An Optimized Two-Step Block Hybrid Method with Symmetric Intra-Step Points for Second Order Initial Value Problems

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Abstract—**We derive a two-step block hybrid method for solving general second-order initial value problems by optimizing the local truncation errors. We test the optimized block method's efficiency on both scalar and system initial value problems of the linear and non-linear type, and the results obtained compared with similar schemes.**

Index Terms—**Interpolation, collocation, hybrid methods.**

I. Introduction

RESEARCHERS proposed the hybrid linear multistep methods to overcome the Dahlquist Barrier theorem by imposing intrastep points during formulation. These modified linear multistep schemes find applications in robotics, electric circuits, vibrating strings etc. This paper focuses on implicit block hybrid methods, specifically derived for second-order initial value problems (IVP) of the form.

$$y''(x) = f(x, y, y'), \quad y(a) = \eta_0, \quad y'(a) = \eta_1. \tag{1}$$

Block hybrid methods for the direct solution of IVPs have been of interest to researchers within the past decade (see Anake et al. [3], Jator and Adeyefa [10], Adeyeye and Omar [2]). Biala et al. [6] formulated a Simpson's type, block hybrid method with two intra-step stiff systems points. Ramos and Singh [16] presented an A-stable two-step block hybrid method for first-order IVPs. Jator and Agyingi [11] derived a backward differentiation scheme specifically for large stiff problems. The two off-grid points imposed are zeros of the second degree Chebyshev polynomial of the first kind.

Researchers have published a considerable amount of literature on two-step block hybrid methods for (1). Awari[4], through interpolation and collocation of a monomial function, derived a class of Numerov type two-step block method. Kayode and Adeyeye [13] utilized the Chebyshev polynomial as the basis in deriving a two-step two-point hybrid method. Awoyemi *et al.* [5] through Taylor's series approach, developed a two-step four-point hybrid method for directly solving IVPs of form (1). Jator [8] presented a class of two-step hybrid method with two non-step points for the direct solution of the general second-order initial value problem. James *et al.* [7] proposed a block hybrid method with four equidistant off-step points. In [1], Chebyshev polynomial was utilized as the basis function in formulating a two-step hybrid method with equally spaced points. Jator [9] presented a piece-wise continuous hybrid third derivative formula specified for second-order initial value problems. Ramos *et al.* [15] proposed an optimized two-step hybrid block method by optimizing the local truncation error (LTE).

This study presents an optimized two-step block hybrid method (OTSBHM) with four distinct symmetric off-step points derived by optimizing the LTE for solving (1). For comparisons, we formulate specific Block Two-Step Hybrid Methods (BTSHM) based on Equispaced Points (BTSHM-EP) and Bhaskara Points (BTSHM-BP). We conduct numerical experiments to test the proposed methods' accuracy and compare with the similar schemes derived from Table I's points.

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II. Formulation, specification and Analysis of the method

A. Formulation of the OTSBHM for second order IVP's

The IVP is solved over an interval $x \in [a, b]$, where $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$. The step length is given as $h = x_{n+1} - x_n$ for $n = 0, 1, \ldots, N$. The continuous method is based on approximating the exact solution $y(x)$ at grid points $x_i = x_0 + ih$ by a polynomial of the form

$$y(x) \approx Y(x) = \sum_{i=0}^{n_i+n_c-1} c_i x^i \tag{2}$$

where c_i are unknown coefficients, n_i is the number of interpolation points and n_c is the number of collocation points. The formulation of the continuous method for second order IVPs starts with the introduction of the basis function given by (2). For our two-step block hybrid method, we interpolate at x_n and x_{n+2}. Through numerical experimentation, it was determined that optimal hybrid method can only be found through interpolating at these points. The collocation is imposed at $x_n,\ x_{n+2}$ plus prescribed intra-step points defined as $x_{p_\nu} = x_n + hp_\nu$. Here we assume that $0 < p_\nu < 2$, $\nu = 1, 2, \ldots, m$, where m is the number of intra-step points. Consequently, a system of $n_c + n_i + 1$ equations with $n_c + n_i + 1$ unknowns is obtained from

$$Y(x_{n+i}) = y_{n+i},\ i = 0, 2, \tag{3}$$

$$Y''(x_{n+i}) = f_{n+i},\ i = 0, 1, 2, \tag{4}$$

TABLE I
INTRA-STEP POINTS FOR SIMILAR TWO-STEP HYBRID BLOCK METHODS

Method	Points
BTSHA-EP	$0, 1/3,\ 2/3,\ 1,\ 4/3,\ 5/3,\ 2$
BTSHA-BP	$0,\ 5/37,\ 1/2,\ 1,\ 3/2,\ 69/37,\ 2$
OTSBHM	$0,\ p_1,\ p_2,\ 1,\ 2-p_1,\ 2-p_2,\ 2$

$$Y''(x_{n+p_\nu}) = f_{n+p_\nu}, \nu = 1, 2, \cdots m. \tag{5}$$

In addition, every intra-step point p is associated with another point $\hat{p} = 2 - p$. The assumption is based on the symmetric nature of collocation points. For the two-step methods considered in this section, the symmetry is about x_{n+1}. The symbolic representation of the intra-step points and other similar points is provided in Table I. Solving (3)-(5) for the unknowns c_i, $i = 0, 1, \ldots, n_i + n_c$, and on substituting in (2) gives a continuous approximation of the form

$$\begin{aligned} Y(x) =& \alpha_0(x)y_n + \alpha_1(x)y_{n+2} + h^2 \sum_{j=0}^{2} \beta_j(x) f_{n+j} \\ &+ h^2 \sum_{\nu}^{m} \beta_{p_\nu}(x) f_n + p_\nu. \end{aligned} \tag{6}$$

