Chebyshev Collocation Methods for Volterra Integro-differential Equations of Pantograph Type

Tianfu Ji, Jianhua Hou, and Changqing Yang

***Abstract*—A numerical scheme based upon the Chebyshev polynomials and collocation method is modified and developed to deal with a class of Volterra integro-differential equations. First, we construct the operational matrices of derivative and pantograph. Then these obtained matrices are then utilized to convert the problems to a system of algebraic equations. Furthermore, we establish a detailed convergence analysis in the weighted square norm. Finally, three numerical experiments are implemented and discussed to confirm the applicability and accuracy of the introduced computational scheme.**

***Index Terms*—Volterra integro-differential equation, Chebyshev polynomial, Collocation method, Operational matrix.**

I. Introduction

IN this work, we will focus on developing an efficient numerical scheme for a class of integro-differential equations

$$\begin{aligned} z'(t) =&\alpha(t)z(t) + \beta(t)z(qt) + g(t) \\ &+ \int_0^t k_1(t,s)z(s)ds \\ &+ \int_0^{qt} k_2(t,s)z(s)ds, \quad 0 < q < 1, \end{aligned} \tag{1}$$

subject to the initial condition

$$z(0) = z_0, \tag{2}$$

where, $t \in [0,T]$, the variable coefficients $\alpha(t), \beta(t)$ and $g(t)$ are known functions, q is a real constant and also the kernel functions $k_1(t,s)$ is defined on $D = \{(t,s) : 0 \le t \le T, 0 \le s \le t\}$ and $k_2(t,s)$ is defined on $D_q = \{(t,s) : 0 \le t \le T, 0 \le s \le qt\}$. Assume the functions $\alpha(t), \beta(t), g(t) \in C^m([0,T])$, $k_1(t,s) \in C^m(D)$ and $k_2(t,s) \in C^m(D_q)$ for some $m \ge 0$. Then the problem 1 exists a unique solution $y(t) \in C^{m+1}([0,T])$ [1]. Volterra integro-differential equations have been employed for modeling various natural and social phenomena, for instance, population dynamics, spread of epidemics, chemical kinetics and so on [2]. There has been a considerable amount of further study on constructing and analyzing numerical schemes for various classes of fractional differential equations.

In [3]–[5] the authors developed classical Runge-–Kutta methods for this class of integro-differential equations. Gan [6] used θ-method to deal with the delay type nonlinear integro-differential equations. The authors of [7], [8] solved the problems with pantograph delay by means of finite element methods. As is now well known, various spectral methods play a significant role in solving all kinds of integro-differential equations, see([9]–[11]). Some papers have used the methods for the considered model 1. For instance, in [12]–[14] the authors employed Legendre polynomials to obtain the approximate solution of the integro-differential equations with delay. Meanwhile, Wei and Chen [15] extended the above technique for this class of equations with proportional delays and developed a error analysis in detail. Afterwards Zhao et al [16] employed the similar technique to solve the nonlinear problem with non-vanishing delays. Also, the Sinc function and collocation method were adopted to deal with the given problem in [17].

The motivation of this work is to develop an improved Chebyshev collocation approach to approximate the solution of Volterra integro-differential equations with pantograph delay. In this work, we first formulate the operational matrices in the physical space. Then we construct the discrete numerical scheme in which the variable coefficients were not approximated. A detailed error analysis is discussed in the weighted square norm.

Manuscript received January 13, 2020; revised June 27, 2020.

Tianfu Ji is an associate professor in the Department of Science, Liangyungang Technical College, Liangyungang, Jiangsu, 222000 China (e-mail:346249152@qq.com).

Jianhua Hou is a lecturer in the Department of Science, Jiangsu Ocean University, Lianyungang 222005, Jiangsu, China (e-mail:2004000087@jou.edu.cn).

Changqing Yang is an associate professor in the Department of Science, Jiangsu Ocean University, Lianyungang 222005, Jiangsu, China (corresponding author e-mail:cdruby@126.com).

II. Some properties of Shifted Chebyshev polynomials

The Chebyshev polynomials are important in many areas of numerical analysis The standard Chebyshev polynomials of the first kind are defined as the following formula

$$T_i(x) = \cos(i \arccos(x)), \quad i \in \mathbb{N}_0; \quad x \in [-1,1],$$

where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. $\mathbb{N}$ denotes a set of positive integers. On the interval $[0,T]$, the shifted Chebyshev polynomials are defined by the change of variable $x = 2t/T - 1, \quad 0 \le t \le T$. Let the shifted Chebyshev polynomials denote by $T_{Li}^*(t)$, which can be obtained as follows

$$T_i^*(t) = T_i(2t/T - 1), \quad i \in \mathbb{N}_0,$$

which satisfy the recurrence relation:

$$T_{i+1}^*(t) = 2\,(2t/T - 1)\, T_i^*(t) - T_{i-1}^*(t),$$

where $T_0^*(t) = 1, T_1^*(t) = 2t/L - 1$. Similarly to the standard form, $T_i^*(t)$ also satisfy the orthogonality property:

$$\sum_{k=0}^{N}{}'' T_i^*(t_k) T_j^*(t_k) = \begin{cases} 0, & i \ne j \text{ and } i,j \le N \\ N, & i = j = 0 \text{ or } N \\ N/2, & i = j < N \end{cases} \tag{3}$$

where

$$t_k = \frac{T}{2}(1 - \cos(k\pi/N)), \quad k \in \mathbb{N}_0.$$

are the shifted Chebyshev Gauss-Lobatto points.

