Partial Chain Graphs

SHAHISTHA, K ARATHI BHAT* and G SUDHAKARA

***Abstract*—Chain graphs and threshold graphs are often referred to as extremal graphs, in the context that, they have the largest spectral radius among all the bipartite graphs (former one) and all the connected graphs (latter one). Nesting in the neighborhood of vertices in the above said extremal graphs have gained the attention of various researchers. Motivated by this structure, we generalize and define a new class of graphs named 'partial chain graphs' and study the properties. We also give the expression for rank, determinant and permanent of these graphs, from which permanent and determinants of well-known wheel graphs, fan graphs, and friendship graphs can be derived.**

***Index Terms*—Rank, Determinant, Permanent, Wheel graph, Fan graph.**

I. Introduction

THROUGHOUT the article, we denote a bipartite graph with the bipartition $V(G) = V_1 \cup V_2$ by $G(V_1 \cup V_2, E)$. A bi-star graph $B(p, q)$ is graph obtained by making the central (apex) vertices of two star graphs $K_{1,p-1}$ and $K_{1,q-1}$ adjacent. For a bipartite graph, the adjacency matrix can be written as $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, where B is called the biadjacency matrix. For other graph and matrix theoretic terminologies used here, we refer [1] and [2], respectively .

Some parameters associated with graph matrices often illuminate the graph structure. The determinant, permanent, rank, and Eigenvalues are a few of the powerful linear algebraic tools, which have been used extensively to study graphs. In specific, the parameters associated with the adjacency matrices of graphs are studied more extensively. For a graph G, we write $rank(G)$, $det(G)$, and $per(G)$ for rank, determinant and permanent of adjacency matrix of G. The expressions for $det(G)$, and $per(G)$ are available in the literature in terms of the elementary spanning subgraphs. A subgraph H of a graph G is said to be elementary if every component of H is a cycle or an edge. An elementary spanning subgraph of a graph is also called Sachs subgraph or a perfect 2-matching. The following theorem gives the expressions for determinant and permanent of a graph([3]).

Theorem 1.1: Let G be a graph on n vertices. Then

$$det(G) = \sum_H (-1)^{n-k_1(H)-k_2(H)} 2^{k_2(H)}$$

$$per(G) = \sum_H 2^{k_2(H)}$$

where H is an elementary spanning subgraph of G, $k_1(H)$ and $k_2(H)$ are the number of components in H which are edges and cycles respectively.

A chain graph is a bipartite graph with the property that neighborhood of vertices of each partite set form a chain with respect to set inclusion. The color classes of a chain graph $G(V_1 \cup V_2, E)$ can be partitioned into h non-empty cells $V_{1,1}, V_{1,2}, \ldots, V_{1,h}$ and $V_{2,1}, V_{2,2}, \ldots, V_{2,h}$ such that $N_G(u) = V_{2,1} \cup ... \cup V_{2,h-i+1}$, for any $u \in V_{1,i}$, $1 \leq i \leq h$. If $m_i = |V_{1,i}|$ and $n_i = |V_{2,i}|$, then we write $G = DNG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$. Due to this nesting property, the chain graphs are also called Double Nested Graphs (DNGs). The interesting facts concerned with chain graphs are available in the literature [4], [5], [6], [7], [8], [9] and [10].

A split graph is a graph which admits a partition of its vertex set into two parts, say W_1 and W_2, so that the vertices of W_1 induce a co-clique, while the vertices of W_2 induce a clique. All other edges, the cross edges, join a vertex in W_1 with a vertex in W_2 ([7]). A threshold graph is a split graph where the subsets of vertices of W_1 and W_2 can be further partitioned into h cells $W_1 = W_{1,1} \cup W_{1,2} \cup \cdots \cup W_{1,h}$ and $W_2 = W_{2,1} \cup W_{2,2} \cup \cdots \cup W_{2,h}$ satisfying the following nesting property: For each vertex $u \in W_{1,i}$, $1 \leq i \leq h$, $N_G(u) = W_{2,1} \cup ... \cup W_{2,h-i+1}$. If $|W_{1,i}| = m_i$ and $|W_{2,i}| = n_i$, then we write $G = NSG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$. The readers are referred to [11], [12], [13], [14], [15] and [16] for more results on threshold graphs. The chain graphs and threshold graphs are often referred to as extremal graphs due to the fact that, they have the largest spectral radius among all the bipartite graphs (former one) and all the connected graphs (latter one) with prescribed order and size. Further, any threshold graph can be obtained from a chain graph G by replacing one color class of G by a clique, and keeping all other edges unchanged.

