On TDS-PCG Iteration Method with Circulant Preconditioners for Solving the Space Fractional Coupled Nonlinear Schrödinger Equations

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***Abstract*—The goal of this paper is to solve the complex symmetric linear systems generated from the discretization of the space fractional coupled nonlinear Schrödinger (CNLS) equations, whose coefficient matrix is equal to the sum of a symmetric positive definite Toeplitz matrix and a Hermitian positive definite complex diagonal matrix. In order to make the best use of the full Toeplitz structure of the coefficient matrix, a new Toeplitz and diagonal splitting (TDS) is given and the corresponding TDS iteration method is proposed to solve the discretized linear systems, then two circulant preconditioners based on Strang's and T. Chan's circulant approximation, are proposed to accelerate the convergence of the preconditioned conjugated gradient (PCG) method for solving the first linear sub-system in the TDS method. Theoretical analysis and numerical experiments demonstrate that the TDS method is unconditional convergence and the TDS-PCG inner-outer iteration method with two circulant preconditioners to solve the discretization linear systems of the space fractional CNLS equations is feasible and efficient.**

***Index Terms*—space fractional coupled nonlinear Schrödinger equations, Toeplitz matrix, Toeplitz and diagonal splitting (TDS) iteration method, circulant preconditioner, PCG method.**

I. Introduction

In this paper, the following space fractional coupled nonlinear Schrödinger (CNLS) equations involving the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ $(1<\alpha<2)$ are considered,

$$\begin{cases} \mathrm{i}u_t-\gamma(-\Delta)^{\frac{\alpha}{2}}u+\rho(|u|^2+\beta|v|^2)u=0, \\ \mathrm{i}v_t-\gamma(-\Delta)^{\frac{\alpha}{2}}v+\rho(|v|^2+\beta|u|^2)v=0, \\ a\le x\le b, \quad 0<t\le T, \end{cases} \tag{1}$$

with the initial value conditions

$$u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad a\le x\le b,$$

and homogeneous Dirichlet boundary conditions

$$u(a,t)=u(b,t)=0, \ v(a,t)=v(b,t)=0, \ 0<t\le T.$$

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Here, $\mathrm{i}=\sqrt{-1}$ is the imaginary unit, $\gamma>0$ is the group velocity dispersion, $\rho>0$ describes the self-focussing of a signal for the pulses in the birefringent media, and $\beta\ge 0$ is the cross-phase modulation and defines the integrability of the CNLS equations [1]–[4]. The fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ is equivalent to the following Riesz-type fractional derivative

$$(-\Delta)^{\frac{\alpha}{2}}u(x,t)=\frac{1}{2\cos\frac{\pi\alpha}{2}}\left[{}_{-\infty}D_x^{\alpha}u(x,t)+{}_xD_{+\infty}^{\alpha}u(x,t)\right],$$

where ${}_{-\infty}D_x^{\alpha}u(x,t)$ and ${}_xD_{+\infty}^{\alpha}u(x,t)$ are the left- and right-side Riemann-Liouville fractional derivatives respectively [5].

The nonlinear Schrödinger equation is one of the most important equations in quantum mechanics and nonlinear optics [6]. In 2000, by generalizing the path integral method from the Brownian motion to the L*é*vy-α process, Laskin [7], [8] developed the fractional quantum mechanics and obtained the space fractional Schrödinger equations, which can be used to describe a wide class of physical nonlinear phenomena, such as hydro dynamics [9] and the dynamics of Bose-Einstein condensate [10]. In the special case where $\alpha=2$, the system (1) becomes the classical coupled nonlinear Schrödinger equations, which can be used to describe the evolution of microscopic particles. When $\beta=0$, this system is reduced to two unrelated fractional nonlinear Schrödinger equations. And this system is decoupled and becomes the fractional Schrödinger equations for describing free particles when $\rho=0$.

Since it is more challenging or sometimes even impossible to obtain the analytical solution of the fractional partial differential equations, numerical methods to solve CNLS become very important and a large amount of work has been intensively conducted [11]–[14]. These methods can be grossly divided into two groups.