Since (1) contains the first derivative, the first derivative is given as,

$$\begin{aligned} Y'(x) =& (\alpha_0^{'}(x)y_n + \alpha_2^{'}(x)y_{n+2} + h^2 \sum_{j=0}^{2} \beta_j(x) f_{n+j} \\ &+ h^2 \sum_{\nu=1}^{m} \beta_{p_\nu}(x) f_n + p_\nu), \end{aligned} \tag{7}$$

where the following conditions are imposed on the starter equation (7)

$$Y'(x) = \delta(x), Y'(a) = \delta_0. \tag{8}$$

B. Specification of the OTSBHM for second order IVP's

We derive the OTSBHM by evaluating (6) and (7) at the main and intra-step points to yield methods (9)–(20)

$$\begin{aligned} &\frac{h^2}{1680(p-2)p(q-2)q} f_{n+2}(-14p^2(5q^2-10q+3) \\ &+ 28p(5q^2-10q+3) - 42q^2 + 84q - 29) \\ &- \frac{h^2}{840(p-1)^2(q-1)^2} f_{n+1}(14p^2(25q^2-50q+22) \\ &- 28p(25q^2-50q+22) + 308q^2 - 616q + 279) \\ &- \frac{h^2}{1680(p-2)p(q-2)q} f_n(14p^2(5q^2-10q+3) \\ &- 28p(5q^2-10q+3) + 42q^2 - 84q + 29) \\ &- \frac{h^2}{1680(p-2)(p-1)^2 p(p^2-2p-(q-2)q)} (42q^2 - 84q \\ &+ 29) f_{n-p+2} \\ &- \frac{h^2}{1680(p-2)(p-1)^2 p(p^2-2p-(q-2)q)} (42q^2 - 84q \\ &+ 29) f_{n+p} \\ &- \frac{h^2(42p^2-84p+29) f_{n-q+2}}{1680(q-2)(q-1)^2 q(-p^2+2p+(q-2)q)} \\ &- \frac{h^2(42p^2-84p+29) f_{n+q}}{1680(q-2)(q-1)^2 q(-p^2+2p+(q-2)q)} \\ &+ \frac{1}{2}(y_n - 2y_{n+1} + y_{n+2}) = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &- \frac{h^2(p-2)^3}{1680(q-2)q} f_{n+2}(13p^3 - 22p^2 - 6p(7q^2-14q+2) \\ &+ 14q^2 - 28q + 8) \\ &- \frac{h^2}{1680(q-2)q} p^3 f_n(13p^3 - 56p^2 + p(-42q^2 + 84q + 56) \\ &+ 70(q-2)q) \\ &- \frac{h^2 p(p-2)}{1680(p-1)^2(p^2-2p-(q-2)q)} (5p^4 - 20p^3 + \\ &p^2(-14q^2+28q+12) + 4p(7q^2-14q+4) + 4(7q^2 - \\ &14q+4)) f_{n+p} \\ &+ \frac{h^2 p(p-2)}{840(p-1)^2(q-1)^2} f_{n+1}(13p^6 - 78p^5 + p^4(-42q^2 \\ &+ 84q + 124) + 24p^3(7q^2-14q+1) + 8p^2(7q^2 \\ &- 14q + 6) - 32p(14q^2-28q+11) + 32(7q^2 \\ &- 14q + 6)) \\ &+ \frac{h^2}{1680(p-1)^2(p^2-2p-(q-2)q)} (35p^6 - 210p^5 \\ &- 70p^4(q^2-2q-6) + 280p^3(q^2-2q-1) - 280p^2(q \\ &- 2)q + 16(7q^2-14q+4)) f_{n-p+2} \\ &- \frac{h^2 p(p-2)}{1680(q-2)(q-1)^2 q(-p^2+2p+(q-2)q)} (13p^6 \\ &+ 2p^5(11q-50) + p^4(276-110q) + 144p^3(q-2) \\ &+ 8p^2(q-2) - 96p(q-2) + 32(q-2)) f_{n-q+2} \\ &- \frac{h^2 p(p-2)}{1680(q-2)(q-1)^2 q(-p^2+2p+(q-2)q)} \\ &(13p^6 - 2p^5(11q+28) + 2p^4(55q+28) - 144p^3 q - 8p^2 q \\ &+ 96pq - 32q) f_{n+q} \\ &+ \frac{1}{2}(py_n - (p-2)y_{n+2} - 2y_{n-p+2}) = 0, \end{aligned} \tag{10}$$

