_2m\Big)\theta_n^3 + \frac{1}{48m^3}\Big((24k_3(6-11m)m^2 \\ &+ 24k_1k_2m(-8 + 9m(1+m))) \\ &+ k_1^3(48 + m(8 - m(111 + 65m)))\Big)\theta_n^4 \\ &+ \frac{\delta\theta_n^5}{384m^4} + O[\theta_n]^6 \end{aligned} \quad (14)$$

where $\delta = (-192k_1^2k_2m(1+m)(-12+m(26+17m)) + 96k_1k_3m^2(-24+m(63+41m))+192m^2(2k_4(4-13m)m+k_2^2(-8+m(17+9m))) + k_1^4(-384+m(-154+m(1733+m(2278+751m)))))$

With the help of the Taylor's Series expansion $\Psi(z)$ can be express as follows:

$$\Psi(z) = \theta_n^{2m}\frac{\Psi^{(m)}(\alpha)}{m!}\Big(D - \frac{D(m-1)(k_1^2(8+7m)-16k_2m)}{4(k_1(-2+m))}\theta_n + \frac{D\upsilon_1\theta_n^2}{96k_1^2(m-2)^2m} + \frac{D\upsilon_2\theta_n^3}{384k_1^3(m-2)^2m^2} + \frac{D\upsilon_3\theta_n^4}{92160k_1^4(m-2)^3m^3} + O[\theta_n]^5\Big) \tag{15}$$

where

$$D = k_1^m\left(\frac{1}{m}-\frac{1}{2}\right)^m$$

$$\begin{aligned}\upsilon_1 &= 768k_2^2(m-1)^3m^2 + 96k_1k_3(m-2)m^2(11m-6)\\ &- 96k_1^2k_2m(8+m(-9+m(-12+m(-4+7m))))\\ &+ k_1^4(576+m(-464+m(-347+m(-499\\ &+ m(155+147m)))))\end{aligned}$$

$$\begin{aligned}\upsilon_2 &= 4096k_2^3(m-1)^4m^3\\ &+ 1536k_1k_2k_3(m-1)^2m^3(11m-6)\\ &- 96k_1^3k_3m^2(48+m(m(m(77m-26)-101)-158)\\ &- 384k_1^2m^3(-2k_4(-2+m)(-4+13m)\\ &+ k_2^2(8+m(-29+m(-7+2m(-2+7m)))))\\ &+ 16k_1^4k_2m(384+m(-368+m(-1367\\ &+ m(-738+m(68+m(386+147m))))))\\ &- k_1^6(2048+m(424+m(-5872+m(-6531\\ &+ m(-1896+m(1412+7m(232+49m)))))))\end{aligned}$$

$$\begin{aligned}\upsilon_3 &= (983040k_2^4(-3+m)(-1+m)^5m^4\\ &+ 737280k_1k_2^2k_3(-2+m)(-1+m)^3m^4(11m-6)\\ &- 15360k_1^2(-1+m)m^4(-3k_3^2(6-11m)^2(m-2)\\ &- 48k_2k_4(-2+m)(-1+m)(-4+13m)\\ &+ 8k_2^3(m-1)(8+m(-53+m(-9+14(m-1)m))))\\ &- 92160k_1^3(-2+m)m^4(-c5(-2+m)(-10+57m)\\ &+ k_2k_3(24+m(-183+m(-12+m(-4+77m)))))\\ &+ 960k_1^5k_3(-2+m)m^2(2304\\ &+ m(-7392+m(-14978+m(-6235+m(2081\\ &+ m(4267+1617m)))))) - 960k_1^6k_2m(-3840\\ &+ m(3784+m(28752+m(20517\\ &+ m(-4255+m(-14961+m(-6472\\ &+ 7m(99+m(225+49m)))))))))\\ &+ 7680k_1^4m^2(-6k_4(-2+m)m(32\\ &+ m(-236+m(-97+m(-18+91m))))\\ &+ k_2^2(-192+m(272+m(2867+m(1741+m(-1294\\ &+ m(-1295+m(-394+m(176+147m)))))))))\\ &+ k_1^8(-921600+m(-327168+m(4726048\\ &+ m(7099968+m(2392485\\ &+ m(-2745672+m(-3002403+m(-799128\\ &+ 5m(75811+147m(352+49m))))))))))\end{aligned}$$