II. Partial chain graphs

Motivated by the nesting property of chain and threshold graphs, we define a new class of graphs, whose vertex set can be partitioned into two subsets such that at least one of the partite sets is independent and has the nesting property. Formally, we define the same as follows.

Definition 2.1: A graph G is said to be a partial chain graph if its vertex set can be partitioned into two subsets V_1 and V_2 such that the following conditions are satisfied.

i. At least one of the partite sets is independent.
ii. If a partite set V_i $(i = 1, 2)$ is independent, then neighborhoods of vertices of V_i form a chain with respect to the operation of set inclusion. If not, $\{V_j \cap N_G(v)\} \neq \phi$ $(j \neq i)$ for every vertex $v \in V_i$.

Clearly, if V_i is not independent, then the neighborhoods

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of its vertices do not form a chain. Further, when both the partite sets are independent, we get a chain graph. When V_1 is independent and $\langle V_2 \rangle = K_n$ for some $n \geq 1$, we get a threshold graph. Partial chain graphs can be regarded as a generalized version of these extremal graphs namely, chain and threshold graphs. We get a partial chain graph from a bipartite chain graph by adding one or more edges joining the vertices of any one of the partite sets.

Let G be a partial chain graph with partition of the vertex set $V(G) = V_1 \cup V_2$ such that V_1 is independent. Due to the nesting property of neighborhoods, it is possible to further partition each of $V_i (i = 1, 2)$ into h cells $V_1 = V_{1,1} \cup V_{1,2} \cup \cdots \cup V_{1,h}$ and $V_2 = V_{2,1} \cup V_{2,2} \cup \cdots \cup V_{2,h}$ such that $N_G(u) = V_{2,1} \cup V_{2,2} \cup \cdots \cup V_{2,h-i+1}$ for all $u \in V_{1,i}, 1 \leq i \leq h$. Suppose $m_i = |V_{1,i}|$ and $n_i = |V_{2,i}|$, then we write

$$G = PCG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h).$$

where $|V_1| = \sum_{i=1}^{h} m_i$ and $|V_2| = \sum_{i=1}^{h} n_i$. The structure induced by the partite set V_2 (which need not be independent) is not taken into account in the above said approach and the notation. Unlike the extremal graphs discussed above, $G = PCG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$ does not represent a single graph, but a family of graphs G_f with nesting as said above. Thus, we write $G_f = PCG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$ (instead of just G).

Example 2.1: The graphs G_1 and G_2 (Figure 1) are the partial chain graphs in the family $G_f = PCG(2, 1, 1; 1, 1, 3)$.

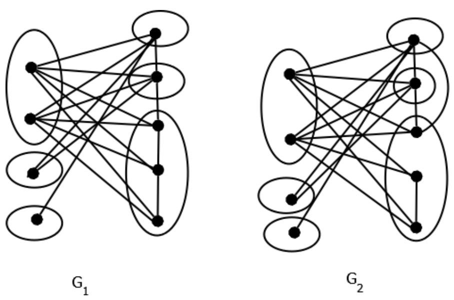

Fig. 1. The graph $G_1, G_2 \in G_f = PCG(2, 1, 1; 1, 1, 3)$

Clearly, V_1 is independent. But, the structure of $\langle V_2 \rangle$ in both the two graphs are distinct, but only the edges joining V_1 and V_2 are identical.

It is evident that the bipartite chain graph $DNG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h) \in G_f$ and the threshold graph $NSG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h) \in G_f$. In particular, any graph G in $G_f = PCG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$ can be obtained from the chain graph $DNG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$ by adding one or more edges between the vertices of V_2.