III. The operational matrix of derivative

A function $z(t) \in C[0,T]$ can be approximated as

$$z_N(t) = {\sum_{k=0}^{N}}'' c_k T_k^*(t). \quad (4)$$

Thanks to the discrete orthogonality (3), we can directly obtain the coefficient c_k in (6) by the explicit formula

$$c_k = \frac{2}{N} {\sum_{i=0}^{N}}'' z(t_i) T_k^*(t_i), \ k = 0, 1, \cdots, N. \quad (5)$$

First of all, according to (4), (5), we rewrite $z_N(t)$ in matrix form as

$$z_N(t) = T(t) \cdot P \cdot Z, \quad (6)$$

where

$$T(t) = \left[T_0^*(t), T_1^*(t), \cdots, T_{N-1}^*(t), T_N^*(t)\right],$$

$$P = \begin{bmatrix} \frac{1}{2N}T_0^*(t_0) & \frac{2}{2N}T_0^*(t_1) & \cdots & \frac{1}{2N}T_0^*(t_N) \\ \frac{1}{N}T_1^*(t_0) & \frac{2}{N}T_1^*(t_1) & \cdots & \frac{1}{N}T_1^*(t_N) \\ \frac{1}{N}T_2^*(t_0) & \frac{2}{N}T_2^*(t_1) & \cdots & \frac{1}{N}T_2^*(t_N) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2N}T_N^*(t_0) & \frac{2}{2N}T_N^*(t_1) & \cdots & \frac{1}{2N}T_N^*(t_N) \end{bmatrix},$$

and

$$Z = \left[z(t_0), z(t_1), z(t_2), \cdots, z(t_N)\right]^T.$$

So, the values of derivatives of $z_N(t)$ are simple computed by

$$Z^{(1)} = D^{(1)} \cdot Z, \quad (7)$$

where

$$Z^{(1)} = \left[z'(t_0), z'(t_1), z'(t_2), \cdots, z'(t_N)\right]^T,$$

$D^{(1)}$ is the operational matrix of derivative (see [18]).

IV. The operational matrix of pantograph

In order to construct an operational matrix of pantograph let

$$Z_q = Q \cdot Z \quad (8)$$

where Q is the operational matrix of pantograph and

$$Z_q = \left[z(qt_0), z(qt_1), z(qt_2), \cdots, z(qt_N)\right]^T,$$

$$Q = R_q \cdot P,$$

where

$$R_q = \begin{bmatrix} T_0^*(qt_0) & T_1^*(qt_0) & \cdots & T_N^*(qt_0) \\ T_0^*(qt_1) & T_1^*(qt_1) & \cdots & T_N^*(qt_1) \\ T_0^*(qt_2) & T_1^*(qt_2) & \cdots & T_N^*(qt_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0^*(qt_N) & T_1^*(qt_N) & \cdots & T_N^*(qt_N) \end{bmatrix}.$$

Now, we consider to handle the part of integral term in (1).

$$\int_0^t k_1(t,s) z(s) ds \approx \int_0^t k_1(t,s) z_N(s) ds$$

Then, applying (6) we get

$$\int_0^t k_1(t,s) z_N(s) ds = \int_0^t k_1(t,s) T(s) P Z ds.$$

So, we approximate the first integral term

$$K_1 = F \cdot Z, \quad (9)$$

where

$$K_1 = [k_0, k_1, k_2, \cdots, k_N]^T,$$

and its elements k_i, $i = 0, 1, 2, \cdots, N$ are

$$\begin{aligned} k_0 &= \textstyle\int_0^{t_0} k_1(t,s) y_N(s) ds, \\ k_1 &= \textstyle\int_0^{t_1} k_1(t,s) y_N(s) ds, \\ &\vdots \\ k_N &= \textstyle\int_0^{t_N} k_1(t,s) y_N(s) ds, \end{aligned}$$

and

$$F = G \cdot P \cdot Z,$$

where

$$G = [G_{ij}]_{(N+1)\times(N+1)},$$

and

$$G_{ij} = \int_0^{t_i} k_1(t_i, s) T_{Lj}^*(s) ds, \quad i, j = 0, 1, 2, \cdots, N.$$

In our numerical scheme, the element G_{ij} of the matrix G are handled with Gauss quadrature formula,

$$\begin{aligned} G_{ij} &= \int_0^{t_i} k_1(t_i, s) T_{Lj}^*(s) ds \\ &= \frac{t_i}{2} \int_{-1}^{1} k_1\left(t_i, \frac{t_i x + t_i}{2}\right) T_{Lj}^*\left(\frac{t_i x + t_i}{2}\right) dx \\ &\approx \frac{t_i}{2} \sum_{r=0}^{N} k_1\left(t_i, \frac{t_i x_r + t_i}{2}\right) T_{Lj}^*\left(\frac{t_i x_r + t_i}{2}\right) \omega_r, \end{aligned}$$

where $\{\omega_r\}_{r=0}^N$ are Chebyshev weights. Similarly, the second part of integral term in (1) can be approximated

$$\int_0^{qt} k_2(t,s) z(s) ds \approx \int_0^{qt} k_2(t,s) z_N(s) ds.$$

Using (6) and Gauss quadrature formula the following operational matrix F_q is obtained:

$$K_2 = F_q Z, \quad (10)$$

where

$$K_2 = \left[\int_0^{qt_0} k_2(t,s) z_N(s) ds, \cdots, \int_0^{qt_N} k_2(t,s) z_N(s) ds\right]^T$$

and

$$F_q = G_q \cdot P \cdot Z.$$

where

$$G_q = [G_q]_{ij} = \int_0^{qt_i} k_2(t_i, s) T_{Lj}^*(s) ds, i, j = 0, 1, \cdots, N$$

and

$$[G_q]_{ij} \approx \frac{qt_i}{2} \sum_{r=0}^{N} k_2\left(qt_i, \frac{qt_i x_r + qt_i}{2}\right) T_{Lj}^*\left(\frac{qt_i x_r + t_i}{2}\right) \omega_r.$$

V. Numerical scheme

In this section, we apply the obtained the operational matrices to solve the given problem 1. For this purpose, we first apply the matrix forms (7),(8), (9) and (10) to the problem (1). Then, we obtain the following equation in the form of matrix

$$(D^{(1)} - A - BQ - F - F_q)Z = G \tag{11}$$

where

$$A = diag[\alpha(x_0), \alpha(x_1), \alpha(x_2), \cdots, \alpha(x_N)],$$
$$B = diag[\beta(x_0), \beta(x_1), \beta(x_2), \cdots, \beta(x_N)],$$
$$G = [g(x_0), g(x_1), \cdots, g(x_N)]^T.$$

