Discriminating Signals by Arrival Time under the Influence of Additive and Multiplicative Random Distortions

Oleg V. Chernoyarov, Alexander V. Zakharov, Alexander A. Makarov and Ilya A. Buravlev

Abstract—**In this paper, the quasi-optimal and optimal algorithms are considered for discriminating the pulse signals by arrival time in the presence of the fast Gaussian modulating interference and additive Gaussian white noise. The case is examined when the time positions of the discriminable signals are a priori unknown and they take values from different clock intervals, so that such signals do not overlap in time. For the introduced algorithms, the analytical expressions are obtained for both the first two moments of the decision statistics and the probabilities of errors occurring while discriminating the signals by arrival time. By statistical simulation methods, one confirms the good agreement of these expressions with the corresponding experimental data presenting a wide range of values of the parameters of both the useful signals and the modulating interference. The comparison of the performance of the quasi-optimal and optimal algorithms in terms of the discrimination error probability is also carried out. It is established that the optimal algorithm can provide a significant gain in the discrimination quality.**

Index Terms—**Pulse signal, modulating interference, Gaussian process, maximum likelihood method, discrimination algorithm, decision statistic, local Markov approximation method, average error probability**

I. Introduction

In radio engineering systems, to transmit discrete and continuous information, a sequence of rectangular pulses is applied as a carrying oscillation [1]-[4]. For continuous information transmission in such systems, time-pulse carrier modulation is often used by which the time positions of the separate sequencing pulses are changed (modulated) within the specified clock intervals according to the transmitted message. Simultaneously, discrete (digital) information can be transmitted using positional modulation by which the transmitted pulse carrying discrete information is located within one of the several predetermined clock intervals. In this case, the task of receiving (demodulating) a discrete component of a discrete-continuous message is reduced to determining the clock intervals within which the transmitted signals are localized. At the same time, reception (demodulation) of the continuous component of the message is reduced to measuring the arrival time (time position) of the received signals relative to the beginning of the clock intervals within which these signals are localized.

In practice, the information signal reception and processing is usually performed against additive broadband noise that is always present in real radio-physical information transmission systems and masks the received signals. In addition, there may be a modulating (multiplicative) interference that occurs in the channels of information transmission and leads to random changes (modulation) of the parameters of the received signals (its amplitude, phase, etc.). In this case, under the influence of a modulating interference, the received signal waveform is distorted and becomes random. This leads to errors in receiving and demodulating the information signals and, therefore, to decreasing the reliability of the received information.

In this paper, the problem is considered of receiving a discrete (digital) message component transmitted by means of a pulse carrier positional modulation. Such a task is reduced to discriminating the signals by their localization within different clock intervals. In this case, the specific signal time position within each clock interval is insignificant (it is a spurious parameter) and may be a priori unknown.

II. The Problem Statement

One starts with considering the problem of discriminating the two pulses by arrival time. It is then presupposed that the received signals $s_j(t,\lambda_{0j})$, $j=1,2$ are distorted by a modulating interference. As an adequate model of such signals, one uses a model of the form [3]-[6]

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O. V. Chernoyarov is a Professor of the Department of Electronics and Nanoelectronics, National Research University “MPEI”, Moscow, Russia, on leave from the Maikop State Technological University, Maikop, Russia (e-mail: chernoyarovov@mpei.ru).

A. V. Zakharov is an Associate Professor of the Department of Radio Physics, Voronezh State University, Voronezh, Russia (e-mail: al.vc.zakharov@gmail.com).

A. A. Makarov is a Research Engineer of the Department of Electronics and Nanoelectronics, National Research University “MPEI”, Moscow, Russia (corresponding author, phone: +7-495-362-7168; fax: +7-495-362-89-38; e-mail: al.an.makarov@mail.ru).

I. A. Buravlev is a Senior Teacher of the Voronezh State Technical University, Voronezh, Russia (e-mail: burr2005@yandex.ru).

$$s_j(t,\lambda_{0j}) = a_0[1+k\xi_0(t)]\,\mathrm{I}\left(\frac{t-\lambda_{0j}}{\tau_0}\right), \qquad (1)$$

where a_0 is the signal amplitude in the absence of a modulating interference; τ_0 is the signal duration; λ_{0j}, $j=1,2$ are the arrival times of the signals corresponding to their midpoints; $\xi_0(t)$ is the dimensionless stationary random process with a zero mathematical expectation and a unit dispersion, based on it, the law of a signal amplitude modulation resulting from the influence of a modulating interference is set; k is the modulation factor determining the amplitude modulation depth; $\mathrm{I}(x)=1$ under $|x|\le 1/2$, and $\mathrm{I}(x)=0$ under $|x|>1/2$ present the function determining the rectangular signal (1) waveform in the absence of a modulating interference (i.e., in the case when $k=0$).

Now it is suggested that the time positions λ_{0j}, $j=1,2$ of the signals $s_j(t,\lambda_{0j})$ (1) are a priori unknown, but they do not coincide and their values are from the set non-overlapping intervals $\lambda_{01}\in[\Lambda_{11},\Lambda_{21}]$ and $\lambda_{02}\in[\Lambda_{12},\Lambda_{22}]$, where $\Lambda_{21}<\Lambda_{12}$. In addition, the signals $s_1(t,\lambda_{01})$ and $s_2(t,\lambda_{02})$ (1) are considered as non-overlapping in time for all the possible values of $\lambda_{01}\in[\Lambda_{11},\Lambda_{21}]$, $\lambda_{02}\in[\Lambda_{12},\Lambda_{22}]$. In other words, the intervals $[T_{11},T_{21}]$ and $[T_{12},T_{22}]$ of a time localization of the signals $s_1(t,\lambda_{01})$ and $s_2(t,\lambda_{02})$ do not overlap, so that the following condition is satisfied:

$$T_{21}<T_{12} \text{ or } \Lambda_{21}+\tau_0<\Lambda_{12}, \qquad (2)$$

where $T_{11}=\Lambda_{11}-\tau_0/2$, $T_{21}=\Lambda_{21}+\tau_0/2$, $T_{12}=\Lambda_{11}-\tau_0/2$, $T_{22}=\Lambda_{22}+\tau_0/2$. Then the signals (1) are orthogonal, i.e., $\int_{-\infty}^{\infty} s_1(t,\lambda_{01})s_2(t,\lambda_{02})\mathrm{d}t=0$.

The stationary random process $\xi_0(t)$ describing the modulating interference is considered as the Gaussian one that is valid in many cases because the central limit theorem of probability theory holds [4]-[6]. Then, for the full statistical description of the random process $\xi_0(t)$, it is sufficient to specify its spectral density $G_0(\omega)$ [4]-[6] that can be represented in a general form as follows

$$G_0(\omega)=(\gamma_0/2)g(\omega/\Omega_0). \qquad (3)$$

Here the notations are: $\Omega_0=\int_0^{\infty}G_0^2(\omega)\mathrm{d}\omega/\max G_0^2(\omega)$ is the effective spectral density width, $\gamma_0/2=\max G_0(\omega)$ is an intensity of the random process $\xi_0(t)$, while the function $g(x)$ describes the shape of the spectral density (3) and is normalized so that $\int_0^{\infty}g^2(x)\mathrm{d}x=1$, $\max g(x)=1$. As the random process $\xi_0(t)$ (1) dispersion is equal to 1, then, taking into account (3), the following relation is satisfied:

$$\gamma_0=2\pi/\Omega_0\Theta, \qquad (4)$$

where $\Theta=\int_0^{\infty}g(x)\mathrm{d}x$ is the parameter defined by the shape of the spectral density (3) of the modulating interference $\xi_0(t)$.

In this case, one assumes that the distortions of the shape of the signals (1) caused by the modulating interference $\xi_0(t)$ are "fast", so that the correlation time $\tau_c=2\pi/\Omega_0$ of the modulating interference $\xi_0(t)$ is much less than the signal (1) duration τ_0, i.e. the condition

$$\mu_0=\tau_0/\tau_c=\tau_0\Omega_0/2\pi>>1 \qquad (5)$$

is fulfilled. From (5), it follows that $\tau_0\Omega_0>>1$, i.e. the modulating interference $\xi_0(t)$ influence leads to a significant spectrum spreading of the signals (1).