The first are centred on how to discretize the fractional Laplacian operator. In 2013, Wang et al. [15] established the Crank-Nicolson difference scheme for the CNLS equations. Subsequently, they proposed a linearly implicit conservative difference scheme and a modified Crank-Nicolson difference scheme for the CNLS equations in [16], [17]. In 2019, Wang and Mei [4] constructed a Crank-Nicolson Legendre spectral Galerkin method and introduced a linearized iterative scheme to compute the nonlinear problem. These discretized schemes satisfy the mass and energy conservation. In addition, the conservative difference scheme [18] and finite element method [19] also have been proposed to solve the strongly CNLS

equations.

And the second are centred on how to solve the discretized linear system, whose coefficient matrix is usually full large-scaled, and which is also the purpose of this article. Recently, according to the Crank-Nicolson difference scheme [15] and the Hermitian and skew-Hermitian splitting (HSS) method [20], Ran and Wang proposed the partially inexact HSS (PIHSS) [1], HSS-like [2] methods and the preconditioner combined with asymmetric Gauss-Seidel splitting [3] to solve the linear system originated from the difference discretization of the space fractional CNLS equations. The primary computing in the methods is to solve a complex diagonal linear system and a Toeplitz linear system which is performed by fast solver. Further, in order to avoid the complex value arithmetic, Wang et al. [21] proposed a preconditioned modified Hermitian and skew-Hermitian splitting iteration (PMHSS) method to solve the CNLS equations.

In order to make the most use of the Toeplitz structure and non-sparse property of the discretized linear system, a new Toeplitz and diagonal splitting (TDS) iteration method is proposed, and two circulant preconditioners based on Strang's and T. Chan's approximation are further constructed to accelerate the convergence speed of preconditioned conjugate gradient (PCG) method to solve the discretized linear systems of the space fractional CNLS equations (1).

The rest of the present paper is organized as follows. In Section II, we make a brief review of the well-known Crank-Nicholson discretization for CNLS and show the Toeplitz-plus-diagonal structure of the coefficient matrix. In Section III, the TDS iteration method and the TDS-PCG iteration method with circulant preconditioners are proposed to solve the discretized linear systems, and the convergence properties are studied. In Section IV, the numerical experiments are given to show the effectiveness of the proposed TDS and TDS-PCG methods. Finally, some concluding remarks are made in Section V.

II. Discretization of the space fractional CNLS equations

In this section, we first review the difference scheme of the space fractional CNLS equations, and show the large-scaled full complex symmetric Toeplitz structure of the discretized linear systems, as seen in references [1]–[3], [16].

Let $h=\frac{b-a}{M+1}$ and $\tau=\frac{T}{N}$ be the sizes of spatial grid and time step, where N and M are positive integers, then the temporal and spatial partitions are defined as $t_n=n\tau$ for $n=0,1,\cdots,N$ and $x_j=a+jh$ for $j=0,1,\cdots,M+1$, respectively. Let $u_j^{(n)}\approx u(x_j,t_n)$ and $v_j^{(n)}\approx v(x_j,t_n)$ denote the corresponding numerical solutions. By the fractional centered difference formula, the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ in the truncated bounded domain as

$$(-\Delta)^{\frac{\alpha}{2}}u(x_j)=-\frac{\partial^\alpha}{\partial|x|^\alpha}u(x_j)=\frac{1}{h^\alpha}\sum_{k=1}^{M}c_{j-k}u_k+o(h^2),$$

$$c_k=\frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function. Moreover, the coefficients c_k satisfy the following properties:

$$c_0\geq 0,\quad c_k=c_{-k}\leq 0,\quad \sum_{k=-\infty,k\neq 0}^{+\infty}|c_k|=c_0\geq 0.$$