For the sake of simplicity, we use the matrix C to denote $D^{(1)} - A - BQ - F - F_q$. Then, the fundamental matrix equation for (11) is reduced to

$$CZ = G,$$

where

$$C = [c_{ij}]_{(N+1)\times(N+1)}.$$

Then, we incorporate the initial value $z(0) = z_0$ in (2). Thus, we have the following system of algebraic equations with unknown $z_1, z_2, \cdots, z_N$.

$$z(0)\begin{bmatrix} c_{01} \\ c_{02} \\ c_{03} \\ \vdots \\ c_{0N} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1N} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2N} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & c_{N3} & \cdots & c_{NN} \end{bmatrix} \begin{bmatrix} z(t_1) \\ z(t_2) \\ z(t_3) \\ \vdots \\ z(t_N) \end{bmatrix} = \begin{bmatrix} g(t_1) \\ g(t_2) \\ g(t_3) \\ \vdots \\ g(t_N) \end{bmatrix}$$

By solving the above system of algebraic equations and using (6), we obtain the approximation $z_N(t)$.

VI. Some key lemmas

We present some useful lemmas and notations, which are necessary in a later section. Let $L^2_{\omega^{\alpha,\beta}}(I)$ be the space of measurable functions in $I := (-1, 1)$ and $\omega^{\alpha,\beta}(x)$ is the weight function. The norm of $L^2_{\omega^{\alpha,\beta}}(I)$ is defined as

$$\|u\|_{L^2_{\omega^{\alpha,\beta}}} = (u, u)^{\frac{1}{2}}_{\omega^{\alpha,\beta}}.$$

For $m \in \mathbb{N}_0$, define

$$H^m_{\omega^{\alpha,\beta},} = \left\{ v : \partial^k_x v \in L^2_{\omega^{\alpha,\beta}}(I), \quad 0 \le k \le m \right\},$$

with the semi-norm and norm as

$$|v|_{m,\omega^{\alpha,\beta}} = \|d^m v/dx^m\|_{\omega^{\alpha,\beta}},$$

$$\|v\|_{m,\omega^{\alpha,\beta}} = \left(\sum_{k=0}^{m} |v|^2_{k,\omega^{\alpha,\beta}} \right)^{\frac{1}{2}},$$

and

$$|v|_{H^{m;N}_{\omega^{\alpha,\beta}}} = \left(\sum_{k=min(m,N+1)}^{m} \|d^k v/dx^k\|^2_{L^2_{\omega^{\alpha,\beta}}} \right)^{\frac{1}{2}}.$$

Particularly, $\omega(x) = \omega^{-\frac{1}{2},-\frac{1}{2}}(x)$. In addition, let us denote $\mathbb{P}_N$ the set of all real polynomials of degree less then $N \in \mathbb{N}$. Moreover, the Lagrange interpolation polynomial of u is

$$I_N u = \sum_{i=0}^{N} u(x_i) F_i(x), \quad u \in C[-1, 1]$$

which satisfies

$$I_N u(x_i) = u(x_i), \quad i \in \mathbb{N}_0,$$

where $F_i(x)$ and x_i are the Lagrange polynomials and Chebyshev Gauss-Lobatto points, respectively.

Lemma 1: (see [19]) For a function $u \in H^{m,N}_{\omega}(I)$, with $m \in \mathbb{N}$. We have the estimation

$$\|u - I_N u\|_{L^2_\omega} \le C N^{-m} |u|_{H^{m;N}_\omega}, \tag{12}$$

where ω is the Chebyshev weight function.
By Lemma 1, we have the following relation:

$$\begin{aligned} &\left| \int_{-1}^{1} u(x)\phi(x)\omega(x) - (u, \phi)_N \right| \\ &\le C N^{-m} |u|_{H^{m;N}_\omega(I)} \|\phi\|_{L^2_\omega(I)}, \end{aligned} \tag{13}$$

where $\phi \in \mathbb{P}_N$ and $(u, \phi)_N$ is discrete inner product.

Lemma 2: (see [20]) Assume that the function u is bounded, then

$$\sup_N \left\| \sum_{i=0}^{N} u(x_i) F_i(x) \right\|_{L^2_\omega(I)} \le C \max_{x\in[-1,1]} |u(x)|,$$

where C is a constant independent of u.

Lemma 3: (see [12], [15]) Suppose $0 \le R_1, R_2 < +\infty$, Assume that the functions $E(x)$ and are nonnegative integrable and satisfy

$$E(x) \le R_1 \int_{-1}^{x} E(t)dt + R_2 \int_{-1}^{x} E(qt + q - 1)dt + H(x),$$

then

$$\|E\|_{L^p} \le C\|H\|_{L^p}, \quad p \ge 1.$$

VII. Convergence analysis

In order to facilitate error analysis, the equation (1) is converted to an equivalent form defined on $[-1, 1]$. Hence, we map the defined interval $[0, T]$ to $[-1, 1]$ through the coordinate transform

$$t = T(1 + x)/2, \quad x = 2t/T - 1$$

. Then, we transform (1) and (2) into

$$\begin{aligned} u'(x) =& A(x)u(x) + B(x)u(qx + q - 1) + G(x) \\ &+ \frac{T}{2} \int_0^{\frac{T}{2}(1+x)} K_1\left(\frac{T}{2}(1 + x), s\right) z(s)ds \\ &+ \frac{T}{2} \int_0^{\frac{qT}{2}(1+x)} K_2\left(\frac{T}{2}(1 + x), \tau\right) z(\tau)d\tau, \end{aligned} \tag{14}$$

$$u(-1) = u_{-1} = z_0, \tag{15}$$

where

$$u(x) = z\left(\frac{T}{2}(1 + x)\right), \quad A(x) = \frac{T}{2}\alpha\left(\frac{T}{2}(1 + x)\right),$$

$$B(x) = \frac{T}{2}\beta\left(\frac{T}{2}(1 + x)\right), \quad G(x) = \frac{T}{2}g\left(\frac{T}{2}(1 + x)\right).$$