In order to simplify the procedure for synthesizing the discrimination algorithm, it is convenient to represent the signal (1) distorted by the modulating interference as the sum $s_j(t,\lambda_{0j})=s_{0j}(t,\lambda_{0j})+s_{mj}(t,\lambda_{0j})$, $j=1,2$, where

$$s_{0j}(t,\lambda_{0j})=a_0\,\mathrm{I}\left(\frac{t-\lambda_{0j}}{\tau_0}\right) \qquad (6)$$

is the undistorted signal (1) component coinciding with the rectangular pulse having the amplitude a_0 and the duration τ_0, while

$$s_{mj}(t,\lambda_{0j})=a_0k\xi_0(t)\,\mathrm{I}\left(\frac{t-\lambda_{0j}}{\tau_0}\right)=\xi(t)\,\mathrm{I}\left(\frac{t-\lambda_{0j}}{\tau_0}\right) \qquad (7)$$

is the distorted signal (1) component that is a segment of the stationary centered Gaussian random process $\xi(t)=a_0k\xi_0(t)$ with the dispersion $\sigma_\xi^2=a_0^2k^2$ and the spectral density

$$G(\omega)=\sigma_\xi^2G_0(\omega). \qquad (8)$$

It should be noted that, according to (3), the representation $G(\omega)=(\gamma/2)g(\omega/\Omega_0)$ is also applicable, where $\gamma/2=\max G(\omega)=a_0^2k^2\gamma_0/2$ is the intensity of the random process $\xi(t)$.

Let the signals $s_j(t,\lambda_{0j})$ (1) are observed against the additive Gaussian white noise $n(t)$ with the spectral density $G_N(\omega)=N_0/2$, while the additive noise $n(t)$ and the modulating interference $\xi_0(t)$ are statistically independent.

White noise is an adequate model of a broadband fluctuation noise, including the internal thermal noise of electronic devices [4]-[7]. The white noise model is widely used in practice to describe the real noise whose spectral density is constant or varies little within the bandwidth of the receiver [4]-[7].

Thus, the additive mix

$$x(t) = s_j(t, \lambda_{0j}) + n(t), \quad t \in [T_1, T_2], \quad j = 1, 2 \tag{9}$$

of a one of the two signals $s_j(t, \lambda_{0j})$ (1) and the white noise $n(t)$ arrives to the receiver input, but it is not known a priori which one of the signals, $s_1(t, \lambda_{01})$ or $s_2(t, \lambda_{02})$, is present in the realization of the observed data (9). Here $[T_1, T_2]$ is the observation interval that includes the time localization intervals $[T_{11}, T_{21}]$ and $[T_{12}, T_{22}]$ of the signals (1).

In view of the latter, the problem of discriminating the signals distorted by a modulating interference can be formulated as follows. Based on the received realization $x(t)$ (9) and available prior information on the signal and noise properties, it is necessary to determine which of the signals, $s_1(t, \lambda_{01})$ or $s_2(t, \lambda_{02})$, is present at the receiver input. In other words, it is necessary to determine within which of the time intervals – $[T_{11}, T_{21}]$ and $[T_{12}, T_{22}]$ – the received signal (1) is present.

Now one can proceed to a direct study of the algorithms for discriminating the signals (1) distorted by a modulating interference and examine the performance characteristics of these algorithms.

III. The Quasi-Optimal Discrimination Algorithm

A. *The Synthesis of the Discrimination Algorithm*

In practice, the synthesis of signal discrimination devices is often carried out without taking into account a modulating interference, i.e., in the synthesis it is presupposed that the modulating interference is absent. This allows us to simplify the structure of the synthesized discrimination algorithm and to reduce hardware and computational costs of its practical implementation. However, the resulting discrimination algorithm is no longer optimal in the presence of a modulating interference. This algorithm is called a quasi-optimal one.

Thus one considers a symmetric signal system built on the assumption that the prior probabilities p_1 and p_2 of receiving each of the two signals $s_1(t, \lambda_{01})$ and $s_2(t, \lambda_{02})$ are the same and equal to $p_1 = p_2 = 1/2$. In addition, the energies of the undistorted signals (6) are then also the same and equal to

$$E = a_0^2 \tau_0 . \tag{10}$$

Therefore, in order to synthesize the quasi-optimal discrimination algorithm, the maximum likelihood (ML) method [5]-[7] is applied.

According to the ML method and under the assumption that the modulating interference is absent ($k = 0$), in order to discriminate the signals (6) it is necessary to generate the functionals [5], [7]

$$M_j(\lambda_j) = \frac{2}{N_0} \int_{T_{1j}}^{T_{2j}} x(t) s_j(t, \lambda_j) \mathrm{d}t = \frac{2a_0}{N_0} \int_{\lambda_j - \tau_0/2}^{\lambda_j + \tau_0/2} x(t) \mathrm{d}t , \quad j = 1, 2 \tag{11}$$

as the functions of all the possible values $\lambda_j \in [\Lambda_{1j}, \Lambda_{2j}]$ of the time positions of the signals. It should be noted that each of the functionals – $M_j(\lambda_j)$, $j = 1, 2$ (11) – is the logarithm of the functional of the likelihood ratio (FLR) specified (up to an insignificant constant term) for the case of receiving the undistorted signal $s_{0j}(t, \lambda_{0j})$ (6) against the white noise $n(t)$ [5], [7].

Based on the functionals $M_j(\lambda_j)$ (11), one can determine the values

$$M_j^* = \sup M_j(\lambda_j), \quad \lambda_j \in [\Lambda_{1j}, \Lambda_{2j}], \quad j = 1, 2 \tag{12}$$

of the absolute maxima of these functionals within the limits of the corresponding prior intervals $\lambda_j \in [\Lambda_{1j}, \Lambda_{2j}]$, and then one compares them with each other. If $M_1^* > M_2^*$, then the decision is made in favor of the H_1 hypothesis on the presence of the 1-st signal $s_1(t, \lambda_{01})$ in the observed data $x(t)$ (9). If $M_1^* < M_2^*$, then the decision is made in favor of H_2 hypothesis on the presence of the 2-nd signal $s_2(t, \lambda_{02})$. Thus, the quasi-optimal algorithm for discriminating the signals (1), taking into account the expressions (11), (12), can be represented in the form

$$M_1^* > M_2^* \Rightarrow H_1, \qquad M_1^* < M_2^* \Rightarrow H_2 . \tag{13}$$

The block diagram of the quasi-optimal discrimination device corresponding to the algorithm (10)-(12) is shown in Fig. 1. Here the nonations are: I is an integrator, DL is the line for the delay time τ_0, SUB is the substractor, PD is the peak detector, and RS is the resolver.

The discrimination device presented in Fig. 1 operates in the following way. The realization $x(t)$ is fed to the input of the integrator I. At the integrator output, the signal $J(t) = \int_0^t x(t')\mathrm{d}t'$ is generated, while at the SUB output –

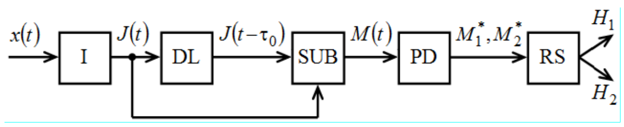


Fig. 1. Block diagram of the quasi-optimal discrimination device.

the signal $M(t)=\int_{t-\tau_0}^{t} x(t')\,\mathrm{d}t' = J(t)-J(t-\tau_0)$. The peak detector PD determines the magnitude of the absolute maximum of the signal $M(t)$ within the time interval $[T_{11}+\tau_0, T_{21}]$ and takes it as the M_1^* value. Then the peak detector determines the magnitude of the absolute maximum of the signal $M(t)$ within the time interval $[T_{12}+\tau_0, T_{22}]$ and takes it as the M_2^* value. Then, the RS resolver performs a comparison between the values of M_1^* and M_2^*. If $M_1^* > M_2^*$, then the decision is made in favor of the H_1 hypothesis on the presence of the 1-st signal $s_1(t,\lambda_{01})$. And, vice versa, if $M_1^* < M_2^*$, then the decision is made in favor of the H_2 hypothesis on the presence of the 2-nd signal $s_2(t,\lambda_{02})$.

One can pass now to the evaluation of the performance of the algorithm (13) for discriminating the signals $s_j(t,\lambda_{0j})$ (1) distorted by a modulating interference and observed against the additive white noise $n(t)$. It is obvious that the performance of the algorithm (13) is determined by the statistical properties of the functionals (determining decision statistics) $M_j(\lambda_j)$ (11). Therefore, to start with, one should study their probabilistic characteristics.