The following unconditionally stable implicit difference scheme is proposed for the space fractional CNLS equations (1):

$$\begin{cases} \mathrm{i}\dfrac{u_j^{(n+1)}-u_j^{(n-1)}}{2\tau}-\dfrac{\gamma}{h^\alpha}\displaystyle\sum_{k=1}^{M}c_{j-k}\left(\frac{u_k^{(n+1)}-u_k^{(n-1)}}{2}\right) \\ \qquad +\rho\left(\left|u_j^{(n)}\right|^2+\beta\left|v_j^{(n)}\right|^2\right)\dfrac{u_j^{(n+1)}-u_j^{(n-1)}}{2}=0, \\ \mathrm{i}\dfrac{v_j^{(n+1)}-v_j^{(n-1)}}{2\tau}-\dfrac{\gamma}{h^\alpha}\displaystyle\sum_{k=1}^{M}c_{j-k}\left(\frac{v_k^{(n+1)}-v_k^{(n-1)}}{2}\right) \\ \qquad +\rho\left(\left|v_j^{(n)}\right|^2+\beta\left|u_j^{(n)}\right|^2\right)\dfrac{v_j^{(n+1)}-v_j^{(n-1)}}{2}=0, \end{cases} \tag{3}$$

where $j=1,2,\cdots,M, n=1,2,\cdots,N-1$. It is proved that the difference scheme (3) conserves the discrete mass and energy, and is unconditional stable and convergent [16], [17].

According the initial boundary value conditions, we have

$$\begin{cases} u_j^{(0)}=u_0(x_j), & v_j^{(0)}=v_0(x_j), \\ u_0^{(n)}=u_{M+1}^{(n)}=0, & v_0^{(n)}=v_{M+1}^{(n)}=0. \end{cases}$$

In addition, the first step can be obtained with some second or higher order time integrators.

Let $\eta=\frac{1}{2\tau}$, $\mu=\frac{\gamma}{2h^\alpha}$,

$$u^{(n+1)}=\left[u_1^{(n+1)},u_2^{(n+1)},\cdots,u_M^{(n+1)}\right]^T,$$
$$b^{(n+1)}=\left[b_1^{(n+1)},b_2^{(n+1)},\cdots,b_M^{(n+1)}\right]^T$$

and

$$v^{(n+1)}=\left[v_1^{(n+1)},v_2^{(n+1)},\cdots,v_M^{(n+1)}\right]^T,$$
$$\tilde{b}^{(n+1)}=\left[\tilde{b}_1^{(n+1)},\tilde{b}_2^{(n+1)},\cdots,\tilde{b}_M^{(n+1)}\right]^T,$$

where

$$b_j^{(n+1)}=\mathrm{i}\eta u_j^{(n-1)}-\mu\sum_{k=1}^{M}c_{j-k}u_k^{(n-1)}-d_j^{(n+1)}u_j^{(n-1)},$$
$$d_j^{(n+1)}=\frac{\rho}{2}\left(|u_j^{(n)}|^2+\beta|v_j^{(n)}|^2\right),\quad j=1,2,\cdots,M$$

and

$$\tilde{b}_j^{(n+1)}=\mathrm{i}\eta v_j^{(n-1)}-\mu\sum_{k=1}^{M}c_{j-k}v_k^{(n-1)}-\tilde{d}_j^{(n+1)}v_j^{(n-1)},$$
$$\tilde{d}_j^{(n+1)}=\frac{\rho}{2}\left(|v_j^{(n)}|^2+\beta|u_j^{(n)}|^2\right),\quad j=1,2,\cdots,M.$$

Then, the difference schemes (3) can be rewritten in the following matrix vector forms:

$$\begin{cases} A^{(n+1)}u^{(n+1)}=b^{(n+1)}, \\ A^{(n+1)}v^{(n+1)}=\tilde{b}^{(n+1)}, \end{cases}\quad n=1,2,\cdots,N-1, \tag{4}$$

where the coefficient matrix $A^{(n+1)}$ is in the following form

$$A^{(n+1)}=\mathrm{i}\eta I+D^{(n+1)}+T_\alpha,$$
$$D^{(n+1)}=\mathrm{diag}\left(d_1^{(n+1)},d_2^{(n+1)},\cdots,d_M^{(n+1)}\right).$$

Here, I is the identity matrix, $D^{(n+1)}$ is the diagonal matrix, and T is the Toeplitz matrix

$$T_\alpha = \mu \begin{bmatrix} c_0 & c_{-1} & \cdots & c_{2-M} & c_{1-M} \\ c_1 & c_0 & \cdots & c_{3-M} & c_{2-M} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{M-2} & c_{M-3} & \cdots & c_0 & c_{-1} \\ c_{M-1} & c_{M-2} & \cdots & c_1 & c_0 \end{bmatrix}_{M\times M}. \quad (5)$$

According the fact $\gamma > 0$ and the properties of the coefficients c_k given in (2), the Toeplitz matrix T in (5) is symmetric and strictly diagonally dominant, so it is symmetric positive definite. And the matrix $D^{(n+1)}$ is non-negative diagonal for $\rho > 0$, $\beta \geq 0$. Thus, the coefficient matrix $A^{(n+1)}$ is complex symmetric and positive definite.

