A New Family of Smoothing Exact Penalty Functions for the Constrained Optimization Problem

Bingzhuang Liu

***Abstract*—In present paper, a new family of smoothing exact penalty functions is given for a general nonlinear inequality constrained optimization problem. We study the simple smoothed penalty algorithm and its convergenceis under the generalized MFCQ condition.**

***Index Terms*—Penalty function, Constrained optimization, Smoothing penalty function, MFCQ condition.**

I. Introduction

MANY problems in industry design, engineering, management science and data science can be modeled as a nonlinear optimization problem (see [1], [2], [3]). Here we study the following problem

$$(P) \qquad \begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, i = 1, \cdots, m, \end{array} \tag{1}$$

where $f_i : R^n \to R, i = 0, 1, \cdots, m,$ are continuously differentiable functions. In this paper, we assume that

$$\inf_{x \in R^n} f_0(x) > 0. \tag{2}$$

The above assumption is common since if it is not satisfied, then we can take the place of $f_0(x)$ by $e^{f_0(x)} + 1$.

Many penalty function methods have been proposed to solve the problem (P) in the literatures. The classical l_1 exact penalty function [4] is given as

$$L_1(x, \beta) = f_0(x) + \beta \sum_{i=1}^{m} f_i^+(x), \tag{3}$$

where $\beta > 0$ is a penalty parameter, and

$$f_i^+(x) = \max\{0, f_i(x)\}, \; i = 1, ..., m.$$

Another kind of exact penalty function is L_p penalty function, where the penalty term is constructed by $\|z\|_p$ $(0 < p < 1)$, that is

$$L_p(x, \beta) = f_0(x) + \beta \sum_{i=1}^{m} [f_i^+(x)]^p. \tag{4}$$

In [5], the authors used this kind of penalty function to establish an exact penalty global optimization algorithm model.

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Bingzhuang Liu is a college lecturer in School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong, 255000, P.R. China. (e-mail: lbzlyj@126.com)

In [6], Huang and Yang studied the nonconvex multiobjective optimization problems by a $p-$th power penalty function

$$L_{p,1}(x, \beta) = \{f_0^p(x) + \beta \sum_{i=1}^{m} [f_i^+(x)]^p\}^{\frac{1}{p}}, \tag{5}$$

which was also studied by Robinov *et al* [7] [8].

We say that the penalty functions $L_1(x, \beta)$, $L_p(x, \beta)$ and $L_{p,1}(x, \beta)$ are exact in the sense that when the penalty parameter is sufficiently large, the optimal solution of the corresponding penalty problem is feasible for the primal problem (P)(which can be seen in Han [9] and Di Pillo [10]).

The exact penalty functions are nondifferentiable at the edge of the feasible set. This defect prevents the use of efficient optimization methods which are based on the Gradient-type or Newton-type algorithms. To overcome the above defect, many researchers (see BeaTal and Teboulle [11], Herty *et al* [12] and Pinar and Zenios [13]) attempted to establish the corresponding smoothed penalty function methods by using the smooth approximation functions.

In [14], Auslender, Cominetti and Haddou studied the smoothed penalty methods. In [15], Gonzaga and Castillo studied the smoothed penalty methods for nonlinear inequality constrained optimization problems. In Xu *et al* [16] and Lian [17], they studied the smoothing penalty functions for the inequality constrained problems.

To study the complementarity problems, Chen and Mangasarian [18] also constructed the smoothing penalty functions by integrating probability distributions. Herty *et al* [12] investigated the smoothing penalty functions for the optimization problems with box and equality constraints. Wu *et al* [19] studied the smoothing penalty functions to study the global optimization problem.

Meng *et al* [20] proposed two smoothing penalty functions for

$$L_{\frac{1}{2}}(x, \beta) = f(x) + \beta \sum_{i=1}^{m} \sqrt{f_i^+(x)}. \tag{6}$$

Some similar smoothing techniques for (6) were proposed in [21] and [22].

Moreover, smoothed penalty methods had important roles in solving the special optimization problems such as the network-structured problems and the minimax problems in [13], and the traffic flow network models in [12].

In this paper, for the problem (P), we give the new smoothing exact penalty functions. And we establish the smoothed penalty algorithm and analysis the convergence of the algorithm.