Moreover, by using a linear transformation:

$$s = \frac{T}{2}(1+\theta), \quad \theta \in [-1, x],$$

$$\tau = \frac{T}{2}(1+\eta), \quad \eta \in [-1, qx+q-1],$$

the equation (14) become

$$\begin{aligned} u'(x) = &A(x)u(x) + B(x)u(qx+q-1) + G(x) \\ &+ \int_{-1}^{x} K_1(x,\theta)u(\theta)d\theta \\ &+ \int_{-1}^{qx+q-1} K_2(x,\eta)u(\eta)d\eta, \end{aligned} \tag{16}$$

where

$$K_1(x,\theta) = \frac{T^2}{4}k_1\left(\frac{T}{2}(1+x), \frac{T}{2}(1+\theta)\right)$$
$$K_2(x,\eta) = \frac{T^2}{4}k_1\left(\frac{T}{2}(1+x), \frac{T}{2}(1+\eta)\right).$$

Theorem 1: Let $u_N(x)$ is the approximate solution given by the numerical scheme. Assumed the analytical solution of (16) $u(x) \in H_\omega^m$ sufficiently smooth, then the following relation is given

$$\begin{aligned} &\|u - u_N\|_{L^2_\omega(I)} \\ &\le CN^{-m}\left(M\|u\|_{L^2_\omega(I)} + |u'|_{H_\omega^{m;N}(I)} + |u|_{H_\omega^{m;N}(I)}\right), \end{aligned}$$

where N is sufficiently large and $M = \max_{x\in[-1,1]} |K_1(x,t)|_{H_\omega^{m;N}(I)} + \max_{x\in[-1,1]} |K_2(x,t)|_{H_\omega^{m;N}(I)}$

Proof: Taking the Gauss-Lobatto collocation x_i on $[-1,1]$ in (16) yields

$$\begin{aligned} u'(x_i) = &A(x_i)u(x_i) + B(x_i)u(qx_i+q-1) \\ &+ G(x_i) + \int_{-1}^{x_i} K_1(x,\theta)u(\theta)d\theta \\ &+ \int_{-1}^{qx_i+q-1} K_2(x,\eta)u(\eta)d\eta, \end{aligned} \tag{17}$$

and

$$u(x_i) = \int_{-1}^{x_i} u'(\theta)d\theta + u_{-1}. \tag{18}$$

We use $u_N(x_i)$ to approximate $u(x_i)$ and

$$u_N(x) = \sum_{i=0}^{N} u_N(x_i)F_i(x),$$

where $F_i(x)$ is the Lagrange interpolating polynomials. Substituting the approximate relation into the first integral parts of (17) yields

$$\begin{aligned} \int_{-1}^{x_i} K_1(x_i,\theta)u_N(\theta)d\theta &= \frac{1+x_i}{2}\int_{-1}^{1} K_1(x_i,\tau)u_N(\tau)d\tau \\ &\approx \frac{1+x_i}{2}\sum_{r=0}^{N} K_1(x_i,\tau_r)u_N(\tau_r)\omega_r, \end{aligned}$$

by

$$\theta = \frac{1+x_i}{2}\tau + \frac{x_i-1}{2}, \tau \in [-1,1],$$

and Gauss quadrature formula. In a similar way we deal with

$$\begin{aligned} &\int_{-1}^{qx_i+q-1} K_2(x_i,\theta)u_N(\theta)d\theta \\ &\approx \frac{q(1+x_i)}{2}\sum_{r=0}^{N} K_2(x_i,\tau_r)u_N(\tau_r)\omega_r. \end{aligned}$$

So our numerical scheme in this study can be rewritten as

$$\begin{aligned} u_N'(x_i) = &A(x_i)u_N(x_i) + B(x_i)u_N(qx_i+q-1) + g(x_i) \\ &+ \frac{1+x_i}{2}\sum_{r=0}^{N} K_1(x_i,\tau_r)u_N(\tau_r)\omega_r \\ &+ \frac{q(1+x_i)}{2}\sum_{r=0}^{N} K_2(x_i,\tau_r)u_N(\tau_r)\omega_r, \\ u_N(x_i) = &\int_{-1}^{x_i} u_N'(\theta)d\theta + u_{-1}. \end{aligned} \tag{19}$$

For ease of analysis, the above equation becomes

$$\begin{aligned} &u_N'(x_i) = A(x_i)u_N(x_i) + B(x_i)u_N(qx_i+q-1) + g(x_i) \\ &+ \int_{-1}^{x_i} K_1(x,\theta)u_N(\theta)d\theta + \int_{-1}^{qx_i+q-1} K_2(x,\eta)u_N(\eta)d\eta \\ &- I_1(x_i) - I_2(x_i), \end{aligned} \tag{20}$$

where

$$\begin{aligned} I_1(x_i) = &\int_{-1}^{x_i} K_1(x_i,\theta)u_N(\theta)d\theta \\ &- \frac{1+x_i}{2}\sum_{j=0}^{N} K_1(x_i,\tau_j)u_N(\tau_j)\omega_j, \end{aligned}$$

and

$$\begin{aligned} I_2(x_i) = &\int_{-1}^{qx_i+q-1} K_2(x_i,\theta)u(\theta)d\theta \\ &- \frac{q(1+x_i)}{2}\sum_{j=0}^{N} K_2(x_i,\tau_j)u_N(\tau_j)\omega_j. \end{aligned}$$