It must be noted that the two linear systems in (4) have the same algebraic structure and that the coefficient matrix $A^{(n+1)}$ is an $M \times M$ full Toeplitz-like matrix because T_α is a Toeplitz matrix, the storage of Toeplitz-like matrix $A^{(n+1)}$ can be reduced to $O(M)$ and the matrix-vector multiplication can be obtained in $O(M \log M)$ operations via the fast Fourier Transforms (FFTs).

III. TDS-PCG ITERATION METHOD WITH CIRCULANT PRECONDITIONERS

A. TDS Method

In this section, we only consider the iterative solution of the first linear system in (4), because the second one is the same, and it can be simplified as

$$Au = b, \qquad A = D + T + \mathrm{i}\eta I \in \mathbb{C}^{M\times M},\ u, b \in \mathbb{C}^M, \quad (6)$$

where $\eta > 0$, $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, I is the identity matrix, $D = \mathrm{diag}(d_1, \cdots, d_M)$ is the nonnegative diagonal matrix with $d_i \geq 0, i = 1, \cdots, M$, T is the full symmetric positive definite Toeplitz matrix.

Considering T and $\delta I + T$ ($\delta > 0$) are all large-scaled full symmetric positive definite Toeplitz matrix, Bai et al. [22] proposed the Toeplitz and diagonal splitting (TDS) and then constructed a new iteration method to solve the linear systems originated from spatial fractional diffusion equations. In view of the fact that the circulant matrix is the optimal preconditioner for Toeplitz matrix, and the coefficient matrix A in (6) can be split into its diagonal part $D + \mathrm{i}\eta I$ and Toeplitz part T, we can constructed the following iteration method to compute an approximate solution of the linear system (6).

Algorithm 1. (The TDS iteration method.)
Given an initial guess $u^{(0)}$, for $k = 0, 1, 2, \cdots$, until $\{u^{(k)}\}$ converges, compute

$$\begin{cases} (\delta I + T)u^{(k+\frac{1}{2})} = (\delta I - D - \mathrm{i}\eta I)u^{(k)} + b, \\ (\delta I + D + \mathrm{i}\eta I)u^{(k+1)} = (\delta I - T)u^{(k+\frac{1}{2})} + b, \end{cases} \quad (7)$$

where δ is a given positive constant and I is the identity matrix.

This iteration method is the same as the HSS-like [2] in the form, but the original idea is entirely different, and it is also similar to that proposed by Bai in [22]. Note that the matrix splitting in TDS method is based on the fact that the first sub-linear systems can be solved fast with the benefit of its Toeplitz structure and circulant preconditioner.

Theorem 1. *Let $A \in \mathbb{C}^{M\times M}$ be defined as in (6) and δ be a positive constant. Then the iteration spectral radius $\rho(M(\delta))$ of the TDS method is less than 1, thus the TDS method (7) unconditionally converges to the solution of the linear system (6). And the optimal parameter is $\delta^* = \sqrt{\lambda_{\min}\lambda_{\max}}$ by minimizing the spectral radius $\rho(M(\delta))$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and the maximum eigenvalues of Toeplitz matrix T respectively.*

Proof. The iteration matrix $M(\delta)$ of the TDS method (7) for solving the linear system (6) is given by

$$M(\delta) = (\delta I + D + \mathrm{i}\eta I)^{-1}(\delta I - T)(\delta I + T)^{-1}(\delta I - D - \mathrm{i}\eta I),$$

then its spectral radius $\rho(M(\delta))$ can be reduced by the similarity invariance, and is bounded by

$$\begin{aligned} \rho(M(\delta)) &\leq \max_{\lambda_i \in \lambda(T)} \left| \frac{\delta - \lambda_i}{\delta + \lambda_i} \right| \cdot \max_{1\leq j\leq M} \sqrt{\frac{(\delta - d_i)^2 + \eta^2}{(\delta + d_i)^2 + \eta^2}} \\ &\leq \max_{\lambda_i \in \lambda(T)} \left| \frac{\delta - \lambda_i}{\delta + \lambda_i} \right| \leq 1, \ \forall \delta > 0. \end{aligned}$$

Thus, the TDS method converges to the unique solution of the system of linear equations (6).