$$
\begin{aligned}
&-\frac{h^2(p-2)^3}{1680(q-2)q}f_n(13p^3-22p^2-6p(7q^2-14q+2)\\
&+14q^2-28q+8)\\
&+\frac{h^2p^3}{1680(q-2)q}f_{n+2}(-13p^3+56p^2+14p(3q^2-6q\\
&-4)-70(q-2)q)\\
&-\frac{h^2p(p-2)}{1680(p-1)^2(p-q)(p+q-2)}(5p^4-20p^3\\
&+p^2(-14q^2+28q+12)+4p(7q^2-14q+4)\\
&+4(7q^2-14q+4))f_{n-p+2}\\
&+\frac{h^2p(p-2)}{840(p-1)^2(q-1)^2}f_{n+1}(13p^6-78p^5+p^4(-42q^2\\
&+84q+124)+24p^3(7q^2-14q+1)+8p^2(7q^2-14q\\
&+6)-32p(14q^2-28q+11)+32(7q^2-14q+6))\\
&+\frac{h^2}{1680(p-1)^2(p-q)(p+q-2)}(35p^6-210p^5-70p^4\\
&(q^2-2q-6)+280p^3(q^2-2q-1)-280p^2(q-2)q\\
&+16(7q^2-14q+4))-280p^2(q-2)q+16(7q^2-14q\\
&+4))f_{n+p}+\frac{h^2}{1680(q-2)(q-1)^2q(p-q)(p+q-2)}\\
&p(p-2)(13p^6-2p^5(11q+28)+2p^4(55q+28)-144p^3q\\
&-8p^2q+96pq-32q)f_{n-q+2}\\
&+\frac{h^2}{1680(q-2)(q-1)^2q(p-q)(p+q-2)}p(p-2)\\
&(13p^6+2p^5(11q-50)+p^4(276-110q)+144p^3(q-2)\\
&+8p^2(q-2)-96p(q-2)+32(q-2))f_{n+q}\\
&+\frac{1}{2}(-(p-2)y_n+py_{n+2}-2y_{n+p})=0, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h^2(q-2)^3}{1680(p-2)p}f_{n+2}(14p^2(3q-1)+p(28-84q)-13q^3\\
&+22q^2+12q-8)\\
&+\frac{h^2q^3}{1680(p-2)p}f_n(14p^2(3q-5)-28p(3q-5)\\
&+q(-13q^2+56q-56))\\
&-\frac{h^2q(q-2)}{1680(q-1)^2(-p^2+2p+(q-2)q)}(-14p^2(q^2-2q\\
&-2)+28p(q^2-2q-2)+5q^4-20q^3+12q^2+16q\\
&+16)f_{n+q}\\
&+\frac{h^2q(q-2)}{1680(p-2)(p-1)^2p(p^2-2p-(q-2)q)}(2p(11q^5\\
&-55q^4+72q^3+4q^2-48q+16)+(-13q^2+56q\\
&-56)q^4)f_{n+p}\\
&-\frac{h^2q(q-2)}{1680(p-2)(p-1)^2p(p^2-2p-(q-2)q)}(2p(11q^5\\
&-55q^4+72q^3+4q^2-48q+16)+(13q^2+4q\\
&-4)(q-2)^4)f_{n-p+2}+\frac{h^2q(q-2)}{840(p-1)^2(q-1)^2}f_{n+1}(-14p^2\\
&(3q^4-12q^3-4q^2+32q-16)+28p(3q^4-12q^3-4q^2\\
&+32q-16)+13q^6-78q^5+124q^4+24q^3+48q^2-352q\\
&+192)\\
&+\frac{h^2}{1680(q-1)^2(-p^2+2p+(q-2)q)}(-14p^2(5q^4\\
&-20q^3\\
&+20q^2-8)+28p(5q^4-20q^3+20q^2-8)+35q^6-210q^5\\
&+420q^4-280q^3+64)f_{n-q+2}\\
&+\frac{1}{2}(qy_n-(q-2)y_{n+2}-2y_{n-q+2})=0, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h^2(q-2)^3}{1680(p-2)p}f_n(14p^2(3q-1)+p(28-84q)-13q^3\\
&+22q^2+12q-8)\\
&+\frac{h^2q^3}{1680(p-2)p}f_{n+2}(14p^2(3q-5)-28p(3q-5)\\
&+q(-13q^2+56q-56)\\
&+\frac{h^2q(q-2)}{1680(q-1)^2(p-q)(p+q-2)}(-14p^2(q^2-2q-2)\\
&+28p(q^2-2q-2)+5q^4-20q^3+12q^2+16q\\
&+16)f_{n-q+2}+\frac{h^2q(q-2)}{840(p-1)^2(q-1)^2}f_{n+1}(-14p^2(3q^4\\
&-12q^3-4q^2+32q-16)+28p(3q^4-12q^3-4q^2\\
&+32q-16)+13q^6-78q^5+124q^4+24q^3+48q^2\\
&-352q+192)+\frac{h^2}{1680(q-1)^2(p-q)(p+q-2)}\\
&(14p^2(5q^4-20q^3+20q^2-8)-28p(5q^4-20q^3+20q^2-8)\\
&-35q^6+210q^5-420q^4+280q^3-64)f_{n+q}\\
&+\frac{h^2q(q-2)}{1680(p-2)(p-1)^2p(p-q)(p+q-2)}(2p(11q^5\\
&-55q^4+72q^3+4q^2-48q+16)+(-13q^2+56q\\
&-56)q^4)f_{n-p+2}-\frac{h^2q(q-2)}{1680(p-2)(p-1)^2p(p-q)(p+q-2)}\\
&(2p(11q^5-55q^4+72q^3+4q^2-48q+16)+(13q^2+4q-4)\\
&(q-2)^4)f_{n+p}+\frac{1}{2}(-(q-2)y_n+qy_{n+2}-2y_{n+q})=0, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h^2}{105(p-2)p(q-2)q}f_n(7p^2(5q^2-10q+2)-14p(5q^2\\
&-10q+2)+14q^2-28q+8)\\
&+\frac{2h^2}{105(p-1)^2(q-1)^2}f_{n+1}(7p^2(5q^2-10q+4)-14p(5q^2\\
&-10q+4)+4(7q^2-14q+6))\\
&+\frac{h^2(7p^2-14p+4)f_{n+q}}{105(q-1)^2q(p-q)(p+q-2)}\\
&-\frac{h^2(7p^2-14p+4)f_{n-q+2}}{105(q-2)(q-1)^2(p-q)(p+q-2)}\\
&+\frac{h^2(7q^2-14q+4)f_{n-p+2}}{105(p-2)(p-1)^2(p-q)(p+q-2)}\\
&-\frac{h^2(7q^2-14q+4)f_{n+p}}{105(p-1)^2p(p-q)(p+q-2)}+h\delta_n+\frac{1}{2}(y_n\\
&-y_{n+2})=0, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h^2}{840(p-2)p(q-2)q}f_{n+2}(7p^2(10q^2-20q+7)\\
&-14p(10q^2-20q+7)+49q^2-98q+38)\\
&-\frac{h^2}{840(p-2)p(q-2)q}f_n(7p^2(10q^2-20q+7)\\
&-14p(10q^2-20q+7)+49q^2-98q+38)\\
&+\frac{h^2(49p^2-98p+38)f_{n-q+2}}{840(q-2)(q-1)q(p-q)(p+q-2)}\\
&-\frac{h^2(49p^2-98p+38)f_{n+q}}{840(q-2)(q-1)q(p-q)(p+q-2)}\\
&+\frac{h^2(49q^2-98q+38)f_{n+p}}{840(p-2)(p-1)p(p-q)(p+q-2)}\\
&-\frac{h^2(49q^2-98q+38)f_{n-p+2}}{840(p-2)(p-1)p(p-q)(p+q-2)}+h\delta_{n+1}\\
&+\frac{1}{2}(y_n-y_{n+2})=0, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2h^2(p-1)}{105(q-1)^2}f_{n+1}(3p^4-12p^3+p^2(-7q^2+14q+4)\\
&+2p(7q^2-14q+8)+4(7q^2-14q+6))\\
&-\frac{h^2p^2}{840(p-2)(q-2)q}f_n(24p^4-133p^3-14p^2(4q^2-8q\\
&-17)+35p(5q^2-10q-4)-140(q-2)q)\\
&-\frac{h^2(p-2)^2}{840p(q-2)q}f_{n+2}(24p^4-59p^3-8p^2(7q^2-14q-2)\\
&+p(49q^2-98q+16)-14q^2+28q-8)\\
&-\frac{h^2}{840(p-1)(p^2-2p-(q-2)q)}(10p^4-40p^3\\
&-7p^2(3q^2-6q-4)+6p(7q^2\\
&-14q+4)+4(7q^2-14q+4))f_{n+p}\\
&+\frac{h^2}{840(p-2)(p-1)p(p^2-2p-(q-2)q)}(130p^6-780p^5\\
&+p^4(-189q^2+378q+1572)+4p^3(189q^2-378q-272)\\
&-16p^2(49q^2-98q-2)+8p(7q^2-14q+4)\\
&+16(7q^2-14q+4))f_{n-p+2}\\
&+\frac{h^2}{840(q-2)(q-1)^2q(-p^2+2p+(q-2)q)}(-24p^7\\
&-7p^6(5q-29)+14p^5(15q-47)-140p^4(3q-7)\\
&+280p^3(q-2)+56p^2(q-2)-112p(q-2)\\
&+32(q-2))f_{n-q+2}\\
&+\frac{h^2}{840(q-2)(q-1)^2q(-p^2+2p+(q-2)q)}(-24p^7\\
&+7p^6(5q+19)-14p^5(15q+17)+140p^4(3q+1)\\
&-280p^3q-56p^2q+112pq-32q)f_{n+q}+h\delta_{n-p+2}\\
&+\frac{1}{2}(y_n-y_{n+2})=0, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h^2(p-2)^2}{840p(q-2)q}f_n(24p^4-59p^3-8p^2(7q^2-14q-2)\\
&+p(49q^2-98q+16)-14q^2+28q-8)\\
&+\frac{h^2p^2}{840(p-2)(q-2)q}f_{n+2}(24p^4-133p^3-14p^2(4q^2-\\
&8q-17)+35p(5q^2-10q-4)-140(q-2)q)\\
&+\frac{h^2}{840(p-1)(p^2-2p-(q-2)q)}(10p^4-40p^3\\
&-7p^2(3q^2-6q-4)+6p(7q^2-14q+4)+4(7q^2-14q\\
&+4))f_{n-p+2}-\frac{2h^2(p-1)}{105(q-1)^2}f_{n+1}(3p^4-12p^3+p^2(-7q^2\\
&+14q+4)+2p(7q^2-14q+8)+4(7q^2-14q+6))\\
&-\frac{h^2}{840(p-2)(p-1)p(p^2-2p-(q-2)q)}(130p^6-780p^5\\
&+p^4(-189q^2+378q+1572)+4p^3(189q^2-378q-272)\\
&-16p^2(49q^2-98q-2)+8p(7q^2-14q+4)\\
&+16(7q^2-14q+4))f_{n+p}\\
&+\frac{h^2}{840(q-2)(q-1)^2q(-p^2+2p+(q-2)q)}(24p^7\\
&-7p^6(5q+19)+14p^5(15q+17)-140p^4(3q+1)\\
&+280p^3q+56p^2q-112pq+32q)f_{n-q+2}\\
&+\frac{h^2}{840(q-2)(q-1)^2q(-p^2+2p+(q-2)q)}(24p^7\\
&+7p^6(5q-29)+p^5(658-210q)+140p^4(3q-7)\\
&-280p^3(q-2)-56p^2(q-2)+112p(q-2)\\
&-32(q-2))f_{n+q}+h\delta_{n+p}+\frac{1}{2}(y_n-y_{n+2})=0, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h^2q^2}{840(p-2)p(q-2)}f_n(7p^2(8q^2-25q+20)-14p(8q^2\\
&-25q+20)+q(-24q^3+133q^2-238q+140))\\
&+\frac{2h^2(q-1)}{105(p-1)^2}f_{n+1}(-7p^2(q^2-2q-4)+14p(q^2\\
&-2q-4)+3q^4-12q^3+4q^2+16q+24)\\
&+\frac{h^2(q-2)^2}{840(p-2)pq}f_{n+2}(7p^2(8q^2-7q+2)-14p(8q^2\\
&-7q+2)-24q^4+59q^3-16q^2-16q+8)\\
&+\frac{h^2}{840(q-1)(-p^2+2p+(q-2)q)}(7p^2(3q^2-6q-4)\\
&+p(-42q^2+84q+56)-2(5q^4-20q^3+14q^2\\
&+12q+8))f_{n+q}\\
&+\frac{h^2}{840(p-2)(p-1)^2p(p^2-2p-(q-2)q)}(p(-35q^6\\
&+210q^5-420q^4+280q^3+56q^2-112q+32)\\
&-(q-2)^4(24q^3-11q^2-6q+4))f_{n-p+2}\\
&+\frac{h^2}{840(p-2)(p-1)^2p(p^2-2p-(q-2)q)}(p(35q^6\\
&-210q^5+420q^4-280q^3-56q^2+112q-32)
\end{aligned}
$$