We arrange this paper as follows. In Section II, we propose a unified smoothing approximate approach to the penalty function $L_{p,1}(x,\beta)$ of (5) and give the simple algorithm for the smoothing penalty function. In Section III, the convergence of our algorithm in Section II is discussed with a generalized MFCQ condition. We also consider the same smoothing approach to the penalty function $L_1(x,\beta)$ of (3) in Section IV. Our main conclusion is given in Section V.

II. Smoothing penalty functions

We now consider a function $\phi : R \to R_+$ which satisfies the following properties:

(a1) $\phi(\cdot)$ is a continuously differentiable and convex function, and $\phi'(0) > 0$;

(a2) $\forall t > 0$, $\phi(t) \geq t$;

(a3) $\lim_{t\to-\infty} \phi(t) = a \geq 0$;

(a4) $\lim_{t\to+\infty} \frac{\phi(t)}{t} = 1$.

It can be easily shown that the following functions $\phi_i(\cdot)(i = 1,2,3,4,5)$ satisfy the above properties.

$$\phi_1(t) = \log(1 + e^t);$$

$$\phi_2(t) = \frac{\sqrt{t^2+4}+t}{2};$$

$$\phi_3(t) = \begin{cases} \frac{1}{2}e^t, & \text{if } t \leq 0, \\ \frac{1}{2}e^{-t} + t, & \text{if } t > 0; \end{cases}$$

$$\phi_4(t) = \begin{cases} e^t + 1, & \text{if } t \leq 0, \\ t + 2, & \text{if } t > 0; \end{cases}$$

$$\phi_5(t) = \begin{cases} 0, & \text{if } t < -1, \\ \frac{(t+1)^2}{4}, & \text{if } -1 \leq t \leq 1, \\ t, & \text{if } t > 1. \end{cases}$$

From (a1)-(a4), we know that this class of function $\phi(\cdot)$ satisfies the better approximate properties:

(a5) $\lim_{t\to-\infty} \phi'(t) = 0$ and $\forall t \in R$, $0 \leq \phi'(t) \leq 1$;

(a6) $\forall t \in R$, $r\phi(\frac{t}{r})$ is monotonically increasing, where $r > 0$;

(a7) $\forall t \in R$, $r\phi(\frac{t}{r}) \to t^+$, as $r \to 0^+$.

For (P), we can see that

$$L_{p,1}(x,\beta,r) = \{f_0^p(x) + r^p \sum_{i=1}^{m}[\phi(\frac{\beta^{\frac{1}{p}} f_i(x)}{r})]^p\}^{\frac{1}{p}}. \quad (7)$$

From the property (a7), it is easy to know that as $r \to 0^+$,

$$L_{p,1}(x,\beta,r) \to \{f_0^p(x) + \beta \sum_{i=1}^{m}[f_i^+(x)]^p\}^{\frac{1}{p}},$$

where $0 < p < 1$.

The following algorithm is based on the function $L_{p,1}(x,\beta,r)(0 < p < 1)$.

Algorithm 2.1

Step 0. Let $\beta_0 = 1$, $r_0 = 1$, $0 < \rho < 1$, $\tau > 1$ and $x^0 \in R^n$, and set $k := 0$.

Step 1. Find an

$$x^k \in \arg\min_{x\in R^n} L_{p,1}(x,\beta_k,r_k). \quad (8)$$

Step 2. Let

$$r_{k+1} = \begin{cases} \rho r_k, & \text{if } 0 \leq \|f^+(x^k)\| \leq r_k, \\ r_k, & \text{otherwise;} \end{cases}$$

$$\beta_{k+1} = \begin{cases} \beta_k, & \text{if } \|f^+(x^k)\| = 0, \\ \tau\beta_k, & \text{if } \|f^+(x^k)\| > 0. \end{cases}$$

Step 3. Set $k := k + 1$, and go to Step 1.

Remark 2.1 It is easy to know that

$$\inf_{x\in R^n} L_{p,1}(x,\beta,r) > 0,$$

for any $\beta > 0$ and $r > 0$ by the assumption (2).

To study the convergence of Algorithm 2.1, we denote Ω_0 the feasible set of the problem (P), that is

$$\Omega_0 = \{x \in R^n | f_i(x) \leq 0, i = 1,\cdots,m\}.$$

In this paper we always suppose that $\Omega_0 \neq \emptyset$.

III. Convergence

A. Preparation

We denote $I(\tilde{x}) = \{i \in I | f_i(\tilde{x}) = 0\}$ the index set of active constraints at $\tilde{x} \in \Omega_0$, where $I = \{1,\cdots,m\}$.