B. The Characteristics of the Determining Decision Statistics of the Discrimination Algorithm

As both the modulating interference $\xi_0(t)$ and the additive noise $n(t)$ are Gaussian random processes, by definition, the process $x(t)$ is the Gaussian one too. Therefore, the functionals $M_j(\lambda_j)$ (11) being the linear transformations of the observed data $x(t)$ are also Gaussian random processes. Thus, for a full statistical description of the random processes $M_j(\lambda_j)$, it is sufficiently to specify their mathematical expectations and correlation functions [6], [7].

Then $S_{jl}(\lambda_j)=\langle M_j(\lambda_j)|H_l\rangle$, $j=1,2$, $l=1,2$ are the regular component (mathematical expectation) of the functional $M_j(\lambda_j)$ (11) in the case when the H_l hypothesis on the presence of the signal $s_l(t,\lambda_{0l})$ (1) in the observed data $x(t)$ is valid. Here $\langle\cdot\rangle$ means averaging in terms of all the possible realizations $x(t)$ with the fixed signal and noise parameters. Averaging the functionals $M_j(\lambda_j)$ (11) over all the realizations $x(t)$ under the fixed H_l hypotheses leads to $S_{jl}(\lambda_j)=0$ for $j\neq l$ and

$$S_{jj}(\lambda_j)=A_S \max\left(0, 1-|\lambda_j-\lambda_{0j}|/\tau_0\right), \quad A_S=z_0^2. \tag{14}$$

Here the value of z_0^2 is determined as

$$z_0^2 = 2E/N_0 = 2a_0^2\tau_0/N_0, \tag{15}$$

that is, it is equal to the ratio between the energy E (10) of the undistorted component (6) of the signals (1) and the spectral density $N_0/2$ of the additive noise $n(t)$.

The notations are the following: $N_{jl}(\lambda_j)=M_j(\lambda_j)-\langle M_j(\lambda_j)|H_l\rangle$, $j=1,2$, $l=1,2$ are the random component of the functional $M_j(\lambda_j)$ (11) when the H_l hypothesis on the presence of the signal $s_l(t,\lambda_{0l})$ (1) in the observed data $x(t)$ is correct. The random components $N_{jl}(\lambda_j)$ are the Gaussian random processes under $\langle N_{jl}(\lambda_j)\rangle=0$. When fulfilling (5), similarly to [7], [8], the correlation functions of the random components are equal to

$$\begin{aligned}
&K_{jl}(\lambda_{1j},\lambda_{2j})=\sigma_N^2 \max\left(0, 1-|\lambda_{2j}-\lambda_{1j}|/\tau_0\right), \; j\neq l, \\
&K_{jj}(\lambda_{1j},\lambda_{2j})=\sigma_N^2 \max\left(0, 1-|\lambda_{2j}-\lambda_{1j}|/\tau_0\right)+\left(\sigma_S^2-\sigma_N^2\right)\times \\
&\quad\times \max\Big[0, 1+\min\left(0, |\lambda_{1j}-\lambda_{0j}|/\tau_0, |\lambda_{2j}-\lambda_{0j}|/\tau_0\right)- \\
&\quad\quad -\max\left(0, |\lambda_{1j}-\lambda_{0j}|/\tau_0, |\lambda_{2j}-\lambda_{0j}|/\tau_0\right)\Big],
\end{aligned} \tag{16}$$

where $\sigma_N^2=z_0^2$, $\sigma_S^2=z_0^2[1+q_0 g(0)]$, $g(0)=2G_0(0)/\gamma_0$ is the normalized magnitude of the spectral density of the modulating interference at the frequency $\omega=0$, and

$$q_0=\gamma/N_0=a_0^2 k^2\gamma_0/N_0 \tag{17}$$

is the ratio between the intensity $\gamma/2$ of the distorted component (7) of the signals (1) and the spectral density $N_0/2$ of additive noise $n(t)$.

According to (14), (16), the signal-to-noise ratio (SNR) z^2 at the output of the quasi-optimal discrimination device is equal to [5]-[7]

$$z^2 = S_{jj}^2(\lambda_{0j})/K_{jj}(\lambda_{0j},\lambda_{0j}) = z_0^2/[1+q_0 g(0)], \tag{18}$$

where z_0^2 is defined using (15), and q_0 – using (17).

C. The Characteristics of the Discrimination Algorithm

The performance of the signal discrimination algorithm is characterized by the average discrimination error probability P_E [5]-[7]. The average error probability P_E for the quasi-optimal discrimination algorithm (13) can be represented as $P_E = p_1 P_{21} + p_2 P_{12}$. Here the notations are: p_1 and p_2 are the prior probabilities of receiving the signals $s_1(t,\lambda_{01})$ and $s_2(t,\lambda_{02})$, respectively; P_{21} is the probability of making the decision in favor of the H_2 hypothesis on the reception of the 2-nd signal $s_2(t,\lambda_{02})$ provided that the H_1 hypothesis on the presence of the 1-st signal $s_1(t,\lambda_{01})$ is correct; and P_{12} is the probability of making the decision in favor of the H_1 hypothesis on the reception of the 1-st signal $s_1(t,\lambda_{01})$ provided that the H_2 hypothesis on the presence of the 2-nd signal $s_2(t,\lambda_{02})$ is correct. The probabilities P_{21} and P_{12} are the conditional discrimination error probabilities under the H_1 or H_2 hypotheses being correct, respectively.

One can now obtain the expressions for the conditional error probabilities P_{21}, P_{12}. It should be noted that for the quasi-optimal algorithm (13), these probabilities are defined as $P_{21} = P\left[M_2^* > M_1^* \middle| H_1\right]$, $P_{12} = P\left[M_1^* > M_2^* \middle| H_2\right]$, where $P[A|B]$ means the probability of an event *A*, while the *B* hypothesis is correct. If the condition (2) is satisfied, then for all the possible values of $\lambda_1 \in [\Lambda_{11},\Lambda_{21}]$, $\lambda_2 \in [\Lambda_{12},\Lambda_{22}]$ the functionals $M_1(\lambda_1)$, $M_2(\lambda_2)$ (11) are the integrals from the realization of observed data $x(t)$ (9) over the non-overlapping intervals. Then, when fulfilling (5), the random processes $M_1(\lambda_1)$ and $M_2(\lambda_2)$ (11) at the intervals $\lambda_1 \in [\Lambda_{11},\Lambda_{21}]$ and $\lambda_2 \in [\Lambda_{12},\Lambda_{22}]$ are statistically independent. Consequently, the values M_1^* and M_2^* (12) of the absolute maxima of these processes are statistically independent too. Then the conditional error probabilities for the quasi-optimal discrimination algorithm (13) can be represented as [6]-[8]

$$P_{21} = \int_{-\infty}^{\infty} F_{11}(x)\mathrm{d}F_{21}(x),\; P_{12} = \int_{-\infty}^{\infty} F_{22}(x)\mathrm{d}F_{12}(x), \tag{19}$$

where $F_{jl}(x) = P\left[M_j^* > x \middle| H_l\right]$, $j = 1,2$, $l = 1,2$ are the distribution functions of the values M_j^* (12) of the absolute maxima of the random processes $M_j(\lambda_j)$, while the H_l hypothesis on the reception of the *l*-th signal $s_l(t,\lambda_{0l})$ is correct.

The next step is to find the expressions for the distribution functions $F_{jl}(x)$, $j = 1,2$, $l = 1,2$. One takes into account that the random processes $M_j(\lambda_j)$ (11) are the Gaussian ones characterized by the conditional mathematical expectations $S_{jl}(\lambda_j)$ (14) and the correlation functions $K_{jl}(\lambda_{1j},\lambda_{2j})$ (16). The general expressions for the distribution functions of the absolute maxima of the Gaussian random processes within the arbitrary definitional domains cannot be found even in the case of the stationary processes [6], [8], [9]. In this regard, following [6]-[8], one implies that the SNR z^2 (18) is big enough, so that the condition

$$z^2 >> 1 \tag{20}$$

is satisfied. Then, similarly to [6], [7], it can be shown that to calculate the conditional error probabilities P_{21}, P_{12} (19) while fulfilling (20), it is sufficient to use the approximations of the subintegral functions $F_{jl}(x)$ that are asymptotically exact under increasing *x*. Such approximations can be obtained similarly to [7], by applying the results from [8], [10].