Throughout the article, we use the notion that the family $G_f = PCG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$ of graphs have the bipartition $V(G) = V_1 \cup V_2$ such that V_1 is independent, $|V_1| = \sum_{i=1}^{h} m_i$ and $|V_2| = \sum_{i=1}^{h} n_i$. For any such graph G in the family $G_f = PCG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$, we note the following:

Remark 2.1: By definition, the partite set V_1 has at least one vertex, say v, such that $N_G(v) = V_2$. These vertices are called dominating vertises in V_1.

Remark 2.2: Suppose V_2 is not independent, then it is true that the set $\{V_1 \cap N_G(v) | v \in V_2\}$ forms a chain with respect to set inclusion. Thus V_2 also has at least one dominating vertex.

Remark 2.3: Let m be the number of edges in G. Then

$$\sum_{j=1}^{h} m_j \left(\sum_{i=1}^{h-i+1} n_i \right) \leq m \leq \sum_{j=1}^{h} m_j \left(\sum_{i=1}^{h-i+1} n_i \right) + \frac{k(k-1)}{2}$$

where $k = \sum_{i=1}^{h} n_i$. The lower and upper bounds are attained by the graphs $DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ and $NSG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$, respectively.

The $2-$complement of a partial chain graph $G \in G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ with respect to the $2-$partition $\{V_1, V_2\}$ contains at least one isolated vertex since the dominating vertices of V_1 turns out to be isolated. More on the $2-$complement of a partial chain graph is explained in the following theorem.

Theorem 2.1: Let $G \in G_f$ be a partial chain graph where $G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ and $n \geq 2$. Then, the $2-$complement, G_2^P of G with respect to the partition $P = \{V_1, V_2\}$ is also a partial chain graph. Further, $G_2^P \in H_f$ where $H_f = PCG(m_h, m_{h-1}, \ldots, m_2; n_h, n_{h-1}, \ldots, n_2)$.

Proof: Let $H = G_2^P$. By the definition of $2-$ complement, it is true that, for any vertex $u \in V_1, N_H(u) = V_2 \setminus N_G(u)$. Clearly, $N_H(u) \subseteq N_H(v)$ if and only if $N_G(v) \subseteq N_G(u)$, for all $u, v \in V_1$. Thus neighborhood of vertices of V_1 in H forms a chain with respect to set inclusion and no changes in the structure of $\langle V_2 \rangle$. It can be easily observed that $H \in H_f = PCG(m_h, m_{h-1}, \ldots, m_1; n_h, n_{h-1}, \ldots, n_1)$. ■

When $h = 1$ in Theorem 2.1, then $G \in G_f = PCG(m_1; n_1)$ and there are no edges in G_2^P joining the vertices of V_1 with vertices of V_2. We discuss some more properties in the following theorems.

Theorem 2.2: Let $G \in G_f$ be a partial chain graph where G_f is the family of graphs given by $G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Then $diam(G) \leq 3$.

Proof: If $G = K_2$, then $diam(G) = 1$. Let $G \neq K_2$. Without loss of generality, let $v \in V_1, u \in V_2$ be the dominating vertices in V_1, V_2, respectively. The distance between any two vertices v_i, v_j in V_1 is 2 $(v_i - u - v_j)$. Similarly, distance between any two non-adjacent vertices u_k, u_l in V_2 is 2 $(u_k - v - u_l)$. Further, for any non dominating vertex $v_i \in V_1$, all the vertices $u_k \in V_2$ which are not adjacent to v_i are at distance 3 $(v_i - u - v - u_k)$. Thus $diam(G) = \max_{u,v \in V(G)} d(u, v) \leq 3$. ■

Corollary 2.3: Let $G \in G_f$ be a partial chain graph where $G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Then $rad(G) = 2$.

Theorem 2.4: Bi-star is the only tree which is a partial chain graph.