Using Lagrange interpolating polynomials and (20) yield

$$\begin{aligned} u_N'(x) = &I_N(A(x)u_N(x)) \\ &+ I_N(B(x)u_N(qx+q-1)) \\ &+ I_N(G(x)) + I_N\left(\int_{-1}^{x} K_1(x,\theta)u_N(\theta)d\theta\right) \\ &+ I_N\left(\int_{-1}^{qx+q-1} K_2(x,\eta)u_N(\eta)d\eta\right) \\ &- J_1(x) - J_2(x), \end{aligned} \tag{21}$$

where

$$J_1(x) = \sum_{i=0}^{N} I_1(x_i)F_i(x), \quad J_2(x) = \sum_{i=0}^{N} I_2(x_i)F_i(x).$$

Clearly by (16),

$$\begin{aligned} I_N(u'(x)) = &I_N(A(x)u(x)) \\ &+ I_N(B(x)u(qx+q-1)) \\ &+ I_N(G(x)) + I_N\left(\int_{-1}^{x} K_1(x,\theta)u(\theta)d\theta\right) \\ &+ I_N\left(\int_{-1}^{qx+q-1} K_2(x,\eta)u(\eta)d\eta\right). \end{aligned} \tag{22}$$

By subtracting (21) from(22), we have

$$\begin{aligned} e_N'(x) =& u'(x) - I_N(u'(x)) + I_N(A(x)e_N(x)) \\ &+ I_N(B(x)e_N(qx+q-1)) \\ &+ I_N\left(\int_{-1}^{x} K_1(x,\theta)e_N(\theta)d\theta\right) \\ &+ I_N\left(\int_{-1}^{qx+q-1} K_2(x,\eta)e_N(\eta)d\eta\right) \\ &+ J_1(x) + J_2(x), \end{aligned} \tag{23}$$

where $e_N(x) = u(x) - u_N(x)$, $e_N'(x) = u'(x) - u_N'(x)$. By subtracting (19) from (18), we can obtain

$$u(x_i) - u_N(x_i) = \int_{-1}^{x_i} e_N'(\theta)d\theta. \tag{24}$$

Similarly, using Lagrange interpolating polynomials and (24) yield

$$\begin{aligned} u(x) - u_N(x) =& u(x) - I_N(u(x)) \\ &+ I_N\left(\int_{-1}^{x} e_N'(\theta)d\theta\right). \end{aligned} \tag{25}$$

Consequently, we rewrite (23) as

$$\begin{aligned} e_N'(x) =& A(x)e_N(x) + B(x)e_N(qx+q-1) \\ &+ \int_{-1}^{x} K_1(x,\theta)e_N(\theta)d\theta \\ &+ \int_{-1}^{qx+q-1} K_2(x,\eta)e_N(\eta)d\eta + \sum_{i=1}^{7} J_i(x), \end{aligned} \tag{26}$$

$$e_N(x) = \left(\int_{-1}^{x} e_N'(\theta)d\theta\right) + J_8(x) + J_9(x). \tag{27}$$

where

$$\begin{aligned} J_3(x) &= u'(x) - I_N(u'(x)), \\ J_4(x) &= I_N(A(x)e_N(x)) - A(x)e_N(x), \end{aligned}$$

$$\begin{aligned} J_5(x) &= I_N(B(x)e_N(qx+q-1)) - B(x)e_N(qx+q-1), \\ J_6(x) &= I_N\left(\int_{-1}^{x} K_1(x,\theta)e_N(\theta)d\theta\right) \\ &\quad - \int_{-1}^{x} K_1(x,\theta)e_N(\theta)d\theta, \\ J_7(x) &= I_N\left(\int_{-1}^{qx+q-1} K_2(x,\eta)e_N(\eta)d\eta\right) \\ &\quad - \int_{-1}^{qx+q-1} K_2(x,\eta)e_N(\eta)d\eta, \\ J_8(x) &= u(x) - I_N(u(x)), \\ J_9(x) &= I_N\left(\int_{-1}^{x} e_N'(\theta)d\theta\right) - \int_{-1}^{x} e_N'(\theta)d\theta. \end{aligned}$$

Substituting (27) into the first integral part of (26) and applying the Dirichlet's formula that $\int_{-1}^{x}\int_{-1}^{\tau}\phi(\tau,s)dsd\tau = \int_{-1}^{x}\int_{s}^{x}\phi(\tau,s)d\tau ds$, we obtain

$$\begin{aligned} \int_{-1}^{x} K_1(x,\theta)e_N(\theta)d\theta =& \int_{-1}^{x}\left(\int_{\theta}^{x} K_1(x,\tau)d\tau\right)e'(\theta)d\theta \\ &+ \int_{-1}^{x} K_1(x,\theta)(J_8(\theta)+J_9(\theta))d\theta \end{aligned} \tag{28}$$

Considering the second integral part of (26)

$$\begin{aligned} &\int_{-1}^{qx+q-1} K_2(x,\eta)e_N(\eta)d\eta \\ &= \int_{-1}^{qx+q-1} K_2(x,\eta)\left(\int_{-1}^{\eta} e'(\theta)d\theta\right)d\eta \\ &+ \int_{-1}^{qx+q-1} K_2(x,\eta)(J_8(\eta)+J_9(\eta))d\eta. \end{aligned} \tag{29}$$