Definition 3.1[23] The Mangasarian-Fromovitz constraint qualification (MFCQ) is satisfied at $\tilde{x} \in \Omega_0$, if there exists a vector $h \in R^n$ and for any $i \in I(\tilde{x})$,

$$\nabla f_i(\bar{x})^T h < 0.$$

For the sequence $\{z^k\}_{k\in K}$, where $K \subseteq N$, we denote the index sets as

$$I^+(K) = \{i \in I \mid \limsup_{k\in K,k\to\infty} f_i(z^k) \geq 0\},$$

$$I^-(K) = \{i \in I \mid \limsup_{k\in K,k\to\infty} f_i(z^k) < 0\}.$$

Thus, $I = I^+(K) \cup I^-(K)$.

Definition 3.2 The feasible set Ω_0 satisfies the generalized Mangasarian-Fromovitz constraint qualification(GMFCQ) for the sequence $\{z^k\}_{k\in K}$, if for any sequence $K \subseteq N$, there exist a subsequence $K_0 \subseteq K$ and a vector $h \in R^n$ such that

$$\limsup_{k\in K_0,k\to\infty} \nabla f_i(z^k)^T h < 0, \quad for\ any\ i \in I^+(K_0),$$

where

$$I^+(K_0) = \{i \in I \mid \limsup_{k\in K_0,k\to\infty} f_i(z^k) \geq 0\}.$$

The above two constraint qualifications satisfy the following proposition.

Proposition 3.1 For a $\bar{z} \in \Omega_0$ satisfies MFCQ, if there exists a sequence $\{z^k\}_{k\in K}$ such that

$$\lim_{k\in K,k\to\infty} z_k = \bar{z},$$

then Ω_0 satisfies GMFCQ for the sequence $\{z^k\}_{k\in K}$.

In the following example, $\|x^k\| \to +\infty(k \to \infty)$, so MFCQ is invalid, but GMFCQ may still be valid.

Example 3.1 Let

$$\Omega_0 = \{x \in R^2 | x_1 - x_2 \leq 0, e^{-x_1} - 1 \leq 0\},$$

and

$$x^k = (k,k)^T,$$

where

$$f_1(x^k) = 0,\ f_2(x^k) = e^{-k} - 1,$$

and

$$\limsup_{k\to\infty} f_1(x^k) = 0,\ \limsup_{k\to\infty} f_2(x^k) < 0.$$

Since here $\nabla f_1(x^k)^T = (1,-1)$, we can choose a vector $h = (-1,1)^T$, and obtain that

$$\limsup_{k\to\infty} \nabla f_1(x^k)^T h = -2 < 0.$$

Then Ω_0 satisfies GMFCQ for the sequence $\{x^k\}$.

B. Convergence

In this subsection, we show that under the GMFCQ condition and other proper conditions, for the sequence $\{x^k\}_{k\in N}$ generated by Algorithm 2.1 based on $L_{p,1}(x,\beta,r)(p \in (0,1))$, there exists a k_0, such that when $k \geq k_0$, $x^k \in int\Omega_0$.

Theorem 3.1 Suppose that $\{x^k\}_{k\in N}$ is generated by Algorithm 2.1 based on $L_{p,1}(x,\beta,r)(p \in (0,1))$,

$$x^k \in \arg\min_{x\in R^n} L_{p,1}(x,\beta_k,r_k),$$

and

$$\{\nabla f_i(x^k)\}_{k\in N}(i = 0,1,\cdots,m)$$

is bounded. If Ω_0 satisfies GMFCQ for the sequence $\{x^k\}$, then there is a k_0 such that when $k \geq k_0$,

$$x^k \in int\Omega_0,$$

where $int\Omega_0$ is the interior of Ω_0.

Proof. Suppose that the subsequence $K \subseteq N$ satisfies that

$$x^k \notin int\Omega_0. \tag{9}$$

(Note that when k is sufficiently large, β_k may be a constant.)