For this, similarly to [7], one assumes that the lengths $\Gamma_1 = \Lambda_{21} - \Lambda_{11}$ and $\Gamma_2 = \Lambda_{22} - \Lambda_{12}$ of the prior intervals $[\Lambda_{11},\Lambda_{21}]$ and $[\Lambda_{12},\Lambda_{22}]$ of the unknown time positions of the signals (1) are the same, i.e. $\Gamma_1 = \Gamma_2$. And, following [6]-[8], one concludes that the length of $\Gamma_1 = \Gamma_2$ of the prior intervals $[\Lambda_{11},\Lambda_{21}],[\Lambda_{12},\Lambda_{22}]$ is much longer than the duration τ_0 of the signals, i.e.

$$m = (\Lambda_{21} - \Lambda_{11})/\tau_0 = (\Lambda_{22} - \Lambda_{12})/\tau_0 >> 1. \tag{21}$$

The condition (21) means that the uncertainty about the time positions of the signals (1) is much greater than the duration of these signals. Then, using the results from [8], [10] and taking into account the expressions (14), (16), similarly to [7], one obtains

$$\begin{aligned} F_{12}(x) = F_{21}(x) \approx F_N(x/\sigma_N), \\ F_{11}(x) = F_{22}(x) \approx F_S(x/\sigma_N), \end{aligned} \tag{22}$$

where

$$F_N(u) = \begin{cases} \exp\left[-mu\exp\left(-u^2/2\right)/\sqrt{2\pi}\right], & u \ge 1, \\ 0, & u < 1, \end{cases} \tag{23}$$

$$\begin{gathered} F_S(u) = \Phi\left(\frac{u}{\kappa} - z\right) - \\ -2\exp\left[\frac{\psi^2 z^2}{2} + \psi z\left(z - \frac{u}{\kappa}\right)\right]\Phi\left[\frac{u}{\kappa} - z(\psi+1)\right] + \\ +\exp\left[2\psi^2 z^2 + 2\psi z\left(z - \frac{u}{\kappa}\right)\right]\Phi\left[\frac{u}{\kappa} - z(\psi+1)\right], \\ \kappa = \sigma_S/\sigma_N = \sqrt{1 + q_0 g(0)}, \\ \psi = 2\kappa^2/\left(1+\kappa^2\right) = 2[1 + q_0 g(0)]/[2 + q_0 g(0)], \end{gathered} \tag{24}$$

and $\Phi(x) = \int_{-\infty}^{x} \exp\left(-t^2/2\right)\mathrm{d}t/\sqrt{2\pi}$.

By substituting (22)-(24) into (19), the conditional error probabilities P_{21} and P_{12} can be found and they are equal: $P_{21} = P_{12}$. The next step is to take into account that $p_1 + p_2 = 1$ and that the average error probability is determined as $P_E = P_{21} = P_{12}$. After applying the obtained expressions for the probabilities $P_{21} = P_{12}$ and carrying out the corresponding analytical transformations, for the average error probability P_E of the quasi-optimal discrimination algorithm (13) one gets:

$$P_E = \left[1 - Q(2m)\right]/2\,, \tag{25}$$

$$Q(m) = \int_{-\infty}^{\infty} F_N(u)\,\mathrm{d}F_S(u) = \frac{2\psi z}{\kappa}\exp\left(\frac{\psi^2 z^2}{2} + \psi z^2\right)\times$$
$$\times\int_1^{\infty}\exp\left[-\frac{mu}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)\right]\left\{\exp\left(-\frac{\psi z u}{\kappa}\right)\times\right.$$
$$\times\Phi\left[\frac{u}{\kappa} - z(\psi+1)\right] - \exp\left[\frac{3\psi^2 z^2}{2} + \psi z\left(z - \frac{2u}{\kappa}\right)\right]\times$$
$$\left.\times\Phi\left[\frac{u}{\kappa} - z(2\psi+1)\right]\right\}\mathrm{d}u\,,$$

where z^2 and m are defined based on (18) and (21), respectively. It must be emphasized that the expression (25) is asymptotically exact and its accuracy increases with z^2 and m.

Now one studies the error probability P_E (25) for the quasi-optimal algorithm (13). In Fig. 2a, by dashed lines, one presents the dependences of the average error probability P_E upon the ratio q_0 under $m = 20$, $g(0) = 1$ and under the different fixed values z_0. There, the curve 1 corresponds to $z_0 = 5$, 2 – $z_0 = 7$, 3 – $z_0 = 8$, 4 – $z_0 = 9$, 5 – $z_0 = 10$. In Fig. 2b, by dashed lines, one can see the similar dependences of the error probability P_E upon the ratio z_0 under $m = 20$, $g(0) = 1$ and under the different fixed values q_0. In this case, the curve 1 corresponds to $q_0 = 0.5$, 2 – $q_0 = 1$, 3 – $q_0 = 1.5$, 4 – $q_0 = 2$, 5 – $q_0 = 2.5$. And also, for comparison, the dotted line is presented here demonstrarting the dependence of the probability P_E upon the ratio z_0 under $q_0 = 0$, i.e. when a modulating interference is absent.

Then one suggests that when the value of the fixed noise power is $N_0/2$, the value z_0^2 (15) determines the energy of the undistorted component (6) while the value q_0 (17) defines the intensity of the distorted component (7) of the signal (1). And the value q_0 increases with the intensity of the modulating interference (modulation factor *k*).

From Figs. 2, it follows that the error probability P_E for the quasi-optimal discrimination algorithm (13) decreases with increasing z_0, that is, when the energy of the undistorted component (6) of the signal (1) increases. The influence of the modulating interference (the case when $q_0 > 0$) leads to the average error probability P_E increasing in comparison with the case when $q_0 = 0$ (i.e., a modulating interference is absent). At the same time, the error probability P_E increases with the modulating interference intensity (i.e., under increasing q_0 ratio), and the greater is the undistorted signal component energy (i.e., the higher is the value of z_0), the greater are the values of the probability P_E.

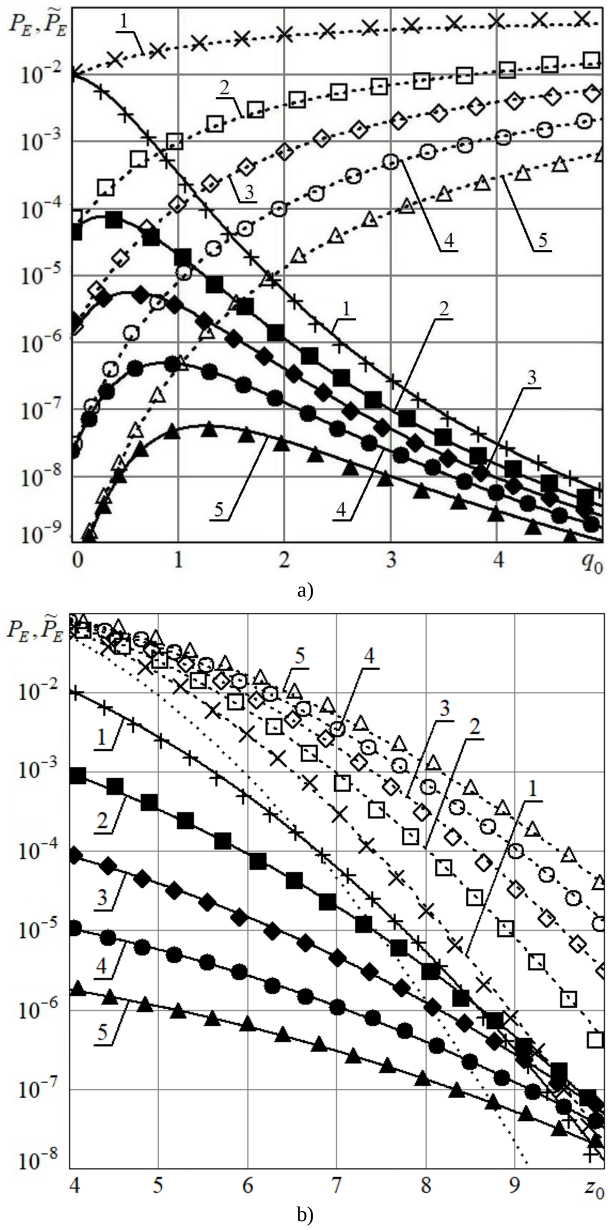


Fig. 2. The average discrimination error probability for the quasi-optimal algorithm.