$$+ (-24q^3 + 133q^2 - 238q + 140)q^4)f_{n+p} + \frac{h^2}{840(q-2)(q-1)q(-p^2+2p+(q-2)q)}(-7p^2(27q^4 - 108q^3 + 112q^2 - 8q - 16) + 14p(27q^4 - 108q^3 + 112q^2 - 8q - 16) + 2(65q^6 - 390q^5 + 786q^4 - 544q^3 + 16q^2 + 16q + 32))f_{n-q+2} + h\delta_{n-q+2} + \frac{1}{2}(y_n - y_{n+2}) = 0, \quad (18)$$

$$\frac{h^2q^2}{840(p-2)p(q-2)}f_{n+2}(-7p^2(8q^2 - 25q + 20) + 14p(8q^2 - 25q + 20) + q24q^3 - 133q^2 + 238q - 140)) + \frac{h^2(q-2)^2}{840(p-2)pq}f_n(-7p^2(8q^2 - 7q + 2) + 14p(8q^2 - 7q + 2) + 24q^4 - 59q^3 + 16q^2 + 16q - 8) + \frac{h^2}{840(q-1)(-p^2+2p+(q-2)q)}(-7p^2(3q^2 - 6q - 4) + 14p(3q^2 - 6q - 4) + 2(5q^4 - 20q^3 + 14q^2 + 12q + 8))f_{n-q+2} - \frac{2h^2(q-1)}{105(p-1)^2}f_{n+1}(-7p^2(q^2 - 2q - 4) + 14p(q^2 - 2q - 4) + 3q^4 - 12q^3 + 4q^2 + 16q + 24) + \frac{h^2}{840(p-2)(p-1)^2p(p^2-2p-(q-2)q)}(p(-35q^6 + 210q^5 - 420q^4 + 280q^3 + 56q^2 - 112q + 32) + (24q^3 - 133q^2 + 238q - 140)q^4)f_{n-p+2} + \frac{h^2}{840(p-2)(p-1)^2p(p^2-2p-(q-2)q)}(p(35q^6 - 210q^5 + 420q^4 - 280q^3 - 56q^2 + 112q - 32) + (24q^3 - 11q^2 - 6q + 4)(q-2)^4)f_{n+p} + \frac{h^2}{840(q-2)(q-1)q(-p^2+2p+(q-2)q)}(7p^2(27q^4 - 108q^3 + 112q^2 - 8q - 16) - 14p(27q^4 - 108q^3 + 112q^2 - 8q - 16) - 2(65q^6 - 390q^5 + 786q^4 - 544q^3 + 16q^2 + 16q + 32))f_{n+q} + h\delta_{n+q} + \frac{1}{2}(y_n - y_{n+2}) = 0, \quad (19)$$