From (9) and the assumption that Ω_0 satisfies GMFCQ for the sequence $\{x^k\}$, we know that there exist a subset $K_0 \subseteq K$ and a vector $h \in R^n$ such that for any $i \in I^+(K_0)$,

$$\limsup_{k\in K_0,k\to\infty} \nabla f_i(x^k)^T h < 0. \tag{10}$$

We denote that

$$I^*(K_0) = \{i \in I \mid f_i(x^k) > 0,\ for\ any\ k \in K_0\} \neq \emptyset. \tag{11}$$

Then by (9), we know that $I^*(K_0) \neq \Phi$, and $I^*(K_0) \subseteq I^+(K_0)$

From the definition of $I^-(K_0)$, we know that there exists a constant $\delta > 0$, such that when k is sufficiently large,

$$\nabla f_i(x^k)^T h \leq -\delta, \quad for\ any\ i \in I^+(K_0), \tag{12}$$

and

$$f_i(x^k) \leq -\delta, \quad for\ any\ i \in I^-(K_0). \tag{13}$$

We know that

$$x^k \in \arg\min_{x\in R^n} L_{p,1}(x,\beta_k,r_k),$$

then

$$\nabla_x L_{p,1}(x^k,\beta_k,r_k) = 0.$$

From the above equation, the general assumptions that $\inf_{x\in R^n} f_0(x) > 0$ and $\phi(\cdot) \geq 0$, we obtain that

$$\nabla f_0(x^k) + f_0(x^k)^{(1-p)} \sum_{i=1}^{m} \beta_k^{\frac{1}{p}} r_k^{(p-1)} \phi(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k})^{(p-1)} \phi'(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k}) \nabla f_i(x^k) = 0.$$

Let $k \in K_0$, from the above equation we obtain that

$$\begin{aligned}&\nabla f_0(x^k)^T h + f_0(x^k)^{(1-p)} \sum_{i\in I^-(K_0)} \beta_k^{\frac{1}{p}} r_k^{p-1} \\ &\phi(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k})^{(p-1)} \phi'(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k}) \nabla f_i(x^k)^T h \\ &+ f_0(x^k)^{(1-p)} \sum_{i\in I^+(K_0)} \beta_k^{\frac{1}{p}} r_k^{(p-1)} \phi(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k})^{(p-1)} \\ &\phi'(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k}) \nabla f_i(x^k)^T h = 0.\end{aligned} \tag{14}$$

For the left side of (14), we have the following statements.

(t1) By the assumption we know that $\{\nabla f_0(x^k)^T h\}_{k\in N}$ is bounded.

(t2) By (13), for any $i \in I^-(K_0)$, we have

$$\lim_{k\in K_0,k\to\infty} \frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k} = -\infty.$$

By the property (a7) of the function $\phi(\cdot)$, we have that

$$\lim_{k\in K_0,k\to\infty} \phi(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k}) = a. \tag{15}$$

On the other hand, by (13) and the convexity of $\phi(\cdot)$, we know

$$\begin{aligned}&r_k^p(\phi(0) - \phi(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k})) \\ &\geq \beta_k^{\frac{1}{p}} r_k^{(p-1)} \phi'(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k})(-f_i(x^k)) \\ &\geq \delta \beta_k^{\frac{1}{p}} r_k^{(p-1)} \phi'(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k}).\end{aligned}$$

It follows that

$$\beta_k^{\frac{1}{p}} r_k^{(p-1)} \phi'(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k}) \to 0 (k \in K_0, k \to \infty). \tag{16}$$

Since that

$$\lim_{k\to\infty} f_0(x^k) \geq \inf_{x\in R^n} f_0(x) > 0,$$

and $\{\nabla f_0(x^k)\}_{k\in N}$ is bounded, by (15) and (16), we have

$$\begin{aligned}&f_0(x^k)^{(1-p)} \sum_{i\in I^-(K_0)} \beta_k^{\frac{1}{p}} r_k^{(p-1)} \phi(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k})^{(p-1)} \\ &\phi'(\frac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k}) \nabla f_i(x^k)^T h \to 0 (k \in K_0, k \to \infty).\end{aligned}$$

(t3) From (11), (12) and the increasing property of $\phi(\cdot)$, it follows that

$$\begin{aligned}&\sum_{i\in I^+(K_0)}\beta_k^{\frac{1}{p}}r_k^{(p-1)}\phi(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})^{(p-1)}\phi'(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})\nabla f_i(x^k)^Th\\ &\leq -\delta\sum_{i\in I^+(K_0)}\beta_k^{\frac{1}{p}}r_k^{(p-1)}\phi(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})^{(p-1)}\phi'(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})\\ &\leq -\delta\sum_{i\in I^*(K_0)}\beta_k^{\frac{1}{p}}r_k^{(p-1)}\phi(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})^{(p-1)}\phi'(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})\\ &\leq -\delta\phi'(0)\sum_{i\in I^*(K_0)}\beta_k^{\frac{1}{p}}r_k^{(p-1)}\phi(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})^{(p-1)}\end{aligned} \tag{17}$$