Using Taylor's expansion, we get

$$\Psi'(z) = \theta_n^{2m}\frac{\Psi^{(m)}(\alpha)}{m!}\Bigg(\frac{k_1\left(\frac{1}{m}-\frac{1}{2}\right)^{m-1}m}{\theta_n^2} + \frac{\left(k_1\left(\frac{1}{m}-\frac{1}{2}\right)\right)^m(m-1)^2m(k_2^2(8+7m)-16k_2m)}{2k_1^2(m-2)^2\theta_n} + O[\theta_n^3]\Bigg) \tag{16}$$

By inserting the equations (15) and (16) in third step of equation (8), we have

$$\theta_{n+1} = \frac{k_1(-2+m)(-1+W(1))\theta_m^2}{2m} + \frac{\mu_1\theta_n^3}{8m^2} + \frac{\mu_2\theta_n^4}{48m^2} + \frac{\mu_3\theta_n^5}{384m^3} + O[\theta_n]^6 \tag{17}$$

where

$$\begin{aligned}\mu_1 &= (-1+m)(-16k_2m+k_1^2(8+7m))(-1+W(1))\\ &+ 2k_1^2(-2+m)mW'(1)\\ \mu_2 &= 24k_3m^2(-6+11m)(-1+W(1))\\ &+ 24k_1k_2m(-(-8+9m(1+m))(-1+W(1))\\ &+ 2(3-2m)mW'(1))\\ &+ k_1^3(48+m(8-56W(1)-54W'(1)\\ &+ m(-111-65m+123W(1)+65mW(1)-36W'(1)\\ &+ 48mW(1)+3(-2+m)W''(1))))\\ \mu_3 &= 96k_1k_3m^2(-(-24\\ &+ m(63+41m))(-1+W(1))+2(14-11m)mW(1))\\ &+ 192m^2(-(2k_4(4-13m)m\\ &+ k_2^2(-8+m(17+9m)))(-1+W(1))\\ &- 8k_2^2(-1+m)mW(1))\\ &+ 96k_1^2k_2m(-8(-3+W(1))\\ &+ m(4W(1)-28-39W(1)+m(-86-34m+94W(1)\\ &+ 34mW(1)-11W(1)+38mW(1)\\ &+ (-5+3m)W^{'}(1))))\\ &+ k_1^4(-384(1+W(1))\\ &+ m(-154+634W(1)+512W'(1)\\ &+ m(1733-1109W(1)+1474W(1)\\ &+ 168W''(1)+m(2278-2614W(1)\\ &- 344W'(1)+150W''(1)\\ &+ 8W^{(3)}(1)-m(-751+751W(1)+1090W'(1)\\ &+ 150W''(1)+4W^{(3)}(1))))))\end{aligned}$$

Putting the conditions $W(1)=1$, $W'(1)=0$ and $W''(1)=\frac{8}{m^2}-\frac{4}{m}$ of Theorem 1 in equation (17), we get

$$\theta_{n+1} = \frac{-C}{96m^4}\theta_n^5 + O[\theta_n]^6 \tag{18}$$

where

$$\begin{aligned}C &= k_1^2(m-2)(96k_2m(m-3)\\ &+ k_1^2(72+m(162+m(mW^{(3)}(1)-66))))\end{aligned}$$

From the error equation (18), we can conclude that new proposed iterative method is of fifth-order. This complete the proof. ■

Particular Cases on Weight Function

We have several choices of weight function from the conditions on Theorem 1. We are considering the following particular cases:

- **Case 1**: The quadratic form of weight function satisfying the condition of Theorem 1 can be represented as:

$$W(h) = 1 - \frac{2}{m} + \frac{4}{m^3} + (\frac{4}{m} - \frac{8}{m^2})h + (\frac{4}{m^2} - \frac{2}{m})h^2 \quad (19)$$

where

$$h = 2^m \frac{\Psi(x)}{\Psi(y)}$$

After substituting $W(h)$ from equation (19) into equation (8), we get the following fifth order method:
NPM1:

$$\begin{aligned} y_n &= x_n + m\frac{\Psi(x_n)}{\Psi'(x_n)} \\ z_n &= x_n - \frac{m}{2^m}\frac{\Psi(y_n)}{\Psi'(x_n)} \\ x_{n+1} &= z_n - m\Bigg(1 - \frac{2}{m} + \frac{4}{m^3} + (\frac{4}{m} - \frac{8}{m^2})h \\ &\quad +(\frac{4}{m^2} - \frac{2}{m})h^2\Bigg)\frac{\Psi(z_n)}{\Psi'(z_n)} \end{aligned} \quad (20)$$

- **Case 2**: The second suggested form of the weight function satisfying the conditions of Theorem 1 is given by:

$$W(h) = 1 + \frac{4(h^3 - 3h + 2)}{3m^2} - \frac{2(h^3 - 3h + 2)}{3m} \quad (21)$$

The corresponding new proposed iterative method using equation (21) is given by:
NPM2:

$$\begin{aligned} y_n &= x_n + m\frac{\Psi(x_n)}{\Psi'(x_n)} \\ z_n &= x_n - \frac{m}{2^m}\frac{\Psi(y_n)}{\Psi'(x_n)} \\ x_{n+1} &= z_n - m\Bigg(1 + \frac{4(h^3 - 3h + 2)}{3m^2} \\ &\quad -\frac{2(h^3 - 3h + 2)}{3m}\Bigg)\frac{\Psi(z_n)}{\Psi'(z_n)} \end{aligned} \quad (22)$$

- **Case 3**: The third suggested form of $W(h)$ satisfying the conditions of Theorem 1 is given by:

$$W(h) = 1 + \frac{4(h-1)^2(2h+1)}{3m^2} - \frac{2(h-1)^2 2h + 1}{3m} \quad (23)$$

The new iterative method of fifth order obtained by using equation (23) in equation (8) is given by:
NPM3:

$$\begin{aligned} y_n &= x_n + m\frac{\Psi(x_n)}{\Psi'(x_n)} \\ z_n &= x_n - \frac{m}{2^m}\frac{\Psi(y_n)}{\Psi'(x_n)} \\ x_{n+1} &= z_n - m\Bigg(1 + \frac{4(h-1)^2(2h+1)}{3m^2} \\ &\quad -\frac{2(h-1)^2 2h + 1}{3m}\Bigg)\frac{\Psi(z_n)}{\Psi'(z_n)} \end{aligned} \quad (24)$$

III. Numerical Comparison

In this section, we provide the performance of newly proposed methods NPM1 (20), NPM2 (22) and NPM3 (24) to solve some test problems. We have used the well-known second-order Newton Method (NM) (3), the third-order Chebyshev's method (CM) (4), the fourth-order Belh and Al-Hamdan [19] method (BAM) (7) for multiple roots, Thukral's [17] fifth-order method (TM1) (5) developed in 2012, and Thukral's [18] fifth-order method (TM2) (6) developed in 2013, to analyze the results of newly proposed methods. The list of considered test functions with their roots (α), multiplicity (m) of the roots and the initial guesses (x_0) in the neighborhood of the roots are furnished in Table I. The results are summarized for methods in Table II to Table VIII after completion of four full iterations $(n = 4)$ for the test functions $\Psi_1(x)$ to $\Psi_7(x)$ respectively. In Table II to Table VIII, we have presented the absolute residual error of the corresponding functions (i.e $|\Psi(x_n)|$), error in the consecutive iterations $\mid x_n - x_{n-1} \mid$. Approximate roots obtained after completion of 4 iterations are also presented in Table II to Table VIII. We presented the computational order of convergence (COC) in Table II to Table VIII after completion of 4-iteration for each test function. The COC is calculated by the following formula [6]:

$$COC = \frac{log \mid \frac{(x_{n+1} - x_n)}{(x_n - x_{n-1})} \mid}{log \mid \frac{(x_n - x_{n-1})}{(x_{n-1} - x_{n-2})} \mid}$$