For the sake of applying Dirichlet's formula we transform the above equation to

$$\begin{aligned} &\int_{-1}^{qx+q-1} K_2(x,\eta)e_N(\eta)d\eta \\ &= q^2\int_{-1}^{x} K_2(x,q\eta+q-1)\left(\int_{-1}^{\eta} e'(q\theta+q-1)d\theta\right)d\eta \\ &+ q\int_{-1}^{x} K_2(x,\phi(q,\eta))(J_8(\phi(q,\eta))+J_9(\phi(q,\eta)))d\eta. \\ &= q^2\int_{-1}^{x}\left(\int_{\eta}^{x} K_2(x,q\eta+q-1)d\theta\right)e'(q\theta+q-1)d\eta \\ &+ q\int_{-1}^{x} K_2(x,\phi(q,\eta))(J_8(\phi(q,\eta))+J_9(\phi(q,\eta)))d\eta, \end{aligned} \tag{30}$$

where $\phi(q,\eta) = q\eta + q - 1$. Substituting (28), (30) into (26) we obtain

$$\begin{aligned} &|e_N'(x)| \le M_1|e_N(x)| \\ &+ (M_2+M_4)\int_{-1}^{x} |e_N'(q\eta+q-1)|d\eta \\ &+ M_3\int_{-1}^{x} |e_N'(\theta)|d\theta + \sum_{i=1}^{7}|J_i(x)| \\ &+ \int_{-1}^{x} K_1(x,\theta)(|J_8(\theta)|+|J_9(\theta)|)d\theta \\ &+ q\int_{-1}^{x} K_2(x,q\eta+q-1)\sum_{i=8}^{9}|J_i(q\eta+q-1)|d\eta, \end{aligned} \tag{31}$$

where $M_1 = \max_{x\in[-1,1]}|A(x)|$, $M_2 = \max_{x\in[-1,1]}|B(x)|$ and

$$\begin{aligned} M_3 &= \max_{x\in D_1}\int_{\theta}^{x}|K_1(x,\tau)|d\tau, \\ M_4 &= q^2\max_{x\in D_2}\int_{\eta}^{x}|K_2(x,q\eta+q-1)|d\eta, \\ D_1 &= (x,\tau) : -1 \le x \le 1, -1 \le \tau \le x, \\ D_2 &= (x,\eta) : -1 \le x \le 1, -1 \le \eta \le qx+q-1. \end{aligned}$$

According to Gronwall inequality, we derive from (26), (27) that

$$\|e'(x)\|_{L^2_\omega(I)} \le C\left(\|e(x)\|_{L^2_\omega(I)} + \sum_{i=1}^{9}\|J_i\|_{L^2_\omega(I)}\right). \tag{32}$$

and

$$\|e_N(x)\|_{L^2_\omega(I)} \le C\left(\|e_N'(x)\|_{L^2_\omega(I)} + \sum_{i=8}^{9}\|J_i(x)\|_{L^2_\omega(I)}\right). \tag{33}$$

Then, by (32) and (33), we have

$$\|e_N(x)\|_{L^2_\omega(I)} \le C\sum_{i=1}^{9}\|J_i\|_{L^2_\omega(I)}. \tag{34}$$

Firstly, it follows from Lemma 2 and (13) that

$$\begin{aligned}\|J_1\|_{L^2_\omega(I)} &\le C \max_{x\in[-1,1]} |I_1(x)| \\ &\le CN^{-m} \max_{x\in[-1,1]} |K_1(x,t)|_{H^{m;N}_\omega(I)} \\ &\quad \cdot (\|u\|_{L^2_\omega(I)} + \|e\|_{L^2_\omega(I)}),\end{aligned} \tag{35}$$

$$\begin{aligned}\|J_2\|_{L^2_\omega(I)} &\le C \max_{x\in[-1,1]} |I_2(x)| \\ &\le CN^{-m} \max_{x\in[-1,1]} |K_2(x,t)|_{H^{m;N}_\omega(I)} \\ &\quad \cdot (\|u\|_{L^2_\omega(I)} + \|e\|_{L^2_\omega(I)}),\end{aligned} \tag{36}$$

Next, by Lemma 1, we have

$$\|J_3\|_{L^2_\omega(I)} \le CN^{-m}|u'|_{H^{m;N}_\omega(I)}, \tag{37}$$

and

$$\|J_8\|_{L^2_\omega(I)} \le CN^{-m}|u|_{H^{m;N}_\omega(I)}. \tag{38}$$

By virtue of Lemma 1 and $m = 1$, we get

$$\begin{aligned}\|J_4\|_{L^2_\omega(I)} &\le CN^{-1} \left\| A'(x) \int_{-1}^{x} e'(\tau)d\tau + A(x)e'(x) \right\|_{L^2_\omega(I)} \\ &\le CN^{-1}\|e'(x)\|_{L^2_\omega(I)}, \\ \|J_5\|_{L^2_\omega(I)} &\le CN^{-1}\|e'(x)\|_{L^2_\omega(I)}, \\ \|J_6\|_{L^2_\omega(I)} &\le CN^{-1}\|e(x)\|_{L^2_\omega(I)}, \\ \|J_7\|_{L^2_\omega(I)} &\le CN^{-1}\|e(x)\|_{L^2_\omega(I)}, \\ \|J_9\|_{L^2_\omega(I)} &\le CN^{-1}\|e(x)\|_{L^2_\omega(I)}.\end{aligned}$$

Therefore, combining the above relations $\|J_i\|_{L^2_\omega(I)}, i = 1, 2, \cdots, 9$ gives the estimate

$$\|u - u_N\|_{L^2_\omega(I)} \le CN^{-m}\left(M\|u\|_{L^2_\omega(I)} + |u'|_{H^{m;N}_\omega(I)} + |u|_{H^{m;N}_\omega(I)}\right),$$

where

$$M = \max_{x\in[-1,1]} |K_1(x,t)|_{H^{m;N}_\omega(I)} + \max_{x\in[-1,1]} |K_2(x,t)|_{H^{m;N}_\omega(I)}.$$

VIII. Numerical Examples and Discussions

Three numerical experiments are conducted to verify the effectiveness of the proposed numerical scheme. All the computations were implemented by using the programming language MATLAB.

Example 8.1: First, consider the pantograph equation

$$z'(t) = z\left(\frac{1}{2}t\right) + \int_0^t z(s)ds + \int_0^{\frac{1}{2}t} z(s)ds + 1 - \frac{3}{2}t,$$

where $t \in [0, T]$ and the initial value $z(0) = 0$. The analytical solution of the above problem is $z(t) = 1 - e^t$. We employ the introduced numerical scheme to handle the example with various values of T and N. In Figure 1, we plot the absolute error function $|e_N(x)|$ for $N = 8, 16$.

Table I provides the computational results on the interval of $[0, 10]$.