For any $i\in I$, we have

$$r_k\phi(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})\to 0(k\to\infty). \tag{18}$$

Thus, by (17) and (18), we know

$$\begin{aligned}&f_0(x^k)^{(1-p)}\sum_{i\in I^+(K_0)}\beta_k^{\frac{1}{p}}r_k^{(p-1)}\phi(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})^{(p-1)}\\ &\phi'(\frac{\beta_k^{\frac{1}{p}}f_i(x^k)}{r_k})\nabla f_i(x^k)^Th\to-\infty(k\in K_0,k\to\infty).\end{aligned}$$

Then we know $-\infty=0$ when $k\to\infty, k\in K_0$, which deduces to a contradiction. ■

IV. Smoothing of the function $L_1(x,\beta)$

We now give a family of smoothing penalty functions to $L_1(x,\beta)$ for the problem (P),

$$L_1(x,\beta,r)=f_0(x)+r\sum_{i=1}^m\phi(\frac{\beta f_i(x)}{r}). \tag{19}$$

The above functions are constructed by $\phi(\cdot)$ given in Section II.

Based on the function $L_1(x,\beta,r)$, we give a simple algorithm for the problem (P) as follows.

Algorithm 4.1
Step 0. Let $\beta_0=1$, $r_0=1$, and $x^0\in R^n$, and set $k:=0$.
Step 1. Find an

$$x^k\in\arg\min_{x\in R^n}L_1(x,\beta_k,r_k). \tag{20}$$

Step 2. Let

$$r_{k+1}=\begin{cases}\frac{1}{2}r_k, & \text{if } 0\leq\|f^+(x^k)\|\leq r_k,\\ r_k, & \text{otherwise;}\end{cases}$$

$$\beta_{k+1}=\begin{cases}\beta_k, & \text{if } \|f^+(x^k)\|=0,\\ 2\beta_k, & \text{otherwise.}\end{cases}$$

Step 3. Set $k:=k+1$, and go to Step 1.

Remark 4.1 The following theorem shows that under the GMFCQ condition and other common conditions, when k is sufficiently large, the sequence $\{x^k\}_{k\in N}$ generated by Algorithm 4.1, will enter the feasible set Ω_0.

Theorem 4.1 Suppose that $\{x^k\}_{k\in N}$ is generated by Algorithm 4.1 based on $L_1(x,\beta,r)$,

$$x^k\in\arg\min_{x\in R^n}L_1(x,\beta_k,r_k),$$

and

$$\{\nabla f_i(x^k)\}_{k\in N}(i=0,1,\cdots,m)$$

is bounded. If Ω_0 satisfies GMFCQ for the sequence $\{x^k\}$, then for sufficiently large k,

$$x^k\in\Omega_0.$$

Proof. Suppose there is a subsequence K such that

$$x^k\notin\Omega_0,\ for\ any\ k\in K. \tag{21}$$

By Step 2 in the algorithm, we know that

$$\lim_{k\to\infty}\beta_k=+\infty. \tag{22}$$

From (21) and the assumption that Ω_0 satisfies GMFCQ for the sequence $\{x^k\}$, it follows that there exist a $K_0\subseteq K$ and $h\in R^n$ such that

$$\limsup_{k\in K_0,k\to\infty}\nabla f_i(x^k)^Th<0,\quad for\ any\ i\in I^+(K_0), \tag{23}$$

and

$$\begin{aligned}&I^*(K_0)=\{i\in I\mid f_i(x^k)>0,\ for\ any\ k\in K_0\}\neq\emptyset,\\ &I^*(K_0)\subseteq I^+(K_0).\end{aligned} \tag{24}$$

By (23) and the definition of $I^-(K_0)$, there exists a $\delta>0$, such that for all sufficiently large $k\in K_0$,

$$\nabla f_i(x^k)^Th\leq-\delta,\quad for\ any\ i\in I^+(K_0), \tag{25}$$

$$f_i(x^k)\leq-\delta,\quad for\ any\ i\in I^-(K_0). \tag{26}$$

Since

$$x^k\in\arg\min_{x\in R^n}L_1(x,\beta_k,r_k),$$

we have

$$\nabla_xL_1(x^k,\beta_k,r_k)=0,$$

that is,

$$\nabla f_0(x^k)+\sum_{i=1}^m\beta_k\phi'(\frac{\beta_kf_i(x^k)}{r_k})\nabla f_i(x^k)=0.$$