We have also given CPU running time for methods in Tables II to VIII. The elapsed CPU-time are computed by selecting $\mid f(x_n) \mid \leq 10^{-1500}$ as the stopping condition. Note that CPU running time is not unique and depends entirely on the computer's specification, but here we present an average of three performances to ensure the robustness of the methods. The star $(\star)$ represents the points where the method is divergent. The results have been carried out with Mathematica 12.2 software on a CPU 2.30 GHz with 4GB of RAM running on the windows 10 on Intel(R) Core(TM) i3-8145U. The numerical result presented in Tables II to Tables VIII suggested that the proposed methods give a better estimate of multiple roots as compared to other existing methods.

IV. Conclusion

We have introduced a new fifth order iterative method for finding multiple roots of non-linear equations that do not require the computation of second or higher derivatives. Convergence analysis proves that the new method preserves the fifth order of convergence. We can easily obtain several new method by considering different weight functions in our scheme (8). After extensive numerical experimentation, we found that our methods have lower residual errors, lower error in two consecutive iteration and stable computational order of convergence as compared to other well known methods. The elapsed CPU-time confirms the highly efficient nature of the proposed methods as compared with the existing methods of same nature. The results obtained are interesting and encouraging. Thus the new methods would be valuable alternative for solving non-linear equations.

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Table I
TEST FUNCTION WITH THEIR INITIAL GUESSES (x_0), ROOTS (α) AND MULTIPLICITY (m)

Test Function $\Psi(x)$	Initial Guesses (x_0)	Roots (α)	Multiplicity (m)
$\Psi_1(x) = (e^{-x^2+x+3} - x + 2)^9$	3.0	2.4905398276083051	9
$\Psi_2(x) = (sin(2cosx) - 1 - x^2 - e^{sinx^3})^3$	-0.60	-0.784895987661212	3
$\Psi_3(x) = (\sqrt{x} - \frac{1}{x} - 1)^7$	2.3	2.1478990357047874	7
$\Psi_4(x) = (4 + 3sinx - 2x^2)^5$	1.90	1.8547101425633862	5
$\Psi_5(x) = (e^{-x} - 2sinx)^5$	3.0	3.1195012582902072	5
$\Psi_6(x) = (log(x^2 + 3x + 5) - 2x + 7)^8$	5.2	5.4690123359101421	8
$\Psi_7(x) = \frac{(x-2)^4}{(x-1)^2+1}$	1.99	2.00	4

Table II
CONVERGENCE BEHAVIOUR OF VARIOUS ITERATIVE METHODS AFTER FOUR FULL ITERATION ON $\Psi_1(x)$

Method	$\mid \Psi_1(x_n) \mid$	$\mid x_n - x_{n-1} \mid$	x_n	COC	CPU Time (sec)
NM	1.6118×10^{-38}	0.0042964	2.4905184205199095	1.9418	0.29522
CM	$\star$	$\star$	$\star$	$\star$	$\star$
BAM	0.34204	0.23099	2.9509917749038942	0.5026	2.92013
TM1	6.0880×10^{-550}	3.8094×10^{-16}	2.4905398276083051	4.0000	0.37768
TM2	$\star$	$\star$	$\star$	$\star$	$\star$
NPM1	1.3493×10^{-1929}	4.14569×10^{-44}	2.4905398276083051	5.0000	0.17601
NPM2	3.8623×10^{-1462}	9.69771×10^{-34}	2.4905398276083051	5.0000	0.24871
NPM3	1.1979×10^{-1210}	3.62985×10^{-28}	2.4905398276083051	5.0000	0.26239