Example 8.2: We consider

$$\begin{aligned}z'(t) =& \frac{1}{2}z(t) + z(\frac{1}{4}t) + \int_0^t e^{t+s}z(s)ds \\ &+ \int_0^{\frac{1}{4}t} sz(s)ds + g(t),\end{aligned}$$

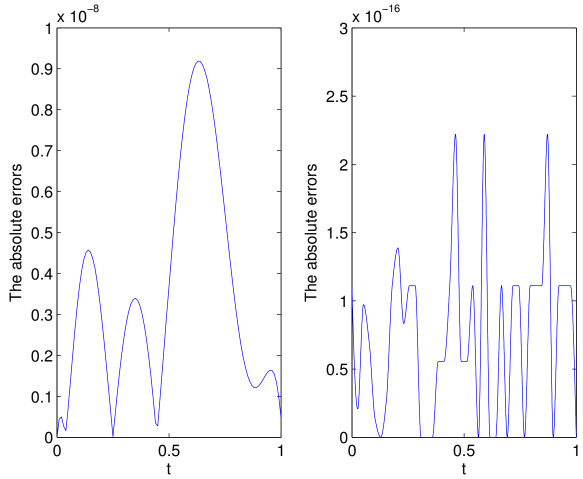

Fig. 1. Absolute error function for $N = 8, 16$ on $[0, 1]$.

TABLE I
ABSOLUTE ERRORS ON THE INTERVAL OF $[0, 10]$ FOR DIFFERENT VALUES OF N.

t	$N = 16$	$N = 20$	$N = 24$	$N = 28$	$N = 32$
1.0	3.8360e-5	5.5962e-7	1.7890e-9	2.1369e-14	4.2000e-16
2.0	1.7532e-4	8.3512e-7	4.1013e-11	5.7216e-14	3.3312e-16
3.0	2.1021e-4	8.0050e-8	2.0405e-9	6.7125e-14	2.2220e-16
4.0	1.2560e-4	7.4765e-7	2.7412e-10	1.3353e-13	7.7731e-16
5.0	2.0596e-5	8.4300e-7	3.4901e-9	1.7291e-15	2.7852e-16
6.0	1.5573e-4	8.5023e-8	5.6554e-9	1.4879e-14	2.2238e-16
7.0	2.2261e-4	1.4791e-6	4.4332e-9	1.4672e-14	4.4469e-16
8.0	1.9180e-4	2.7250e-6	5.3202e-10	7.8362e-15	7.6832e-16
9.0	6.6612e-5	3.3342e-6	3.5920e-9	1.3876e-14	1.2215e-16

where

$$g(t) = \frac{1}{2} - \frac{1}{4}e^{\frac{t}{4}} + \frac{t^2}{32} - \frac{e^{3t}}{2} + e^{2t},$$

and $z(0) = 0$. The analytical solution is $z(t) = e^t - 1$. This numerical example has been considered in [17], [21]. By using the proposed numerical scheme, we calculate the example with various values of N and T. The L^2 errors for different polynomial degree N are plotted in Figure 2. Table II displays a comparison of the maximum absolute errors of the suggested approach and Sine collocation method of [17].

TABLE II
COMPARISON OF L^∞ ERROR FOR EXAMPLE 8.2.

Sinc method		Proposed method	
N	$\|e(t)\|_{L^\infty}$	N	$\|e(t)\|_{L^\infty}$
5	3.6000e-3	4	8.1760e-4
10	2.2328e-4	8	7.7913e-9
20	5.7215e-6	12	2.8315e-14
30	2.8939e-7	16	3.3316e-16
40	2.2096e-7	20	2.2202e-16

Example 8.3: Finally, consider the integral equation with a convolution kernel

$$z(t) = g(t) + \int_0^{q_1 t} \cos(t-s)z(s)ds + \int_0^{q_2 t} \sin(t-s)z(s)ds,$$

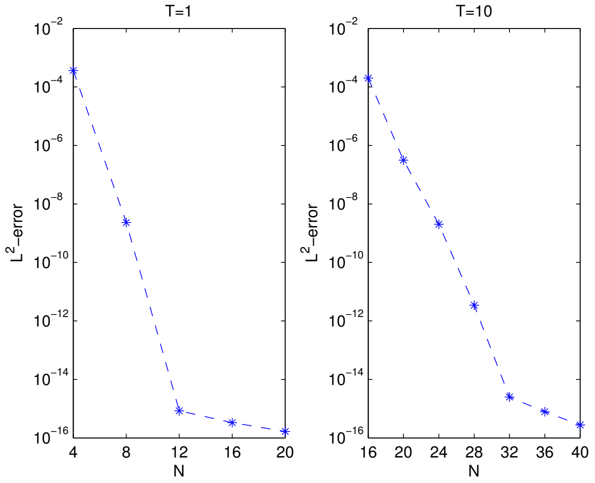

Fig. 2. L^2- errors versus the polynomial degree N for $T = 1, 10$

with $q_1 = 0.05$, $q_2 = 0.95$ and $g(t) = -0.475t\sin t - 0.025t\cos t + 1.75\cos t + 0.25\cos 0.9t - \cos 0.05t\cos 0.95t + 0.25\sin 0.9t - 0.25\sin t$. The given problem has an analytical solution $z(t) = \cos t$. The integral equation has been discussed in [12], [22]. Likewise, we calculate this example on $[0, 1]$ and $[0, 5]$. The computational results obtained by the proposed numerical method and other existing methods, such collocation Legendre spectral method [12] and Chebyshev cardinal functions method [22] are tabulated in Tables III, IV. Clearly, the results reveal that the suggested numerical scheme is valid and that its accuracy is comparable to existing numerical schemes. However, our proposed computational scheme is easier to implement.

TABLE III
COMPARISON OF THE ABSOLUTE ERROR FOR EXAMPLE 8.3.