By using the similar analytical techniques in [2], [20], the optimal parameter δ^* is easily obtained as $\delta^* = \sqrt{\lambda_{\min}\lambda_{\max}}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of Toeplitz matrix T respectively.

We emphasize that the optimal parameter δ^* minimizes the upper bound of the spectral radius $\rho(M(\delta))$, thus it is really the optimal parameter in theory. And the considerable advantage of the TDS method is that the coefficient matrix $\delta I + T$ in the first half-step stays the same, so the optimal parameter δ^* keeps constant with the time step n increasing in linear system (4).

B. TDS-PCG Method With Circulant Preconditioner

The two half-steps at each TDS iterate require exact solutions of the linear systems with the symmetric positive definite Toeplitz matrix $\delta I + T$ and the complex diagonal matrix $\delta I + D + \mathrm{i}\eta I$. As the matrix $\delta I + D + \mathrm{i}\eta I$ is diagonal, the iterate $u^{(k+1)}$ in the second half-step of the TDS method can be directly solved. Thus the iterate $u^{(k+\frac{1}{2})}$ in the first half-step is an overwhelming part of the TDS method (7).

Given the coefficient matrix $\delta I + T$ is large-scaled, full and symmetric positive definite, the iterate $u^{(k+\frac{1}{2})}$ in the first half-step of the TDS method can be solved inexactly by Krylov subspace methods, for example, PCG method. In order to speed up its convergence rate, some circulant approximate matrices can be used as well preconditioners. And then this results in the following TDS-PCG iteration method with circulant preconditioner for solving the system of linear equations (6).

Algorithm 2. (The TDS-PCG iteration method.)
Given an initial guess $u^{(0)}$, for $k = 0, 1, 2, \cdots$, until $\{u^{(k)}\}$ converges, compute $u^{(k+\frac{1}{2})}$ approximately from

$$(\delta I + T)u^{(k+\frac{1}{2})} = (\delta I - D - \mathrm{i}\eta I)u^{(k)} + b$$

by employing the PCG with $u^{(k)}$ as the initial guess, and then solve $u^{(k+1)}$ exactly from

$$(\delta I + D + \mathrm{i}\eta I)u^{(k+1)} = (\delta I - T)u^{(k+\frac{1}{2})} + b$$

by using the direct matrix inversion, where δ is a given positive constant and I is the identity matrix.

In 1986, Strang [23] and Olkin [24] proposed independently the use of PCG method with circulant matrices as preconditioners to solve Toeplitz linear systems. The advantage of the circulant matrix is that it can be diagonalized by the Fourier matrix F_M, i.e.,

$$F_M^* \Lambda_M F_M^* = C_M, \quad (F_M)_{j,k} = \frac{1}{\sqrt{M}} \exp \frac{2\pi j k \mathrm{i}}{M},$$

where Λ_M is a diagonal matrix holding the eigenvalues of the circulant matrix C_M, and the subscript index M represents the matrix dimension.

For Toeplitz matrix $T = (c_{i-j})_{1\le i,j\le M}$, whose first row and first column are $[c_0, c_{-1}, \cdots, c_{2-M}, c_{1-M}]$ and $[c_0, c_1, \cdots, c_{M-2}, c_{M-1}]$, respectively, the Strang's preconditioner $S = (s_{i-j})_{1\le i,j\le M}$ [23] and the T. Chan's preconditioner $P = (p_{i-j})_{1\le i,j\le M}$ [25] are two classic circulant matrices, whose diagonal elements are defined as

$$\text{(Strang)} \quad s_k = \begin{cases} c_k, & 0 \le k < M/2, \\ 0, & k = M/2 \text{ is even}, \\ c_{k-M}, & M/2 < k \le M-1, \\ s_{k+M}, & 0 < -k \le M-1, \end{cases}$$

and

$$\text{(T. Chan)} \ p_k = \begin{cases} \dfrac{(M-k)c_k + k c_{k-M}}{M}, & 0 \le k \le M-1, \\ p_{M+k}, & 0 < -k \le M-1. \end{cases}$$