Table III
CONVERGENCE BEHAVIOUR OF VARIOUS ITERATIVE METHODS AFTER FOUR FULL ITERATION ON $\Psi_2(x)$

Method	$\mid \Psi_2(x_n) \mid$	$\mid x_n - x_{n-1} \mid$	x_n	COC	CPU Time (sec)
NM	5.5373×10^{-36}	9.2526×10^{-7}	-0.78489598766184054	1.9993	0.90442
CM	6.5734×10^{-125}	9.7745×10^{-15}	-0.78489598766121254	3.0000	2.72026
BAM	1.0413×10^{-53}	7.8107×10^{-10}	-0.78489598766121253	2.0000	1.37674
TM1	2.8554×10^{-569}	4.0717×10^{-48}	-0.78489598766121254	4.0000	1.33434
TM2	3.4010×10^{-34}	1.8379×10^{-6}	-0.78489598766368997	1.9990	2.64414
NPM1	4.4312×10^{-1152}	2.2181×10^{-77}	-0.78489598766121254	5.0000	0.18628
NPM2	9.1785×10^{-1093}	1.8854×10^{-73}	-0.78489598766121254	5.0000	0.90422
NPM3	9.6502×10^{-1047}	2.1094×10^{-70}	-0.78489598766121254	5.0000	0.88324

Table IV
CONVERGENCE BEHAVIOUR OF VARIOUS ITERATIVE METHODS AFTER FOUR FULL ITERATION ON $\Psi_3(x)$

Method	$\mid \Psi_3(x_n) \mid$	$\mid x_n - x_{n-1} \mid$	x_n	COC	CPU Time (sec)
NM	1.5922×10^{-14}	0.026353	2.1671468885892880	0.9859	0.03692
CM	3.5328×10^{-16}	0.013481	2.1368124108266634	1.0026	2.12999
BAM	4.9369×10^{-24}	0.0035838	2.1487385207132503	0.9989	0.13112
TM1	1.5455×10^{-2590}	1.0447×10^{-92}	2.1478990357047874	4.0000	0.08776
TM2	7.9725×10^{-13}	0.015248	2.1146779036122280	1.0010	0.32087
NPM1	4.3220×10^{-2467}	6.8132×10^{-71}	2.1478990357047874	5.0000	0.03541
NPM2	1.7645×10^{-4644}	3.8512×10^{-133}	2.1478990357047874	5.0000	0.03512
NPM3	3.6953×10^{-2042}	8.1647×10^{-59}	2.1478990357047874	5.0000	0.03191

Table V
CONVERGENCE BEHAVIOUR OF VARIOUS ITERATIVE METHODS AFTER FOUR FULL ITERATION ON $\Psi_4(x)$

Method	$\mid \Psi_4(x_n) \mid$	$\mid x_n - x_{n-1} \mid$	x_n	COC	CPU Time (sec)
NM	6.6065×10^{-133}	3.2638×10^{-14}	1.8547101425633862	2.0000	0.19552
CM	1.9430×10^{-12}	0.0022004	1.8552605165568522	0.99417	8.85616
BAM	5.9941×10^{-2211}	3.5090×10^{-111}	1.8547101425633862	4.0000	0.47856
TM1	6.5504×10^{-2234}	2.4938×10^{-112}	1.8547101425633862	4.0000	0.19611
TM2	6.5207×10^{-133}	3.2595×10^{-14}	1.8547101425633862	2.0000	0.37127
NPM1	6.1597×10^{-4595}	1.3568×10^{-184}	1.8547101425633862	5.0000	0.18646
NPM2	5.7905×10^{-4551}	7.5751×10^{-183}	1.8547101425633862	5.0000	0.18951
NPM3	4.7121×10^{-4512}	2.6621×10^{-181}	1.8547101425633862	5.0000	0.18779