Let $k\in K_0$, from the above equality we obtain that

$$\begin{aligned}&\frac{\nabla f_0(x^k)^Th}{\beta_k}+\sum_{i\in I^-(K_0)}\phi'(\frac{\beta_kf_i(x^k)}{r_k})\nabla f_i(x^k)^Th\\ &+\sum_{i\in I^+(K_0)}\phi'(\frac{\beta_kf_i(x^k)}{r_k})\nabla f_i(x^k)^Th=0.\end{aligned} \tag{27}$$

Then for the left side of (27), we know the following arguments.

(i) By (22) and the assumption that $\{\nabla f_0(x^k)\}_{k\in N}$ is bounded, we know that

$$\lim_{k\in K_0,k\to\infty}\frac{\nabla f(x^k)^Th}{\beta_k}=0.$$

(ii) For any $i\in I^-(K_0)$, $f_i(x^k)$, then we have

$$\lim_{k\in K_0,k\to\infty}\frac{\beta_kf_i(x^k)}{r_k}=-\infty.$$

From the property (a6) of the function $\phi(\cdot)$ and the assumption that $\{\nabla f_0(x^k)\}_{k\in N}$ is bounded, we have that the second term satisfies that

$$\lim_{k\in K_0,k\to\infty}\sum_{i\in I^-(K_0)}\phi'(\frac{\beta_k f_i(x^k)}{r_k})\nabla f_i(x^k)^T h=0.$$

(iii) By (24), (25) and the properties (a1) and (a5) of $\phi(\cdot)$, we have that

$$\begin{aligned}&\sum_{i\in I^+(K_0)}\phi'(\tfrac{\beta_k f_i(x^k)}{r_k})\nabla f_i(x^k)^T h\\ &\le -\delta\sum_{i\in I^+(K_0)}\phi'(\tfrac{\beta_k f_i(x^k)}{r_k})\\ &\le -\delta\sum_{i\in I^*(K_0)}\phi'(\tfrac{\beta_k^{\frac{1}{p}} f_i(x^k)}{r_k})\\ &\le -\delta m_0\phi'(0),\end{aligned}$$

where m_0 denotes the number of the elements in $I^*(K_0)$.

Set $k\to\infty, k\in K_0$, and take the limit on both sides of (27), we obtain from (i)-(iii) that

$$\delta m_0\phi'(0)\le 0,$$

which deduces to a contradiction. ■

Remark 4.2 Next we give an example and we can see that the above result does not hold for the equality constrained problem.

Example 4.1 We consider

$$\begin{array}{ll}\min & f_0(x)=x_1+x_2^2\\ \text{s.t.} & f_1(x)=x_1-x_2^2=0,\end{array}$$

where $\Omega_0=\{x\in R^2|x_1=x_2^2\}$. the only optimal solution is $\{(0,0)\}$.

It is obvious that

$$\nabla f_1(0,0)=(1,0)^T,$$

and

$$h=(0,s)^T(s\ne 0)$$

satisfy that

$$\nabla f_1(0,0)^T h=0.$$

So MFCQ holds at $(0,0)$. Let

$$L_1(x,\beta,r)=x_1+x_2^2+r\sqrt{\frac{\beta^2}{r^2}(x_1-x_2^2)^2+4}.$$

It is easy to know that for any $x\in R^2$, $\beta\ge 1$ and $r\ge 1$, it holds that $L_1(x,\beta,r)>0$, and for $\beta>1$ and $r>1$, it also holds that

$$\arg\min_{x\in R^2}L_1(x,\beta,r)\ne\emptyset.$$

Let

$$x^k=(x_1^k,x_2^k)=(-\frac{2r_k}{\beta_k\sqrt{\beta_k^2-1}},0)\to(0,0)(k\to\infty).$$

It is obvious that

$$(x_1^k,x_2^k)\notin\Omega_0$$

for any $k>1$. ■

V. Conclusion

In our present paper, the family of smoothing approximate approach for the exact penalty functions for inequality constrained problem is given. We establish the simple smoothed penalty algorithm and discuss its global convergence under proper conditions.

In our future works, we will use this family of smoothing exact penalty functions to study the mathematical program with complementarity constraints (MPCC).

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