For any Toeplitz matrix T_M with a circulant preconditioner C_M, the matrix-vector product $C_M^{-1} T_M v$ can be computed in $O(M \log M)$ operations for any vector v, because the circulant linear systems can be solved efficiently by FFTs and the multiplication $T_M v$ can also be computed by FFTs by embedding T_M into $2M \times 2M$ circulant matrix. Thus, the cost per iterate of PCG method is still $O(M \log M)$ [26].

IV. Numerical experiments

In this section, numerical experiments are carried out to investigate the performance of the new TDS-PCG method with two circulant preconditioners. All numerical experiments are performed in MATLAB 9.4 (R2018a) in double precision on Dell precision tower 7910 with 3.5 GHz CPU (Intel Xeon E5-2637 v4), 32.00× 4 GB RAM.

All tests are stared from the zeros vector, and the outer TDS iterate for the linear system (6) are terminated once the stopping criterion $\|r^{(k)}\|_2 / \|r^{(0)}\|_2 < 10^{-5}$ are satisfied, where $r^{(k)}$ is the residual vector of the k-th iterate. In inner PCG iteration, the current residuals of the inner iterate satisfy $\|p^{(k-1,j)}\|_2 / \|p^{(k-1,0)}\|_2 < 10^{-3}$ where $p^{(k-1,j)}$ is the residual of the j-th inner PCG iterate in the k-th outer TDS iterate. We must note the lower accuracy of the inner iteration could lead to an increase in the number of the outer iterate, but it greatly saves the computation amount of the inner iterate.

We consider the following space fractional CNLS equations with $\gamma = 1,\ \rho = 2,\ \beta = 1,\ 1 < \alpha < 2$, i.e.,

$$\begin{cases} \mathrm{i}u_t + (-\Delta)^{\frac{\alpha}{2}} u + 2(|u|^2 + |v|^2)u = 0, \\ \mathrm{i}v_t + (-\Delta)^{\frac{\alpha}{2}} v + 2(|v|^2 + |u|^2)v = 0, \\ -20 \le x \le 20,\ 0 < t \le 2. \end{cases}$$

And the initial boundary value conditions are taken in the form

$$\begin{cases} u(x,0) = \mathrm{Sech}(x+1)\exp(2x\mathrm{i}), \quad u(-20,t) = u(20,t) = 0, \\ v(x,0) = \mathrm{Sech}(x-1)\exp(-2x\mathrm{i}), \ v(-20,t) = v(20,t) = 0. \end{cases}$$

In our experiments, all of the Toeplitz matrix-vector multiplications Av are done by using FFTs in $O(M \log M)$ operations to reduce the computational cost. And the parameters involved are all theoretically optimal value $\sqrt{\lambda_{\max}\lambda_{\min}}$.

Numerical results are given in Tables I and II. In these tables, we denote "Non" as non-preconditioned method, "Chan" T. Chan's circulant preconditioner, "Strang" Strang's circulant preconditioner, and "HSS" the HSS iteration method proposed in [1]. "M" denotes the number of spatial grid points, "$N = \lfloor ((M+1)^\alpha T v_{M,N})/(b-a)^\alpha \rfloor$" denotes the number of time steps and it is such that the discrete grid ratio of spatial to time $v_{M,N} = 0.5$, where the function $\lfloor x \rfloor$ rounds the elements of x to the nearest integers less than or equal to x. And "CPU" denotes the total CPU time in seconds, "$\mathrm{IT}_{\mathrm{inn}}$" the average number of the PCG method required in every TDS outer iterate and "$\mathrm{IT}_{\mathrm{out}}$" the average number of TDS iterate required for solving FDEs, respectively, i.e.,

$$\mathrm{IT}_{\mathrm{out}} = \frac{1}{N}\sum_{i=1}^{N} \mathrm{TDS}(i), \qquad \mathrm{IT}_{\mathrm{inn}} = \frac{\sum_{i=1}^{N}\sum_j \mathrm{PCG}(i,j)}{\sum_{n=1}^{N} \mathrm{TDS}(i)},$$

where $\mathrm{TDS}(i)$ denotes the number of the TDS iterations required for solving the discretized linear systems at the i-th time step, and $\mathrm{PCG}(i,j)$ denotes the iteration number of the inner PCG method in the j-th TDS outer iterate at the i-th time step.