Proof: Let T be a partial chain graph, which is a tree. Without loss of generality, let v_1, u_1 be the dominating vertices in V_1 and V_2, respectively. Suppose a vertex $v_i \in V_1 (i \neq 1)$ is adjacent to $u_j \in V_2 (j \neq 1)$, then T has a cycle $v_i - u_j - v_1 - u_1 - v_i$, a contradiction. Thus, any vertex v_i in V_1 is adjacent to at most one vertex of V_2, and vice versa. Further, due to nesting of neighborhoods, for any vertices $v_i \in V_1, u_j \in V_2$, $N_T(v_i) = \{u_1\}$ and $N_T(u_j) = \{v_1\}$. Suppose any two vertices $u_j, u_k \in V_2$ in T are adjacent, then T has a cycle $u_j - v_1 - u_k$, contradiction and no two vertices in V_2 are adjacent. Thus, G is a bi-star graph with the central vertices u_1, v_1. ■

We also note that, whenever $|V_2| = 1$, we get a star graph, which can be considered as a special case of bi-star graphs.

III. Rank, determinant, and permanent

As discussed earlier, every partial chain graph can be obtained from a chain graph by the addition of edges between the vertices of V_2. We obtain rank, determinant, and permanent of partial chain graphs which are obtained from special chain graphs like bi-star graphs, complete bipartite graphs, etc.

Lemma 3.1: Let G be any partial chain graph in the family $G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Suppose V_1 has two or more pendant vertices, then $det(G) = per(G) = 0$.

Proof: As the neighborhood of vertices in V_1 forms a chain, all the pendant vertices are adjacent to the same vertex. Thus, if V_1 has more than two pendant vertices, then G has no elementary subgraph which spans all the vertices. Thus $det(G) = per(G) = 0$. ■

Lemma 3.2: Let G be any partial chain graph in the family $G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Suppose $|V_1| > |V_2|$, then $det(G) = 0$

Proof: If $|V_1| > |V_2|$, then at least two vertices of V_1 have the same neighborhood, resulting in identical rows in the adjacency matrix of the graph G. Thus $det(G) = 0$. ■

Theorem 3.3: Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph. If $\sum_{i=1}^{h} m_i = \sum_{i=1}^{h} n_i$, then for all the graphs H in the family $H_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$, $det(H) = det(G)$.

Proof: We know that

$$det(H) = det\left(\begin{array}{c|c} 0 & M \\ \hline M^T & C \end{array}\right)$$

where C is adjacency matrix of $\langle V_1 \rangle$ and M is the biadjacency matrix of the chain graph G. Since C, M are square matrices,

$$\begin{aligned} det(H) &= det(-MM^T) \\ &= det\left(\begin{array}{c|c} 0 & M \\ \hline M^T & 0 \end{array}\right) \\ &= det(G) \end{aligned}$$

■

From the above theorem, we note the following remark.

Remark 3.1: Whenever $|V_1| = |V_2|$, all graphs in the family of graphs $G_f = PCG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ have same determinant irrespective of the structure of $\langle V_2 \rangle$.

Example 3.1: All the three graphs G_1, G_2 and $G_3 \in G_f = PCG(1, 1, 1; 1, 1, 1)$ as shown in Figure 2 have determinant value equal to -1.

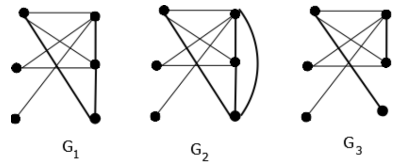

Fig. 2. The graphs $G_1, G_2, G_3 \in G_f = PCG(1, 1, 1; 1, 1, 1)$

Corollary 3.4: Let H be a partial chain graph in the family $H_f = PCG(\underbrace{1, 1, \ldots, 1}_{\text{h times}}; \underbrace{1, 1, \ldots, 1}_{\text{h times}})$. Then

$$\begin{aligned} det(H) &= (-1)^h \\ per(H) &= 1 \\ rank(H) &= 2h \end{aligned}$$

Proof: From Theorem 3.3, it follows that $det(H) = det(DNG(\underbrace{1, 1, \ldots, 1}_{\text{h times}}; \underbrace{1, 1, \ldots, 1}_{\text{h times}}))$. Hence the graph has full rank. Further, the graph H has same elementary spanning subgraphs as that of $DNG(\underbrace{1, 1, \ldots, 1}_{\text{h times}}; \underbrace{1, 1, \ldots, 1}_{\text{h times}})$. That is, H has only one elementary spanning subgraph, given by union of h K_2s, thus $per(H) = 1$. ■

The following theorem discusses the partial chain graphs obtained from bi-star graph in which V_2 is either a cycle or a path.