In order to decrease the discrimination error probability P_E when the modulating interference causes distortions, the optimal discrimination algorithm discussed below can be applied.

IV. The Optimal Discrimination Algorithm

A. The Synthesis of the Discrimination Algorithm

One of the reasons for a decreasing performance of the quasi-optimal algorithm (13) under the influence of a modulating interference is that it is designed to discriminate the signals (6) undistorted by a modulating interference. Thus, one should consider now the discrimination algorithm designed to receive the signals (1) distorted by a modulating interference. This algorithm can be called an optimal one.

In order to synthesize the optimal discrimination algorithm, one applies the ML method [5]-[7]. Contrary to the case of the quasi-optimal algorithm synthesis (13), in the synthesis of the optimal algorithm it is assumed that the signals $s_j(t,\lambda_{0j})$ (1) distorted by a modulating interference are to be discriminated.

According to the ML method, based on the received realization of the observed data $x(t)$ (9), for each of the signals $s_j(t,\lambda_{0j})$, $j=1,2$ it is necessary to generate the FLR logarithm $L_j(\lambda_j)$ as the function of the possible values of $\lambda_j \in [\Lambda_{1j},\Lambda_{2j}]$ of the unknown signal time positions. If the inequality (5) is satisfied then, just to an accuracy of an insignificant constant summand, one obtains [6], [10]

$$L_j(\lambda_j)=\frac{1}{N_0}\int_{\lambda_j-\tau_0/2}^{\lambda_j+\tau_0/2}\left[\frac{y^2(t)}{2}+\frac{2a_0x(t)}{1+q_0g(0)}\right]\mathrm{d}t\,,\quad j=1,2\,. \tag{26}$$

In (26), $y(t)=\int_{-\infty}^{\infty}x(t')h(t-t')dt'$ is the output signal of the filter with the transfer function $H(\omega)$, which is the Fourier transform from the pulse response $h(t)$, satisfying the condition [6], [10]

$$|H(\omega)|^2=\frac{2G(\omega)}{N_0+2G(\omega)}=\frac{q_0g(\omega/\Omega_0)}{1+q_0g(\omega/\Omega_0)}, \tag{27}$$

where $G(\omega)$ (8) is the spectral density of the random process $\xi(t)$ (7) determining the distorted component (7) of the signals (1), $g(x)$ is defined using (3), and q_0 – using (17). It should be noted that the condition (27) specifies only the modulus $|H(\omega)|$ of the transfer function $H(\omega)$, i.e. the filter amplitude-frequency characteristic. And there is no restriction on the phase of the transfer function $H(\omega)$ (filter phase-frequency characteristic), so it can be chosen arbitrarily, following the criterion of a simplicity of the filter implementation.

Further, according to the ML method [5]-[7], it is necessary to generate the values

$$L_j^*=\sup L_j(\lambda_j),\quad \lambda_j\in[\Lambda_{1j},\Lambda_{2j}],\quad j=1,2 \tag{28}$$

of the absolute maxima of the functionals $L_j(\lambda_j)$ (26) within the corresponding intervals $\lambda_j\in[\Lambda_{1j},\Lambda_{2j}]$ and then to compare them with each other. If $L_1^*>L_2^*$, then the decision is made in favor of the H_1 hypothesis on the presence of the 1-st signal $s_1(t,\lambda_{01})$ in the observed data $x(t)$. And, vice versa, if $L_1^*<L_2^*$, then the decision is made in favor of the H_2 hypothesis on the presence of the 2-nd signal $s_2(t,\lambda_{02})$. Thus, the optimal algorithm for discriminating the signals (1), taking into account the expressions (26)-(28), can be represented as

$$L_1^*>L_2^*\Rightarrow H_1,\qquad L_1^*<L_2^*\Rightarrow H_2\,. \tag{29}$$

The block diagram of the optimal discrimination device corresponding to the algorithm (28) is shown in Fig. 3. Here the notations are: F is a linear filter with transfer function $H(\omega)$ satisfying to the condition (26); SQ is the squarer; A is a linear amplifier with the gain $\alpha\sim 4a_0/[1+q_0g(0)]$; SUM is the summator; I is an integrator, DL is the line for the delay time τ_0; SUB is the substractor; PD is the peak detector; and RS is the resolver.

The discrimination device presented in Fig. 3 operates in the following way. The received realization $x(t)$ is fed to the inputs of the linear filter F and the amplifier A. Then, at different outputs the specific signals are generated: at the summator SUM output – the signal $v(t)=y^2(t)+\alpha x(t)$; at the integrator output – the signal $J(t)=\int_0^t v(t')\mathrm{d}t'$; and at the substractor SUB output – the signal $L(t)=\int_{t-\tau_0}^{t}v(t')\mathrm{d}t'$. The peak detector PD determines the magnitude of the absolute maximum of the signal $L(t)$ within the time interval $[T_{11}+\tau_0,T_{21}]$ and takes it as the L_1^* value. Then the peak detector determines the magnitude of the absolute maximum of the signal $L(t)$ within the time interval $[T_{12}+\tau_0,T_{22}]$ and takes it as the L_2^* value. The RS resolver performs a comparison between the values of L_1^* and L_2^*. If $L_1^*>L_2^*$, then the decision is made in favor of the H_1 hypothesis on the presence of the 1-st signal $s_1(t,\lambda_{01})$. In turn, if $L_1^*<L_2^*$, then the decision is made in favor of the H_2 hypothesis on the presence of the 2-nd signal $s_2(t,\lambda_{02})$.

Now one can analyze the performance of the optimal algorithm (29) when discriminating the signals $s_j(t,\lambda_{0j})$ (1) distorted by a modulating interference and observed against the additive white noise $n(t)$. As the performance of the algorithm (29) is determined by the statistical properties of the determining decision statistics $L_j(\lambda_j)$ (26), one examines the probabilistic characteristics of the functionals (26) firstly.

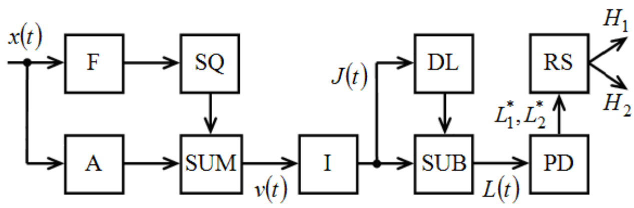


Fig. 3. Block diagram of the optimal discrimination device.

B. The Characteristics of the Determining Decision Statistics of the Discrimination Algorithm

As it is noted previously, the modulating interference $\xi_0(t)$, the additive noise $n(t)$ and the realization of the observable data $x(t)$ (9) are the Gaussian random processes. In addition, according to [6], the random process $y(t)$ (26) is the asymptotically Gaussian one under $\mu_0 \to \infty$. Therefore, if condition (5) is satisfied, then the process $y(t)$ can be considered as the approximately Gaussian one. Consequently, when fulfilling (5), the functionals $L_j(\lambda_j)$ (26) are also the approximately Gaussian random processes. Thus, for a full statistical description of the random processes $L_j(\lambda_j)$, it is sufficient to specify their mathematical expectations and correlation functions [6], [7].

Assuming that H_l, being the hypothesis on the presence of the signal $s_l(t,\lambda_{0l})$ (1) in the observed data $x(t)$, is valid, one can now denote the regular component (the mathematical expectation) of the j-th functional $L_j(\lambda_j)$ (26) as $S_{jl}(\lambda_j) = \langle L_j(\lambda_j) | H_l \rangle$, $j = 1,2$, $l = 1,2$. By averaging the functionals $L_j(\lambda_j)$ (26) over all the possible realizations for various hypotheses H_l, one can find out that the regular components $S_{jl}(\lambda_j)$ are defined from (14) to an accuracy of an insignificant constant summand A_N, and thus it should be presumed that

$$A_N = \mu_0 q_0 \int_0^{\infty} \frac{g(x)\,\mathrm{d}x}{1+q_0 g(x)},$$
$$A_S = \mu_0 q_0^2 \int_0^{\infty} \frac{g^2(x)\,\mathrm{d}x}{1+q_0 g(x)} + \frac{z_0^2\left[2+q_0 g(0)\right]}{2\left[1+q_0 g(0)\right]}. \quad (30)$$

In (30), $g(x)$, μ_0, z_0^2, q_0 are defined from (3), (5), (15), (17), respectively.