$$-\frac{2h^2f_{n+1}}{105(p_1-1)^2(p_2-1)^2}(7p_1^2(5p_2^2 - 10p_2 + 4) - 14p_1(5p_2^2 - 10p_2 + 4) + 4(7p_2^2 - 14p_2 + 6)) - \frac{h^2f_{n+2}}{105(p_1-2)p_1(p_2-2)p_2}(7p_1^2(5p_2^2 - 10p_2 + 2) - 14p_1(5p_2^2 - 10p_2 + 2) + 14p_2^2 - 28p_2 + 8) + \frac{h^2(7p_2^2 - 14p_2 + 4)f_{n-p_1+2}}{105(p_1-1)^2p_1(p_1-p_2)(p_1+p_2-2)} - \frac{h^2(7p_2^2 - 14p_2 + 4)f_{n+p_1}}{105(p_1-2)(p_1-1)^2(p_1-p_2)(p_1+p_2-2)} - \frac{h^2(7p_1^2 - 14p_1 + 4)f_{n-p_2+2}}{105(p_2-1)^2p_2(p_1-p_2)(p_1+p_2-2)} + \frac{h^2(7p_1^2 - 14p_1 + 4)f_{n+p_2}}{105(p_2-2)(p_2-1)^2(p_1-p_2)(p_1+p_2-2)} + h\delta_{n+2} + \frac{1}{2}(y_n - y_{n+2}) = 0. \quad (20)$$

We optimize the supplementary method (20) with LTE

$$\frac{h^9}{793800}\big(-3p_1^2\big(7p_2^2 - 14p_2 + 4\big) + 6p_1\big(7p_2^2 - 14p_2 + 4\big) - 4\big(3p_2^2 - 6p_2 + 2\big)\big)y^{(9)}(x_n) + \frac{h^{10}}{907200}\big(-3p^2\big(7q^2 - 14q + 4\big) + 6p\big(7q^2 - 14q + 4\big) - 4\big(3q^2 - 6q + 2\big)\big)y^{(10)}(x_n) + \frac{h^{11}}{209563200}\big(-11p^4\big(7q^2 - 14q + 4\big) + 44p^3\big(7q^2 - 14q + 4\big) - 11p^2\big(7q^4 - 28q^3 + 277q^2 - 498q + 140\big) + 22p\big(7q^4 - 28q^3 + 249q^2 - 442q + 124\big) - 4\big(11q^4 - 44q^3 + 385q^2 - 682q + 222\big)\big)y^{(11)}(x_n) + \cdots. \quad (21)$$

The first and second term in LTE (21) have the same coefficient given as

$$(-3p_1^2(4 - 14p_2 + 7p_2^2) + 6p_1(4 - 14p_2 + 7p_2^2) - 4(2 - 6p_2 + 3p_2^2)). \quad (22)$$

The third coefficient in the LTE (21) is also

$$-11p_1^4\big(7p_2^2 - 14p_2 + 4\big) + 44p^3\big(7p_2^2 - 14p_2 + 4\big) - 11p_1^2\big(7p_2^4 - 28p_2^3 + 403p_2^2 - 750p_2 + 212\big) + 22p_1\big(7p_2^4 - 28p_2^3 + 375p_2^2 - 694p_2 + 196\big) - 4\big(11p_2^4 - 44p_2^3 + 583p_2^2 - 1078p_2 + 354\big). \quad (23)$$

We equate the coefficients (22) and (23) to zero and simultaneously solve to obtain values of p_1 and p_2 given as

$$p_1 = 1 - \sqrt{\frac{1}{33}\left(15 - 2\sqrt{15}\right)},$$

$$p_2 = 1 - \sqrt{\frac{1}{33}\left(15 + 2\sqrt{15}\right)}.$$

The optimized LTE is obtained by substituting the values p_1 and p_2 in (21), and given as

$$-\frac{h^{13}y^{(13)}(x_n)}{92712069450} + \cdots. \quad (24)$$

C. Analysis of the OTSBHM for second order IVP's

We can write the formula for the block method for the second-order IVP equation in matrix-vector form as

$$AY_n = hBF_n + \Delta_n + h^2CF_n, \quad (25)$$

where A, B and C are coefficient matrices and the vectors Y_n, Δ_n and F_n are defined as

$$Y_n = (y_n, y_{n+p_1}, y_{n+p_2}, \ldots, y_{n+1}, \ldots, y_{n+2-p_1}, y_{n+2-p_2}, y_{n+2})^T, \quad (26)$$

$$\Delta_n = (\delta_n, \delta_{n+p_1}, \delta_{n+p_2}, \ldots, \delta_{n+1}, \ldots, \delta_{n+2-p_1}, \delta_{n+2-p_2}, \delta_{n+2})^T, \quad (27)$$

$$F_n = (f_n, f_{n+p_1}, f_{n+p_2}, \ldots, f_{n+1}, \ldots, f_{n+2-p_1}, f_{n+2-p_2}, f_{n+2})^T. \quad (28)$$

1) Local truncation errors and order of the OTSBHM: To analyze the truncation errors, we define the linear operator $\mathcal{L}$ as

$$\mathcal{L}\big[z(x_n);h\big] = \sum_{\nu}\big[\hat{\alpha}_\nu z(x_n+\nu h) - h\hat{\beta}_\nu z''(x_n+\nu h) - h^2\hat{\gamma}_\nu z'(x_n+\nu h)\big] \quad (29)$$

where $\hat{\alpha}_\nu$, $\hat{\beta}_\nu$ and $\hat{\gamma}_\nu$ are columns of the matrices A, B and C. The orders and error constants of the methods can be obtained through Taylor series about x_n. The $LTE's$ of the OTSBHM are given as

$$\mathcal{L}[y(x_n);h] = \begin{cases} -\dfrac{h^{10}y^{(10)}(x_n)}{159667200} + O(h^{11}) \\ \dfrac{\sqrt{(\frac{17235-562\sqrt{15}}{33})}h^9y^{(9)}(x_n)}{-747130230} + O(h^{10}) \\ \dfrac{\sqrt{(\frac{17235+562\sqrt{15}}{33})}h^9y^{(9)}(x_n)}{747130230} + O(h^{10}) \\ \dfrac{\sqrt{(\frac{17235-562\sqrt{15}}{33})}h^9y^{(9)}(x_n)}{747130230} + O(h^{10}) \\ \dfrac{\sqrt{(\frac{17235-562\sqrt{15}}{33})}h^9y^{(9)}(x_n)}{-747130230} + O(h^{10}) \\ -\dfrac{h^{13}y^{(13)}(x_n)}{92712069450} + O(h^{14}) \\ \dfrac{h^9y^{(9)}(x_n)}{1330560} + O(h^{10}) \\ -\dfrac{h^{13}y^{(13)}(x_n)}{92712069450} + O(h^{13}) \\ -\dfrac{h^9((13+10\sqrt{15})y^{(9)}(x_n))}{83014470} + O(h^{10}) \\ -\dfrac{h^9((13+10\sqrt{15})y^{(9)}(x_n))}{83014470} + O(h^{10}) \\ \dfrac{(10\sqrt{15}-13)h^9y^{(9)}(x_n)}{83014470} + O(h^{10}) \\ \dfrac{(10\sqrt{15}-13)h^9y^{(9)}(x_n)}{83014470} + O(h^{10}). \end{cases}$$

Thus, the OTSBHM has order $p \geq 7$.