Theorem 3.5: Let $G \in G_f = PCG(1, p-1; 1, q-1)$ be a partial chain graph such that either $\langle V_2 \rangle = C_q$ or $\langle V_2 \rangle = P_q$ with one of the pendant vertex being the dominating vertex. Then $rank(G) = \begin{cases} (q+1) & if\ q \equiv 0(mod 4) \\ (q+2) & else \end{cases}$.

Proof: Let $\langle V_2 \rangle = C_q$. After relabeling the vertices of G, the adjacency matrix A of G can be written as

$$A = \left(\begin{array}{c|c} 0_{(p\times p)} & M_{(p\times q)} \\ \hline M^T_{(q\times p)} & A(C_q)_{(q\times q)} \end{array}\right)$$

where $A(C_q)$ is the adjacency matrix of C_q, given by

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

and $M = \left(\begin{array}{c|c} 1_{(1\times 1)} & \mathbf{1}_{(1\times q-1)} \\ \hline \mathbf{1}^T_{(p-1\times 1)} & O_{(p-1\times q-1)} \end{array}\right)$ ($\mathbf{1}$ being the row vector of one's). Consider $AX = 0$ where $X^T = \begin{pmatrix} x_1 & x_2 & \ldots x_p & x_{p+1} & \ldots & x_{p+q} \end{pmatrix}$. This is equivalent to

$$\sum_{j=1}^{q} x_{p+j} = 0 \tag{1}$$

$$x_{p+1} = 0 \tag{2}$$

$$\sum_{i=1}^{p} x_i + x_{p+2} + x_{p+q} = 0 \tag{3}$$

$$x_1 + x_{p+i} + x_{p+i+2} = 0\ for\ i = 1, 2, \ldots, (q-2) \tag{4}$$

$$x_1 + x_{p+q-1} + x_{p+1} = 0 \tag{5}$$

When $q \equiv 0\ (mod\ 4)$, from Equations 1, 4 and 5, we get that for all even j such that $2 \leq j \leq q$, $x_{p+j} = 0$ and for all odd j such that $2 \leq j \leq q$, $x_{p+j} = \begin{cases} k & j \equiv 1\ (mod\ 4) \\ -k & else \end{cases}$ for some arbitrary constant k. But, when $q \not\equiv 0\ (mod\ 4)$, we get $x_{p+j} = 0$ for all $2 \leq j \leq q$. Also, in both the cases $x_1 = x_{p+1} = 0$. Further, for the remaining variables, let $x_i = c_i$ $(2 \leq i \leq p-1)$ for some arbitrary constants c_j. From 3, we get $x_p = -\sum_{i=2}^{p-1} c_i$. Thus, $X^T = \begin{pmatrix} 0 & c_2 & \dots & c_{p-1} & \sum_{i=2}^{p-1} c_i & 0 & k & 0 & -k & \dots \end{pmatrix}$ if $q \equiv 0(mod\ 4)$ and $X^T = \begin{pmatrix} 0 & c_2 & c_3 & \dots & c_{p-1} & -\sum_{i=2}^{p-1} c_i & 0 & \dots & 0 \end{pmatrix}$ otherwise. Thus $nullity(A) = \begin{cases} (p-1) & if\ q \equiv 0(mod\ 4) \\ (p-2) & else \end{cases}$. This implies $rank(A) = \begin{cases} (q+1) & if\ q \equiv 0(mod\ 4) \\ (q+2) & else \end{cases}$.