Now one can introduce the random component of the j-th functional $L_j(\lambda_j)$ (26), while the H_l hypothesis on the presence of the signal $s_l(t,\lambda_{0l})$ (1) in the observed data $x(t)$, as $N_{jl}(\lambda_j) = L_j(\lambda_j) - \langle L_j(\lambda_j) | H_l \rangle$, $j = 1,2$, $l = 1,2$. When the inequality (5) holds, the random components $N_{jl}(\lambda_j)$ are the approximately Gaussian centered random processes, so that $\langle N_{jl}(\lambda_j) \rangle = 0$. At that, similarly to [6]-[8], the correlation functions $K_{jl}(\lambda_{1j},\lambda_{2j}) = \langle N_{jl}(\lambda_{1j}) N_{jl}(\lambda_{2j}) \rangle$, $j = 1,2$, $l = 1,2$ of the random components $N_{jl}(\lambda_j)$ are defined from (16), where it should be presumed

$$\sigma_N^2 = \mu_0 q_0^2 \int_0^{\infty} \frac{g^2(x)\,\mathrm{d}x}{\left[1+q_0 g(x)\right]^2}, \; \sigma_S^2 = \mu_0 q_0^2 + z_0^2\left[1+q_0 g(0)\right]. \quad (31)$$

According to (30), (31), the SNR at the output of the optimal discrimination device is equal to [5]-[7]

$$z^2 = S_{jj}^2(\lambda_{0j}) \big/ K_{jj}(\lambda_{0j},\lambda_{0j}) = \frac{\left\{\mu_0 q_0^2 \int_0^{\infty} \frac{g^2(x)\,\mathrm{d}x}{1+q_0 g(x)} + \frac{z_0^2\left[2+q_0 g(0)\right]}{2\left[1+q_0 g(0)\right]}\right\}^2}{\mu_0 q_0^2 + z_0^2\left[1+q_0 g(0)\right]}. \quad (32)$$

It should be noted that the limits of integration in the expressions for the functionals $L_1(\lambda_1)$ and $L_2(\lambda_2)$ (26) do not overlap under $\lambda_1 \in [\Lambda_{11},\Lambda_{21}]$, $\lambda_2 \in [\Lambda_{12},\Lambda_{22}]$. Then, when fulfilling (2), (5), the random processes $L_1(\lambda_1)$ and $L_2(\lambda_2)$ (26) at the intervals $\lambda_1 \in [\Lambda_{11},\Lambda_{21}]$, $\lambda_2 \in [\Lambda_{12},\Lambda_{22}]$ are statistically independent.

C. The Characteristics of the Discrimination Algorithm

As before, the performance of the optimal signal discrimination algorithm (29) is characterized by the average discriminating error probability P_E [5]-[7]. Following [6]-[8], one presupposes that the SNR z^2 is great enough, so that the condition (20) is satisfied. One can also suggest that the lengths of the prior intervals $[\Lambda_{11},\Lambda_{21}]$ and $[\Lambda_{12},\Lambda_{22}]$ of the unknown time positions λ_{01} and λ_{02} of the signals (1) are the same and they are much longer than the duration of the signals τ_0, i.e. the condition (21) is satisfied. Then, taking into account the notations (30), (31) used to define the statistical characteristics of the functionals $L_j(\lambda_j)$ (26), the expressions (14), (16) allow us to apply the results of Section 2.C and to write the average error probability P_E for the optimal algorithm (29) in the form of (25), where the SNR z^2 is determined from (32), and

$$\kappa^2 = \frac{\sigma_S^2}{\sigma_N^2} = \frac{\mu_0 q_0^2 + z_0^2\left[1+q_0 g(0)\right]}{\mu_0 q_0^2 \int_0^{\infty} \frac{g^2(x)\,\mathrm{d}x}{\left[1+q_0 g(x)\right]^2} + \frac{z_0^2}{\left[1+q_0 g(0)\right]^2}},$$
$$\psi = 2\kappa^2 \big/ \left(1+\kappa^2\right) = \frac{2\left\{1 + z_0^2\left[1+q_0 g(0)\right] \big/ \mu_0 q_0^2\right\}}{\int_0^{\infty} \frac{g^2(x)\left\{1+\left[1+q_0 g(x)\right]^2\right\}\mathrm{d}x}{\left[1+q_0 g(x)\right]^2} + \frac{z_0^2\left\{1+\left[1+q_0 g(x)\right]^3\right\}}{\mu_0 q_0^2\left[1+q_0 g(0)\right]^2}}. \quad (33)$$

Proceeding to the study of the error probability P_E for the optimal algorithm (29), one can state that, in Fig. 2a, by solid lines, one presents the dependences of the average error probability P_E upon the ratio q_0 under $m = 20$,

$\mu_0 = 50$ and under the different fixed values of z_0. The curve 1 corresponds to $z_0 = 5$, 2 – $z_0 = 7$, 3 – $z_0 = 8$, 4 – $z_0 = 9$, 5 – $z_0 = 10$. In Fig. 2b, by solid lines, one can see the similar dependences of the error probability P_E upon the ratio z_0 under $m = 20$, $\mu_0 = 50$ and under the different values of q_0. The curve 1 corresponds to $q_0 = 0.5$, 2 – $q_0 = 1$, 3 – $q_0 = 1.5$, 4 – $q_0 = 2$, 5 – $q_0 = 2.5$. The results shown in Figs. 2 are obtained for the case of the band modulating interference, when

$$g(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases} \tag{34}$$

From Figs. 2, it follows that the error probability P_E for the optimal discrimination algorithm (29) decreases when the ratio z_0 (i.e. the energy E of the undistorted component (6) of the signal (1)) increases. On the other hand, when the ratio q_0 (that is, the intensity of the modulating interference $\xi_0(t)$) increases, the error probability P_E for the optimal algorithm (29) decreases under the fixed ratio z_0, if the value of q_0 is not too low while the value of z_0 is not too high. In other words, the modulating interference $\xi_0(t)$ influence can lead to decreasing the error probability P_E for the optimal algorithm (29). At the same time, the error probability P_E for the quasi-optimal algorithm (13) always increases with the appearance of a modulating interference. This happens because the optimal algorithm (29), in contrast to the quasi-optimal one (13), uses both the undistorted component (6) energy and the energy of the distorted signal component (7) (i.e. the modulating interference $\xi_0(t)$) while discrimination of the signals (1) is carried out.

In Fig. 4a, the solid lines show the dependences of the average error probability P_E for the optimal algorithm (29) upon the ratio q_0 under $m = 10$ and under the different fixed values of z_0 and μ_0. The dashed lines demonstrate the corresponding dependences for the quasi-optimal algorithm (13). The curves 1-3 correspond to $z_0 = 10$, while the curves 4-6 – to $z_0 = 7$. The curves 1, 4 are calculated under $\mu_0 = 40$, the curves 2, 5 – under $\mu_0 = 60$, and the curves 3, 6 – under $\mu_0 = 100$.

From Fig. 4a, it follows that the error probability P_E for the optimal discrimination algorithm (29) decreases when the ratio μ_0 (5) increases, that is, with widening the bandwidth Ω_0 of the modulating interference $\xi_0(t)$. This happens because the average power of the distorted component (7) used in optimal discrimination increases with the ratio μ_0. Moreover, the performance of the quasi-optimal algorithm (13) does not depend upon the value of μ_0.

It is time to introduce the average energy of the signals

$$E_S = \int_{-\infty}^{\infty} \left\langle s_j^2(t, \lambda_{0j}) \right\rangle \mathrm{d}t = E_0 + E_m, \tag{35}$$

where E_0 and E_m are the energies of the undistorted (6) and distorted (7) components of the signals (1) that are defined as

$$\begin{gathered} E_0 = \int_{-\infty}^{\infty} s_{0j}^2(t, \lambda_{0j}) \mathrm{d}t = a_0^2 \tau_0, \\ E_m = \int_{-\infty}^{\infty} \left\langle s_{mj}^2(t, \lambda_{0j}) \right\rangle \mathrm{d}t = \tau_0 \sigma_\xi^2 = a_0^2 \tau_0 k^2. \end{gathered} \tag{36}$$

Here σ_ξ^2 is the dispersion (8) of the random process $\xi(t)$ (7).