2) Zero Stability: In the limit as $h \to 0$, the matrix system (25) reduces to

$$\bar{A}^{(0)}Y_\lambda - \bar{A}^{(1)}Y_{\lambda-1} = 0, \quad (30)$$

where $\bar{A}^{(0)}$ and $\bar{A}^{(1)}$ are square matrices and

$$Y_\lambda = \big(y_{n+p_1},\ldots,y_{n+p_m}\ldots,y_{n+1},y_{n+2-p_1},\cdots,y_{n+p_m}\ldots,y_{n+2}\big), \quad (31)$$

$$Y_{\lambda-1} = \big(y_{n-(2-p_m)},\ldots,y_{n-1},y_{n-p_m},\ldots,y_{n+p_1},y_n\big). \quad (32)$$

In general, for symmetrically arranged intra-step points, the matrices $\bar{A}_{(0)}$ and $\bar{A}_{(1)}$ are defined as

$$\bar{A}^{(1)} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & p_1-1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & p_m-1 \\ 0 & \cdots & 0 & 1-p_m \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1-p_1 \\ 0 & \cdots & 0 & -1 \end{bmatrix}$$

and

$$\bar{A}^{(0)} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -2 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & -p_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -p_2 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -p_m & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & -(2-p_m) & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & -(2-p_2) & 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & -(2-p_1) & 0 & \cdots & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

The first characteristic polynomial for the OTSBHM is $\rho(R) = R^{3+1}(1+R)$. Accordingly, the OTSBHM is zero stable, consistent and convergent.

Problem 1

The first example is an IVP of Bessel type given as

$$x^2y'' + xy' + \big(x^2 - 0.25\big)y = 0,$$

$$y(1) = \sin\sqrt{2/\pi},\; y' = (2\cos 1 - \sin 1)/\sqrt{2\pi},$$

$$Exact: y(x) = \sin x\sqrt{2\pi}$$

This problem is solved using the three similar block hybrid methods with different steps $N = 67, 82, 97, 112$ and 125.

TABLE II
ABSOLUTE ERROR ($|\, y(x_i) - y_i \,|$) AT THE END POINT $x = 8$ FOR PROBLEM 1

N	BTSHA-EP	BTSHA-BP	OTSBHM
67	2.7897×10^{-12}	1.4856×10^{-13}	6.6160×10^{-19}
82	5.6879×10^{-13}	3.0245×10^{-14}	8.6679×10^{-20}
97	1.5064×10^{-13}	8.0032×10^{-15}	5.7713×10^{-21}
112	4.8152×10^{-14}	2.5568×10^{-15}	2.1993×10^{-21}
125	2.0119×10^{-13}	1.0679×10^{-15}	1.0308×10^{-22}

In the Tables II and III, we provide the absolute errors at the endpoint $x = 8$, and the maximum errors respectively. From the results displayed in Tables II and III, the OTSBHM outperforms the BTSHA-EP and BTSHA-BP.

TABLE III
MAXIMUM ERRORS FOR PROBLEM 1 WITH DIFFERENT STEP-LENGTHS N

N	BTSHM-EP [14]	BTSHM-BP [14]	OTSBHM
67	6.1828×10^{-12}	3.2946×10^{-13}	4.2255×10^{-15}
82	1.2694×10^{-12}	6.7517×10^{-14}	6.5778×10^{-16}
97	3.3602×10^{-13}	1.7857×10^{-14}	1.3726×10^{-16}
112	1.0773×10^{-13}	5.7211×10^{-15}	3.5442×10^{-17}
125	4.5011×10^{-14}	2.3896×10^{-15}	1.2513×10^{-17}

Problem 2

The second example is a non-linear IVP given as

$$y^{''} + x(y^{'})^2 = 0,$$

$$y(1) = 1,\ y^{'} = \frac{1}{2},$$

$$Exact : y(x) = 1 + \frac{1}{2}\ln\left(\frac{(2-x)}{(2+x)}\right).$$

Problem 2 is solved using similar three-block hybrid methods considered. Table IV displays the absolute errors at the main points. From the results presented, it is clear that the OTSBHM outperforms the BTSHA-EP and BTSHA-BP methods.

TABLE IV
ABSOLUTE ERROR FOR PROBLEM 2

x	BTSHM-EP[14]	BTSHM-BP [14]	OTSBHM
0.1	2.2205×10^{-16}	2.2205×10^{-16}	2.2205×10^{-16}
0.2	8.8818×10^{-16}	0	0
0.3	1.5543×10^{-15}	2.2205×10^{-16}	0
0.4	2.8866×10^{-15}	0	0
0.5	4.6629×10^{-15}	4.4409×10^{-16}	2.2205×10^{-16}
0.6	8.4377×10^{-15}	2.2205×10^{-16}	2.2205×10^{-16}
0.7	1.5321×10^{-14}	6.6613×10^{-16}	2.2205×10^{-16}
0.8	2.8644×10^{-14}	1.5543×10^{-15}	2.2205×10^{-16}
0.9	5.8398×10^{-14}	3.1086×10^{-15}	0
1.0	1.2679×10^{-13}	6.6613×10^{-15}	0

Problem 3

Consider the inhomogeneous second order IVP given by

$$y^{''} + 100y = 99\sin x,\ y(0) = 1,\ y^{'}(0) = 11,$$

with the exact solution given as $y(x) = \cos(10x) + \sin(10x) + \sin(x)$.

Figures 1, 2 and 3 show the absolute error comparisons of the three similar methods in the domain of the problem 3. Despite sharing an equal number of function evaluation per block, the OTSBHM is superior to the other similar methods. Table V displays the maximum errors using different step lengths.