Table VI
CONVERGENCE BEHAVIOUR OF VARIOUS ITERATIVE METHODS AFTER FOUR FULL ITERATION ON $\Psi_5(x)$

Method	$\mid \mathbf{\Psi_5(x_n)} \mid$	$\mid \mathbf{x_n - x_{n-1}} \mid$	$\mathbf{x_n}$	**COC**	**CPU Time (sec)**
NM	2.6827×10^{-178}	8.3224×10^{-18}	3.1195012582902072	2.0000	0.39374
CM	$\star$	$\star$	$\star$	$\star$	$\star$
BAM	5.1030×10^{-290}	4.0603×10^{-29}	3.1195012582902072	4.0000	0.58213
TM1	6.1902×10^{-3139}	1.7835×10^{-156}	3.1195012582902134	4.0000	0.42446
TM2	2.6800×10^{-178}	08.3216×10^{-18}	3.1133352197958358	2.0000	0.83365
NPM1	7.0828×10^{-4417}	8.9088×10^{-177}	3.1195012582902072	5.0000	0.34443
NPM2	8.3831×10^{-4413}	1.2962×10^{-176}	3.1195012582902072	5.0000	0.32443
NPM3	8.6717×10^{-4409}	1.8758×10^{-176}	3.1195012582902072	5.0000	0.32745

Table VII
CONVERGENCE BEHAVIOUR OF VARIOUS ITERATIVE METHODS AFTER FOUR FULL ITERATION ON $\Psi_6(x)$

Method	$\mid \mathbf{\Psi_6(x_n)} \mid$	$\mid \mathbf{x_n - x_{n-1}} \mid$	$\mathbf{x_n}$	**COC**	**CPU Time (sec)**
NM	4.1145×10^{-310}	3.4926×10^{-19}	5.4690123359101421	2.0000	0.13664
CM	0.00045342	0.21612	5.2474958311885309	0.9984	19.8773
BAM	8.6163×10^{-2287}	5.7893×10^{-71}	5.4690123359101421	4.0000	0.13465
TM1	$.2765 \times 10^{-}5282$	2.3947×10^{-164}	5.4690123359101421	4.0000	0.19679
TM2	5.3897×10^{-8}	0.028053	5.3975433953855991	0.9999	0.28534
NPM1	2.8437×10^{-9579}	3.8726×10^{-239}	5.4690123359101421	5.0000	0.09425
NPM2	9.3836×10^{-9557}	1.4050×10^{-238}	5.4690123359101421	5.0000	0.09345
NPM3	3.8982×10^{-9535}	4.8413×10^{-238}	5.4690123359101421	5.0000	0.08692

Table VIII
CONVERGENCE BEHAVIOUR OF VARIOUS ITERATIVE METHODS AFTER FOUR FULL ITERATION ON $\Psi_7(x)$

Method	$\mid \mathbf{\Psi_7(x_n)} \mid$	$\mid \mathbf{x_n - x_{n-1}} \mid$	$\mathbf{x_n}$	**COC**	**CPU Time (sec)**
NM	3.4668×10^{-165}	6.0416×10^{-21}	2.0000000000000000	2.0000	0.02913
CM	7.8630×10^{-18}	0.00033998	1.9999370280226280	1.0003	0.76482
BAM	6.3614×10^{-2795}	8.5618×10^{-175}	2.0000000000000000	4.0000	0.02447
TM1	1.4691×10^{-2663}	1.1048×10^{-166}	2.0000000000000000	4.0000	0.06258
TM2	3.7442×10^{-14}	0.00057150	1.9994769529708866	1.0001	0.06108
NPM1	4.7734×10^{-6208}	1.0845×10^{-310}	2.0000000000000000	5.0000	0.02344
NPM2	3.7982×10^{-6154}	5.1621×10^{-308}	2.0000000000000000	5.0000	0.02321
NPM3	3.2930×10^{-6109}	8.8148×10^{-306}	2.0000000000000000	5.0000	0.02873

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