The ratio between the average energy E_S (35) of the signals (1) and the spectral density $N_0/2$ of the additive noise is denoted as

$$Z_S^2 = 2E_S/N_0 = z_0^2(1 + k^2). \tag{37}$$

This ratio can be considered as the average SNR at the receiver input. Then

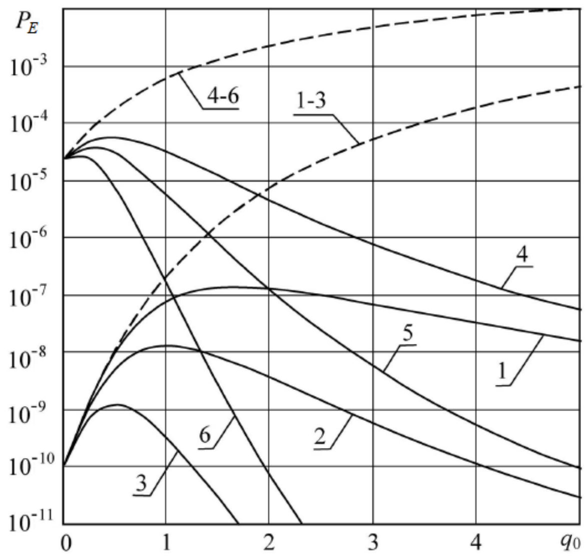
$$z_0^2 = \frac{Z_S^2}{1 + k^2}, \qquad q_0 = \frac{1}{2\mu_0 \Theta} \frac{Z_S^2 k^2}{1 + k^2},$$

where Θ is defined in the same way as in (4).

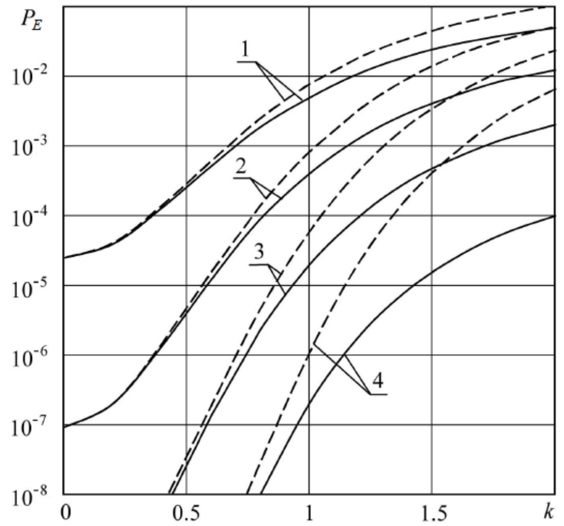
In Fig. 4b, one can see the solid lines describing the dependences of the average error probability P_E of the optimal algorithm (29) upon the modulation factor k under $m = 20$, $\mu_0 = 100$ and under the different fixed values of Z_S (37). The curve 1 corresponds to $Z_S = 7$, 2 – $Z_S = 8.5$, 3 – $Z_S = 10$, 4 – $Z_S = 12$. The corresponding dependences for the quasi-optimal algorithm (13) are also shown by the dashed lines.

The results presented in Figs. 4a, 4b are obtained for the case when the function $g(x)$ is defined according to (34).

From Fig. 4b, it follows that for both the quasi-optimal (13) and optimal (29) algorithms, the discrimination error probability P_E increases with the modulation factor k under the fixed average signal energy E_S (the fixed ratio Z_S). This is so due to the fact that, under the fixed average energy E_S, the energy of the undistorted signal component E_0 decreases with the factor k increasing. Still, however, when the factor k increases, the error probability P_E for the optimal discrimination algorithm (29) always turns out to be significantly less than the one for the quasi-optimal algorithm (13).



a)



b)

Fig. 4. The average discrimination error probability for the optimal algorithm.

From Figs. 2, 4, it also follows that the optimal discrimination algorithm (29) in the presence of a modulating interference $\xi_0(t)$ (the case when $q_0 > 0$ or $k > 0$) provides a lower error probability P_E (i.e., higher performance) than the quasi-optimal algorithm (13). And the gain in the error probability P_E that the optimal algorithm demonstrates in comparison with the quasi-optimal one increases together with the modulating interference intensity (ratio q_0 (17) or factor *k*), or with the undistorted signal component energy E_0 (36) (ratio z_0 (15)), or under increasing the average signal energy E_S (35) (ratio Z_S (37)), or under widening the bandwidth Ω_0 of the modulating interference spectral density (ratio μ_0 (5)).

V. Statistical Simulation of the Discrimination Algorithms

In order to test the performance of the introduced discrimination devices and to establish the limits of applicability of the asymptotically exact formulas for their characteristics, statistical computer simulation of the operation of the algorithms (13), (29) is carried out.

During simulation, within the intervals $\left[\tilde{\Lambda}_{1j}, \tilde{\Lambda}_{2j}\right]$, $j = 1{,}2$ of possible values of the normalized parameters $l_{0j} = \lambda_{0j}/\tau_0$, the samples $M_{r_j j} = M_j\left(\tilde{\Lambda}_{1j} + r_j\Delta_1\right)$, $r_j = 0, 1, \ldots, \left\{\left(\tilde{\Lambda}_{2j} - \tilde{\Lambda}_{1j}\right)/\Delta_1\right\}$ of the functionals (11) are generated with the discretization step Δ_1 as follows

$$M_{r_j j} = \vartheta_j\left[z_0^2 \max\left(0, 1 - \left|\tilde{\Lambda}_{1j} + r_j\Delta_1 - l_{0j}\right|\right) + \right.$$
$$\left. + z_0\Delta_2 \sum_{v=V_{1j}}^{V_{2j}-1} \tilde{\xi}_{jv}\right] + z_0\sqrt{\Delta_2} \sum_{k=K_{1j}}^{K_{2j}-1} \alpha_k . \qquad (38)$$

In (38) the notations are: $\tilde{\Lambda}_{ij} = \Lambda_{ij}/\tau_0$, $i, j = 1{,}2$; $\vartheta_j = 1$, if the signal $s_{0j}(t, \lambda_{0j})$ (6) is present in the observable realization (9), and otherwise – $\vartheta_j = 0$; z_0 is determined from (15); $K_{1j} = \left\{\left(\tilde{\Lambda}_{1j} + r_j\Delta_1 - l_{0j} - 1/2\right)/\Delta_2\right\}$, $K_{2j} = \left\{\left(\tilde{\Lambda}_{1j} + r_j\Delta_1 - l_{0j} + 1/2\right)/\Delta_2\right\}$; $\alpha_k = \int_{k\Delta_2}^{(k+1)\Delta_2} \tilde{n}(\tilde{t})d\tilde{t} / \sqrt{\Delta_2}$ are the independent Gaussian random numbers with the zero mathematical expectations and unit dispersions; $\tilde{n}(\tilde{t}) = n(t)\sqrt{2\tau_0/N_0}$ is the normalized Gaussian white noise with the unit spectral density; $\tilde{t} = t/\tau_0$ is the normalized time; $V_{1j} = \left\{\left[\max\left(0, \tilde{\Lambda}_{1j} + r_j\Delta_1 - l_{0j}\right) - 1/2\right]/\Delta_2\right\}$, $V_{2j} = \left\{\left[\min\left(0, \tilde{\Lambda}_{1j} + r_j\Delta_1 - l_{0j}\right) + 1/2\right]/\Delta_2\right\}$; $\tilde{\xi}_{jv} = \tilde{\xi}\left(l_{0j} + v\Delta_2\right)$, $\tilde{\xi}(\tilde{t}) = \xi(t)\sqrt{2\tau_0/N_0}$; Δ_2 is the discretization step of the process $\tilde{\xi}(\tilde{t})$, and $\{\cdot\}$ is an integer part. For certainty, in the simulation, it is assumed that the spectral density $\tilde{G}(\tilde{\omega}) = q_0 g(\tilde{\omega})$, $\tilde{\omega} = \omega/\Omega_0$ is rectangular by shape, i.e. the function $g(\tilde{\omega})$ satisfies the relation (34).