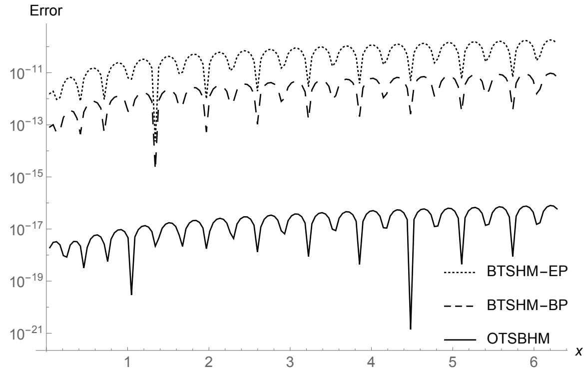

Fig. 1. Absolute error ($|\ y(x_i) - y_i\ |$) for Example 5.1 showing different number of off-step points m and with $h = 2\pi/300$

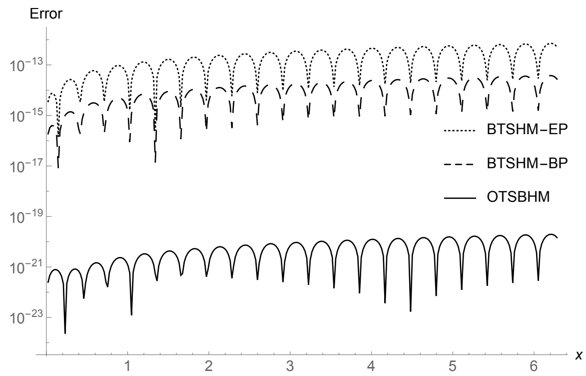

Fig. 2. Absolute error ($|\ y(x_i) - y_i\ |$) for Example 5.1 showing different number of off-step points m and with $h = 2\pi/600$

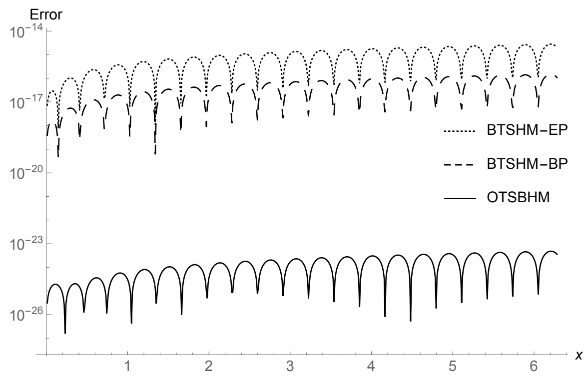

Fig. 3. Absolute error ($|\ y(x_i) - y_i\ |$) for Example 5.1 showing different number of off-step points m and with $h = 2\pi/1200$

TABLE V
MAXERR ($\max_i \mid y(x_i) - y_i \mid$) FOR PROBLEM 3 WITH $h = 2\pi/300,\ 2\pi/600,$ AND $2\pi/1200$

h	BTSHM-EP	BTSHM-BP	OTSBHM
$2\pi/300$	1.8210×10^{-10}	9.5744×10^{-12}	1.4461×10^{-15}
$2\pi/600$	7.1469×10^{-13}	3.7805×10^{-14}	1.9588×10^{-20}
$2\pi/1200$	2.7950×10^{-15}	1.4807×10^{-16}	4.7787×10^{-24}

Problem 4

Consider the stiff IVP system

$$y_1^{''} = (\epsilon - 2)y_1 + (2\epsilon - 2)y_2, \quad y_1(0) = 2,\ y_1^{'}(0) = 0,$$
$$y_2^{''} = (1 - \epsilon)y_1 + (1 - 2\epsilon)y_2, \quad y_2(0) = -1,\ y_2^{'}(0) = 0,$$

with the exact solutions $y_1(x) = 2\cos x$, $y_2(x) = -\cos x$, and $\epsilon = 2500$.

The results presented in Tables VI, VII, VIII, IX and X are the absolute errors generated by solving Problem 4 using the three similar methods with step lengths $h = \pi/2, \pi/3, \pi/4, \pi/5$ and $\pi/12$. As with previous comparisons, both algorithms share equal number of function evaluations per block. The results presented in Tables VI–X underscore the importance of systematically selecting collocation points that minimize the approximation error.

TABLE VI
ABSOLUTE ERROR FOR PROBLEM 4 WITH $h = \pi/2$

x	y_i	BTSHM-EP	BTSHM-BP	OTSBHM
2π	$y_1(x)$	1.01×10^{-9}	4.26×10^{-12}	4.28×10^{-13}
	$y_2(x)$	5.03×10^{-10}	2.13×10^{-12}	2.14×10^{-13}
4π	$y_1(x)$	4.03×10^{-9}	1.71×10^{-11}	1.71×10^{-12}
	$y_2(x)$	2.01×10^{-9}	8.53×10^{-12}	8.56×10^{-13}
6π	$y_1(x)$	9.06×10^{-9}	3.84×10^{-11}	3.85×10^{-12}
	$y_2(x)$	4.53×10^{-9}	1.92×10^{-11}	1.93×10^{-12}
8π	$y_1(x)$	1.61×10^{-8}	6.82×10^{-11}	6.85×10^{-12}
	$y_2(x)$	8.05×10^{-9}	3.41×10^{-11}	3.42×10^{-12}
10π	$y_1(x)$	2.52×10^{-8}	1.07×10^{-10}	1.07×10^{-11}
	$y_2(x)$	1.26×10^{-8}	5.33×10^{-11}	5.35×10^{-12}

TABLE VII
ABSOLUTE ERROR FOR PROBLEM 4 WITH $h = \pi/3$

x	y_i	BTSHM-EP	BTSHM-BP	OTSBHM
2π	$y_1(x)$	1.99×10^{-11}	3.40×10^{-14}	1.80×10^{-18}
	$y_2(x)$	9.96×10^{-12}	1.70×10^{-14}	9.01×10^{-19}
4π	$y_1(x)$	7.97×10^{-11}	1.36×10^{-13}	7.20×10^{-18}
	$y_2(x)$	3.99×10^{-11}	6.80×10^{-14}	3.60×10^{-18}
6π	$y_1(x)$	1.79×10^{-10}	3.06×10^{-13}	1.62×10^{-17}
	$y_2(x)$	8.97×10^{-11}	1.53×10^{-13}	8.11×10^{-18}
8π	$y_1(x)$	3.19×10^{-10}	5.44×10^{-13}	2.88×10^{-17}
	$y_2(x)$	1.59×10^{-10}	2.72×10^{-13}	1.44×10^{-17}
10π	$y_1(x)$	4.98×10^{-10}	8.50×10^{-13}	4.50×10^{-17}
	$y_2(x)$	2.49×10^{-10}	4.25×10^{-13}	2.25×10^{-17}