From these tables, it can be found that all experimented methods can successfully produce approximate solution of the full complex symmetric linear system originated from the discretization of the space fractional CNLS equation (1). When M increases, the number of time steps N is doubled and tripled in order to keep stability, the numbers of the outer and inner iteration steps are slightly decreased and keeps stable, but the amount of total CPU time is greatly increased. Obviously, the results of TDS-PCG with circulant preconditioners are the best, especially when the dimension is larger, the advantage is more obvious.

Meanwhile, it is also found that although the number of the HSS-like method are decreased with the number of the spatial grid M increases, but the CPU time are increasing rapidly. This is probably due to the first linear systems in HSS iteration method may cost much time for the coefficient matrix $\delta I + D + T$ are full and changed along with the time step process.

It must be note that the results of the TDS-PCG with non circulant preconditioner can be regarded as those of the HSS-like method proposed in [2]. Thus, the TDS-PCG

with two proposed circulant preconditioners are the best in iteration steps and CPU time.

V. Concluding Remarks

The coefficient matrix of the linear system generated by numerical discretization of the the space fractional coupled nonlinear Schrödinger equations, is equal to the sum of a symmetric positive definite Toeplitz matrix and a complex diagonal matrix. In order to make full use of the full Toeplitz structure of the coefficient matrix, a new Toeplitz and diagonal splitting (TDS) and the corresponding TDS iteration method are proposed to solve this discretized linear systems. And than two circulant preconditioners based on Strang's and T. Chan's circulant approximations are proposed to accelerate the convergence of the TDS-PCG double-layer iteration method. Theoretical analysis shows that the proposed TDS iteration method is converge unconditionally, and the numerical experiments demonstrate that the TDS-PCG double-layer iteration method with two circulant preconditioners is feasible and efficient, in solving the discretization linear systems of the space fractional CNLS equations.

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TABLE I: NUMERICAL RESULTS FOR $\alpha = 1.2$.

M	N	Non			Strang			Chan			HSS	
		IT_{out}	IT_{inn}	CPU	IT_{out}	IT_{inn}	CPU	IT_{out}	IT_{inn}	CPU	IT	CPU
2^8	18	89.6	14.1	6.1783	73.1	3.8	1.4136	80.9	3.1	1.4158	90.8	1.3081
2^9	42	106.1	14.0	15.8130	101.0	3.9	6.1673	101.2	3.6	6.7589	88.7	16.4472
2^{10}	98	145.6	10.1	60.0518	148.8	3.3	29.5829	148.2	3.3	29.4587	81.9	159.2555
2^{11}	225	176.9	9.5	302.2775	173.6	3.4	149.5520	176.4	3.5	150.4915	52.9	1179.4052
2^{12}	517	188.5	10.1	1815.1800	187.1	3.3	825.1128	189.6	3.4	840.9087	33.9	11754.5146

TABLE II: NUMERICAL RESULTS FOR $\alpha = 1.5$.

M	N	Non			Strang			Chan			HSS	
		IT_{out}	IT_{inn}	CPU	IT_{out}	IT_{inn}	CPU	IT_{out}	IT_{inn}	CPU	IT	CPU
2^8	32	289.0	10.7	19.0140	177.9	3.2	5.4450	231.2	3.0	6.9267	115.7	2.3191
2^9	91	291.7	15.9	104.0163	264.7	4.4	37.8728	271.0	3.8	35.7078	80.8	33.0619
2^{10}	259	372.3	14.2	549.7561	385.7	3.3	208.6113	379.7	3.4	206.1529	51.4	325.2823
2^{11}	733	484.2	13.5	3532.8144	444.6	3.8	1306.8070	500.3	3.2	1336.4785	28.4	3155.3741
2^{12}	2073	524.9	13.9	26019.1957	526.9	3.7	9806.2491	528.8	3.4	9364.2430	16.1	35959.1451

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