	Cardinal functions method		Proposed method	
t	$N = 8$	$N = 16$	$N = 8$	$N = 16$
1	3.29e-4	3.73e-10	8.17e-6	8.76e-13
2	7.52e-4	6.03e-10	2.15e-5	9.13e-13
3	2.81e-4	4.44e-10	1.70e-5	3.50e-12
4	9.00e-4	4.83e-10	2.24e-5	5.19e-12
5	7.13e-4	4.30e-10	5.87e-5	7.70e-12

TABLE IV
COMPARISON OF L^2 AND L^∞ ERRORS ON $[0, 1]$ FOR EXAMPLE 8.3.

	Legendre spectral method		Proposed method	
N	$\|e(t)\|_{L^2_\omega}$	$\|e(t)\|_{L^\infty}$	$\|e(t)\|_{L^2_\omega}$	$\|e(t)\|_{L^\infty}$
8	1.731e-13	1.908e-13	2.746e-13	3.207e-13
12	3.454e-16	4.441e-16	1.065e-16	3.090e-16
16	3.628e-16	4.441e-16	6.881e-16	8.636e-16
20	2.606e-16	4.441e-16	1.776e-16	5.884e-16

IX. CONCLUSION

In this current study, we have developed and discussed the numerical scheme based on shifted Chebyshev polynomials to calculate a class of Volterra integro-differential equations with pantograph delay. First, we constructed the operational matrices of derivative and pantograph. The obtained matrices were applied to approximate the unknown functions. Also, we investigate a rigorous convergence analysis for the numerical scheme. Besides, we implement the proposed numerical method by three experiments. The provided results were exhibited to confirm the validity of the introduced approach. Moreover, the proposed computational scheme is readily modified to handle the nonlinear Volterra integro-differential equations of pantograph type.

REFERENCES

[1] H. Brunner, *Collocation methods for Volterra integral and related functional differential equations*. Cambridge University Press, 2004, vol. 15.

[2] K. E. Atkinson, *The numerical solution of integral equations of the second kind*. Cambridge University Press, 1997.

[3] C. Zhang and S. Vandewalle, "Stability analysis of Runge–Kutta methods for nonlinear Volterra delay-integro-differential equations," *IMA Journal of Numerical Analysis*, vol. 24, no. 2, pp. 193–214, 2004.

[4] W. Wang and S. Li, "Convergence of Runge-Kutta methods for neutral Volterra delay-integro-differential equations," *Frontiers of Mathematics in China*, vol. 4, no. 1, pp. 195–216, 2009.

[5] J. V. Shaalini and A. E. K. Pushpam, "Analysis of composite Runge Kutta methods and new one-step technique for stiff delay differential equations," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 3, pp. 359–368, 2019.

[6] S. Gan, "Dissipativity of θ-methods for nonlinear Volterra delay-integro-differential equations," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 898–907, 2007.

[7] H. Brunner and Q. Hu, "Optimal superconvergence results for delay integro-differential equations of pantograph type," *SIAM Journal on Numerical Analysis*, vol. 45, no. 3, pp. 986–1004, 2007.

[8] W. Ming, C. Huang, and L. Zhao, "Optimal superconvergence results for Volterra functional integral equations with proportional vanishing delays," *Applied Mathematics and Computation*, vol. 320, pp. 292–301, 2018.

[9] H. Brunner, "Recent advances in the numerical analysis of Volterra functional differential equations with variable delays," *Journal of Computational and Applied Mathematics*, vol. 228, no. 2, pp. 524–537, 2009.

[10] ——, "Current work and open problems in the numerical analysis of Volterra functional equations with vanishing delays," *Frontiers of Mathematics in China*, vol. 4, no. 1, pp. 3–22, 2009.

[11] M. Li, L. Wang, and Y. Liu, "Generalized hat functions method for solving fractional integro-differential equations of Bratu-type," *IAENG International Journal of Computer Science*, vol. 44, no. 1, pp. 105–111, 2017.

[12] I. Ali, H. Brunner, and T. Tang, "Spectral methods for pantograph-type differential and integral equations with multiple delays," *Frontiers of Mathematics in China*, vol. 4, no. 1, pp. 49–61, 2009.

[13] C. Sheng, Z. Wang, and B. Guo, "An hp-spectral collocation method for nonlinear Volterra functional integro-differential equations with delays," *Applied Numerical Mathematics*, vol. 105, pp. 1–24, 2016.

[14] R. E. Jid, "Moving least squares and Gauss Legendre for solving the integral equations of the second kind," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 1, pp. 90–97, 2019.

[15] Y. Wei and Y. Chen, "Legendre spectral collocation methods for pantograph Volterra delay-integro-differential equations," *Journal of Scientific Computing*, vol. 53, no. 3, pp. 672–688, 2012.

[16] J. Zhao, Y. Cao, and Y. Xu, "Legendre spectral collocation methods for Volterra delay-integro-differential equations," *Journal of Scientific Computing*, vol. 67, no. 3, pp. 1110–1133, 2016.

[17] ——, "Sinc numerical solution for pantograph Volterra delay-integro-differential equation," *International Journal of Computer Mathematics*, vol. 94, no. 5, pp. 853–865, 2017.

[18] C. Yang, "Modified Chebyshev collocation method for pantograph-type differential equations," *Applied Numerical Mathematics*, vol. 134, pp. 132–144, 2018.

[19] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*. Springer, 2006.

[20] P. Nevai, "Mean convergence of Lagrange interpolation. III," *Transactions of the American Mathematical Society*, vol. 282, no. 2, pp. 669–698, 1984.

[21] S. S. Ezz-Eldien and E. H. Doha, "Fast and precise spectral method for solving pantograph type Volterra integro-differential equations," *Numerical Algorithms*, vol. 81, pp. 57–77, 2019.

[22] M. Heydaria, G. B. Loghmania, and S. M. Hosseini, “Operational matrices of Chebyshev cardinal functions and their application for solving delay differential equations arising in electrodynamics with error estimation,” *Applied Mathematical Modelling*, vol. 37, no. 14, pp. 7789 – 7809, 2013.