TABLE VIII
ABSOLUTE ERROR FOR PROBLEM 4 WITH $h = \pi/4$

x	y_i	BTSHM-EP	BTSHM-BP	OTSBHM
2π	$y_1(x)$	2.88×10^{-13}	4.97×10^{-16}	1.94×10^{-21}
	$y_2(x)$	1.14×10^{-13}	2.49×10^{-16}	9.72×10^{-22}
4π	$y_1(x)$	9.13×10^{-13}	1.99×10^{-15}	7.78×10^{-21}
	$y_2(x)$	4.57×10^{-13}	9.95×10^{-16}	3.89×10^{-21}
6π	$y_1(x)$	2.05×10^{-12}	4.48×10^{-15}	1.75×10^{-20}
	$y_2(x)$	1.03×10^{-12}	2.24×10^{-15}	8.75×10^{-21}
8π	$y_1(x)$	3.65×10^{-12}	7.96×10^{-15}	3.11×10^{-20}
	$y_2(x)$	1.83×10^{-12}	3.98×10^{-15}	1.56×10^{-20}
10π	$y_1(x)$	5.71×10^{-12}	1.24×10^{-14}	4.86×10^{-20}
	$y_2(x)$	2.85×10^{-12}	6.22×10^{-15}	2.43×10^{-20}

TABLE IX
ABSOLUTE ERROR FOR PROBLEM 4 WITH $h = \pi/5$

x	y_i	BTSHM-EP	BTSHM-BP	OTSBHM
2π	$y_1(x)$	6.83×10^{-15}	1.64×10^{-17}	9.50×10^{-24}
	$y_2(x)$	3.42×10^{-15}	8.21×10^{-18}	4.75×10^{-24}
4π	$y_1(x)$	2.73×10^{-14}	6.57×10^{-17}	3.80×10^{-23}
	$y_2(x)$	1.37×10^{-14}	3.28×10^{-17}	1.90×10^{-23}
6π	$y_1(x)$	6.15×10^{-14}	1.48×10^{-16}	8.55×10^{-23}
	$y_2(x)$	3.07×10^{-14}	7.39×10^{-17}	4.27×10^{-23}
8π	$y_1(x)$	1.09×10^{-13}	2.63×10^{-16}	1.52×10^{-22}
	$y_2(x)$	5.47×10^{-14}	1.31×10^{-16}	7.60×10^{-23}
10π	$y_1(x)$	1.71×10^{-13}	4.10×10^{-16}	2.37×10^{-22}
	$y_2(x)$	8.54×10^{-14}	2.05×10^{-16}	1.19×10^{-22}

TABLE X
ABSOLUTE ERROR FOR PROBLEM 4 WITH $h = \pi/12$

x	y_i	BTSHM-EP	BTSHM-BP	OTSBHM
2π	$y_1(x)$	6.16×10^{-21}	1.69×10^{-23}	7.48×10^{-33}
	$y_2(x)$	3.08×10^{-21}	8.44×10^{-24}	3.74×10^{-33}
4π	$y_1(x)$	2.47×10^{-20}	6.75×10^{-23}	2.99×10^{-32}
	$y_2(x)$	1.23×10^{-20}	3.38×10^{-23}	1.50×10^{-32}
6π	$y_1(x)$	5.55×10^{-20}	1.52×10^{-22}	6.73×10^{-32}
	$y_2(x)$	2.77×10^{-20}	7.60×10^{-23}	3.37×10^{-32}
8π	$y_1(x)$	9.86×10^{-20}	2.70×10^{-22}	1.20×10^{-31}
	$y_2(x)$	4.93×10^{-20}	1.35×10^{-22}	5.98×10^{-32}
10π	$y_1(x)$	1.54×10^{-19}	4.22×10^{-22}	1.87×10^{-31}
	$y_2(x)$	7.71×10^{-20}	2.11×10^{-22}	9.35×10^{-32}

Problem 5

We consider the nonlinear Fehlberg problem

$$y_1^{''} = -4x^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}},$$

$$y_2^{''} = \frac{2y_1}{\sqrt{y_1^2 + y_2^2}} - 4x^2 y_2,$$

subject to the conditions

$$y_1\Big(\sqrt{\frac{\pi}{2}}\Big) = 0,\ y_1^{'}\Big(\sqrt{\frac{\pi}{2}}\Big) = -2\sqrt{\frac{\pi}{2}},$$

$$y_2\Big(\sqrt{\frac{\pi}{2}}\Big) = 1,\ y_2^{'}\Big(\sqrt{\frac{\pi}{2}}\Big) = 0,$$

with the exact solutions $y_1(x) = 2\cos(x^2)$, $y_2(x) = \sin(x^2)$. This problem was chosen to demonstrate the three similar block methods' performance on a nonlinear system with variable coefficients and was solved in [12] and [17].

TABLE XI
ABSOLUTE ERRORS ($|\ y(x_i) - y_i\ |$) AT THE END POINT $x = 10$ FOR PROBLEM 5

N	y_i	BTSHM-EP	BTSHM-BP	OTSBHM
200	$y_1(x)$	2.5716×10^{-6}	1.1898×10^{-7}	2.8919×10^{-10}
	$y_2(x)$	2.5436×10^{-6}	1.1876×10^{-7}	2.1970×10^{-10}
400	$y_1(x)$	1.0536×10^{-8}	5.4164×10^{-10}	5.8505×10^{-14}
	$y_2(x)$	1.0797×10^{-8}	5.5543×10^{-10}	4.7740×10^{-14}
800	$y_1(x)$	4.1702×10^{-11}	2.1797×10^{-12}	7.1054×10^{-15}
	$y_2(x)$	4.3007×10^{-11}	2.2389×10^{-12}	1.2879×10^{-14}
1600	$y_1(x)$	1.6964×10^{-13}	5.8842×10^{-15}	6.5503×10^{-15}
	$y_2(x)$	1.7997×10^{-13}	1.5543×10^{-14}	1.1546×10^{-14}
3200	$y_1(x)$	7.8826×10^{-15}	1.4988×10^{-14}	6.9941×10^{-15}
	$y_2(x)$	1.1435×10^{-14}	2.3426×10^{-14}	1.0214×10^{-14}

We compare these methods because they share the same computational effort: the schemes have an equal number of function evaluations. The results in Table XI shows that despite sharing the same number of points and using different step lengths, the OTSBHM is superior to the BTSHM-EP and BTSHM-BP.

III. CONCLUSION

We have presented an optimized two-step block hybrid method and implemented simultaneously to solve (1). We tested the schemes on both scalar and system initial value problems of the linear and non-linear type. Tables II-XI show the details of the numerical results. It is clear from the results that the type of intra-step points imposed during formulation affects the accuracy of implicit two-step hybrid block methods.

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