The samples $\tilde{\xi}_{jv}$ of the normalized random process $\tilde{\xi}(\tilde{t})$ are generated by the moving summation method [14], as it is described in [8], [15]:

$$\tilde{\xi}_{jv} = \frac{1}{\pi}\sqrt{\frac{q_0}{\Delta_2}} \sum_{m=0}^{2p} H_{mp}\beta_{m+k} , \qquad (39)$$

where $H_{mp} = \sin[\pi\mu_0\Delta_2(m-p)]/\pi(m-p)$, and β_m are independent Gaussian numbers with the parameters $\sim N(0{,}1)$. In (38), (39), the discretization step Δ_2 is chosen based on the condition [8], [15]

$$\sqrt{2[1-R(\Delta_2/2)]} < \varepsilon << 1, \quad (40)$$

where $R(\tilde{t}) = \sin(\pi\mu_0\tilde{t})/\pi\mu_0\tilde{t}$ is the correlation coefficient of the process $\tilde{\xi}(\tilde{t})$. In (40), the discretization error ε is taken equal to 0.05, so that one can assume that $\Delta_2 = 0.05/\mu_0$.

In the sum (39), the number of terms is chosen according to the condition [15]

$$\sqrt{\left|\sigma_{\tilde{\xi}}^2 - \sigma_{\tilde{\xi}_{jv}}^2\right| / \sigma_{\tilde{\xi}}^2} < \delta. \quad (41)$$

Here $\sigma_{\tilde{\xi}}^2 = 2\mu_0 q_0$, $\sigma_{\tilde{\xi}_{jv}}^2 = \left(q_0/\pi^2\Delta_2\right)\sum_{m=0}^{2p} H_{mp}^2$ are the dispersions of the process $\tilde{\xi}(\tilde{t})$ and its generated samples, respectively, while $\delta << 1$ is the maximum allowable relative deviation of the dispersion of the generated sample from the dispersion of the simulated process. The value $\delta = 0.05$ can be considered acceptable. Then, according to (41), it should be assigned that $p = 130$ in (39). Now, if one takes $\Delta_1 = 0.01$, then the stepwise approximations built based on the samples (38) approximate the continuous realizations of the functionals (11) with a mean-square error that does not exceed 10%.

The generation of the Gaussian numbers α_k, β_m is carried out using sequences of the independent random numbers uniformly distributed within the interval $[0,1]$ by applying the Cornish-Fisher method, as it is described in [8], [15].

From the sequences of the generated samples $M_{r_1 1}$, $M_{r_2 2}$, the maximum ones are selected: $M_{1\max} = \max_{r_1} M_{r_1 1}$, $M_{2\max} = \max_{r_2} M_{r_2 2}$. Further, the values of $M_{1\max}$ and $M_{2\max}$ are compared and, according to the decision rule (13), the decision is made in favor of one of the hypotheses – H_1 or H_2. If, when processing *N* realizations (9) within which the signal $s_{01}(t,\lambda_{01})$ (6) is presented ($\vartheta_1 = 1$, $\vartheta_2 = 0$), the decision in favor of the hypothesis H_2 is made in n_1 cases, then the estimate $\tilde{P}_{21}$ of the probability estimate P_{21} (19) is found as $\tilde{P}_{21} = n_1/N$. Similarly, from *N* observations of the realizations (9) within which the signal $s_{02}(t,\lambda_{02})$ (6) is present ($\vartheta_1 = 0$, $\vartheta_2 = 1$), one defines the estimate $\tilde{P}_{12}$ of the probability P_{12} (19), and then – the estimate $\tilde{P}_E = (\tilde{P}_{21} + \tilde{P}_{12})/2$ of the probability P_E (25).

When simulating the discrimination algorithm (29), the continuous realizations of the functionals (26) are replaced by their stepwise approximations built based on the samples $L_{r_j j} = L_j(\tilde{\Lambda}_{1j} + r_j\Delta_1)$ generated with the discretization step Δ_1, while the spectral density of the modulating interference is the rectangular one (34). Following [15], one gets

$$L_{r_j j} = \frac{\Delta_2}{4}\sum_{k=K_{1j}}^{K_{2j}-1} \tilde{y}_{kj}^2 + \frac{1}{1+q_0} M_{r_j j}, \quad r_j = \overline{0, \{(\tilde{\Lambda}_{2j} - \tilde{\Lambda}_{1j})/\Delta_1\}}. \quad (42)$$

Here

$$y_{kj} = \vartheta_j \sum_{w=W_{1j}}^{W_{2j}-1} \left(\tilde{\xi}_{jw} + z_0\right) H_{kw} + \frac{1}{\sqrt{\Delta_2}} \sum_{w=k-p}^{k+p-1} \alpha_w H_{kw},$$

$W_{1j} = \max(-W, k-p)$, $W_{2j} = \min(W, k+p)$, $W = \{1/2\Delta_2\}$, a $\tilde{\Lambda}_{ij}$, K_{ij} $i,j = 1,2$, $M_{r_j j}$, ϑ_j, α_w are determined in the same way as in (38), while $\tilde{\xi}_{jw}$, H_{kw} – as in (39). It can be shown [15] that when choosing $\Delta_1 = 0.01$, $\Delta_2 = 0.05/\mu_0$, $p = 130$, the root-mean-square error does not exceed 10% while the functionals (26) are approximated by means of (42). Based on the generated realizations (42), the estimates of the probabilities of the erroneous decisions are found by a technique similar to that described in section devoted to the simulation of the algorithm (13).

Some results of the statistical simulation of the algorithms (13), (29) are presented in Fig. 2, under $\tilde{\Lambda}_{11} = 1/2$, $\tilde{\Lambda}_{21} = m + 1/2$, $\tilde{\Lambda}_{12} = m + 3/2$, $\tilde{\Lambda}_{22} = 2m + 3/2$, $m = 20$, $l_{01} = (\tilde{\Lambda}_{11} + \tilde{\Lambda}_{21})/2$, $l_{02} = (\tilde{\Lambda}_{21} + \tilde{\Lambda}_{22})/2$, $\mu_0 = 50$. To obtain each experimental value, 10^{10} realizations (38) or (42) are processed. In this case, the boundaries of the confidence intervals deviate from the experimental values by no more than 10..15%, and the probability of this deviation is not greater than 0.9. In Fig. 2a, by crosses and light squares, rhombuses, circles and triangles, one draws the experimental values $\tilde{P}_E$ of the average error probability (25) for the quasi-optimal discrimination algorithm (13) depending upon the value of q_0, while $z_0 = 5$, 7, 8, 9, 10. Here, by pluses and dark squares, rhombuses, circles, triangles, one also draws the corresponding experimental values $\tilde{P}_E$ of the average error probability (25), (32)-(34) for the optimal discrimination algorithm (29). In Fig. 2b, the experimental values $\tilde{P}_E$ of the probabilities (25) and (25), (32)-(34) are plotted using the similar symbols and demonstrating the dependencies upon the values of z_0 under q_0 equal to 0.5, 1, 1.5, 2, 2.5.

It follows from Fig. 2 that both proposed algorithms for discriminating non-overlapping pulse signals can be used in practical applications. The algorithm (13) is applicable, if the design of the discrimination device is an extremely simple one, the requirements for the level of the discrimination error probability are not too high, and it is possible to provide a sufficiently large ratio z_0 (15). In turn, the algorithm (29) can be recommended, if it is required to provide a very low level of the discrimination error probability, while the ratio (15) may not be too large. In addition, the formulas (25) and (25), (32)-(34) approximate well the experimental values of the average discrimination error probability in a wide range of values of the parameters of both the useful signals and the modulating interference, at least, under $q_0 \geq 0$, $z_0 \geq 4$.

VI. CONCLUSION

The quasi-optimal and optimal algorithms are introduced for discriminating the pulse signals distorted by a modulating interference. By the statistical simulation methods, their operability and efficiency are established. The analytical expressions are obtained for the discrimination error probabilities when using the considered algorithms, and this makes it possible to theoretically evaluate their performance in each specific case. It is shown that the application of the optimal algorithm to discriminate the signals distorted by a modulating interference leads to a lower error probability in comparison with the results of application of the quasi-optimal algorithm. However, the quasi-optimal discrimination algorithm is simpler than the optimal discrimination one, and it does not require a priori information on the properties of the modulating interference. And the hardware implementation of the devices specified above can be effectively implemented by means of both digital signal processors [12] and field-programmable gate arrays [13].

The expressions obtained for the discrimination error probabilities allow us to make a choice between the discrimination devices presented in Figs. 1, 3 and others, depending on the available prior information as well as the requirements to the discrimination accuracy and to the simplicity of the device practical implementation.

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