The proof is similar when $\langle V_2 \rangle = P_q$. ■

Theorem 3.6: Let $G \in G_f = PCG(1, p-1; 1, q-1)$ be a partial chain graph such that either $\langle V_2 \rangle = C_q$ or $\langle V_2 \rangle = P_q$ with one of the pendant vertex being the dominating vertex. Then $det(G) = per(G) = 0$ for all $p \geq 3$. Further, when $p = 2$

$$det(G) = \begin{cases} 0 & if\ q \equiv 0(mod\ 4) \\ \frac{(q-1)}{2} & if\ q \equiv 1(mod\ 4) \\ 1 & if\ q \equiv 2(mod\ 4) \\ -\frac{(q+1)}{2} & if\ q \equiv 3(mod\ 4) \end{cases}$$

$$per(G) = \begin{cases} \frac{q^2}{4} & if\ q\ is\ even \\ \frac{q^2-1}{4} & else \end{cases}$$

Proof: If $p \geq 3$, then V_1 has at least two pendant vertices. Then by Lemma 3.1, $det(G) = per(G) = 0$. We consider the case when $p = 2$ and $\langle V_2 \rangle = C_q$. We note that every elementary spanning subgraph of G contains at least one K_2 whose end vertices are a full degree vertex of V_2 and pendant vertex of V_1. The elementary spanning subgraphs of G are given by

$C_k \cup \left(\frac{q-k+2}{2}\right) K_2$ for each odd number k such that $3 \leq k \leq q$, if q is odd and

$\cup K_2$ and $C_k \cup \left(\frac{q-k+2}{2}\right) K_2$ for each even k such that $4 \leq k \leq q$, if q is even.

If q is odd:

There are $\left(\frac{q-k+2}{2}\right)$ number of $C_k \cup \left(\frac{q-k+2}{2}\right) K_2$ for each odd integer $3 \leq k \leq q$. Thus

$$\begin{aligned} per(G) &= \sum_{\substack{k\ is\ odd \\ 3 \leq k \leq q}} 2\left(\frac{q-k+2}{2}\right) \\ &= (q-1) + (q-3) + \dots + 4 + 2 \\ &= 2\left(1 + 2 + \dots + \frac{(q-1)}{2}\right) \\ &= \frac{(q^2-1)}{4} \end{aligned}$$

On evaluation of determinant, the sign corresponding to each of the elementary spanning subgraph is considered. Since q is odd, for each odd number $3 \leq k \leq q$, the sign corresponding to $C_k \cup \left(\frac{q-k+2}{2}\right) K_2$ is given by $(-1)^{(q+2)-1-\frac{(q-k+2)}{2}} = (-1)^{\frac{q+k}{2}}$. Thus

$$det(G) = \sum_{\substack{k\ is\ odd \\ 3 \leq k \leq q}} 2(-1)^{\frac{(q+k)}{2}} \left(\frac{q-k+2}{2}\right)$$

But, we note that when $q \equiv 1(mod\ 4)$, $(-1)^{\frac{q+k}{2}} = \begin{cases} -1 & if\ k \equiv 1(mod\ 4) \\ 1 & if\ k \not\equiv 1(mod\ 4) \end{cases}$. Thus

$$\begin{aligned} det(G) &= (q-1) - (q-3) + \dots + 4 - 2 \\ &= 2\left(-1 + 2 - 3 + 4 - \dots + \frac{(q-1)}{2}\right) \\ &= 2\underbrace{(1 + 1 + \dots + 1)}_{\frac{q-1}{4}\ \text{times}} \\ &= \frac{(q-1)}{2} \end{aligned}$$

Similarly, if $q \not\equiv 1(mod\ 4)$, then $(-1)^{\frac{q+k}{2}} = \begin{cases} 1 & if\ k \equiv 1(mod\ 4) \\ -1 & if\ k \not\equiv 1(mod\ 4) \end{cases}$. Thus

$$\begin{aligned} det(G) &= (-(q-1) + (q-3) + \dots + 4 - 2) \\ &= 2\left(-1 + 2 - 3 + 4 - \dots - \frac{(q-1)}{2}\right) \\ &= 2\left(-1 + \underbrace{(-1 - 1 - \dots - 1)}_{\frac{q-3}{4}\ \text{times}}\right) \\ &= -\frac{(q+1)}{2} \end{aligned}$$

If q is even:

Along with $\left(\frac{q-k+2}{2}\right)$ number of $C_k \cup \left(\frac{q-k+2}{2}\right) K_2$ for each even integer $4 \leq k \leq q$, G also has an elementary spanning subgraph given by union of K_2s. Thus

$$\begin{aligned} per(G) &= \frac{q}{2} + \sum_{\substack{k\ is\ even \\ 4 \leq k \leq q}} 2\left(\frac{q-k+2}{2}\right) \\ &= \frac{q}{2} + (q-2) + (q-4) + \dots + 4 + 2 \\ &= \frac{q}{2} + 2\left(1 + 2 + \dots + \frac{(q-2)}{2}\right) \\ &= \frac{q^2}{4} \end{aligned}$$

Since q is even, for each even number $4 \leq k \leq q$, the sign corresponding to $C_k \cup \left(\frac{q-k+2}{2}\right) K_2$ is given by $(-1)^{\frac{q+k}{2}}$ and sign corresponding to union of K_2s is $(-1)^{\frac{(q+2)}{2}}$. Thus

$$det(G) = \sum_{\substack{k\ is\ even \\ 4 \leq k \leq q}} 2(-1)^{\frac{(q+k)}{2}} \left(\frac{q-k+2}{2}\right) + (-1)^{\frac{(q+2)}{2}} \frac{q}{2}$$

If $q \equiv 0(mod\ 4)$, then we note that $(-1)^{\frac{q+k}{2}} = \begin{cases} 1 & if\ k \equiv 0(mod\ 4) \\ -1 & if\ k \not\equiv 0(mod\ 4) \end{cases}$ and the sign corresponding to the union of K_2s is (-1). Thus

$$\begin{aligned} det(G) &= ((q-2)-(q-4)+\cdots-4+2)-\frac{q}{2} \\ &= 2\left(1-2+3-4+\cdots+\frac{(q-2)}{2}\right)-\frac{q}{2} \\ &= 2\left(1+\underbrace{(1+1+\cdots+1)}_{\frac{q-4}{4}\text{ times}}\right)-\frac{q}{2} \\ &= 0 \end{aligned}$$

Similarly, if $q \not\equiv 0(mod\ 4)$, $(-1)^{\frac{q+k}{2}} = \begin{cases} -1 & if\ k \equiv 0(mod\ 4) \\ 1 & if\ k \not\equiv 0(mod\ 4) \end{cases}$ and the sign corresponding to the union of K_2s is $(+1)$. Thus

$$\begin{aligned} det(G) &= (-(q-2)+(q-4)+\cdots-4+2)+\frac{q}{2} \\ &= 2\left(1-2+3-4-\cdots-\frac{(q-2)}{2}\right)+\frac{q}{2} \\ &= 2\left(\underbrace{(-1-1-\cdots-1)}_{\frac{q-2}{4}\text{ times}}\right)+\frac{q}{2} \\ &= 1 \end{aligned}$$

When $\langle V_2 \rangle = P_q$, the expressions for permanents and determinants remains the same as that of the graph where $\langle V_2 \rangle = C_q$. This is because both the graphs have same set of elementary spanning subgraphs as the extra edge which converts the path P_q to C_q do not make any difference to the elementary spanning subgraphs. ■

The following corollary gives the determinant and permanent of fan graphs.

Corollary 3.7: Let $F_{1,n-1} = P_{n-1} + K_1$ be a fan graph on n vertices. Then

$$det(F_{1,n-1}) = \begin{cases} 0 & if\ n \equiv 0(mod4) \\ \frac{(1-n)}{2} & if\ n \equiv 1(mod4) \\ -1 & if\ n \equiv 2(mod4) \\ \frac{(n+1)}{2} & if\ n \equiv 3(mod4) \end{cases}$$

$$per(F_{1,n-1}) = \begin{cases} \frac{n^2}{4} & if\ n\ is\ even \\ \frac{n^2-1}{4} & else \end{cases}$$

Proof: Let $G \in G_f = PCG(1,1;1,n-1)$ and $\langle V_2 \rangle = P_n$. The fan graph $F_{1,n-1}$ can be obtained from the partial chain graph G by removing the pendant vertex of V_1 and the full degree vertex of V_2. Since there is one to one correspondence between the elementary spanning subgraphs of G and $F_{1,n-1}$, the terms in the summations of $per(G)$ and $per(F_{1,n-1})$ remain the same. Thus $per(G) = per(F_{1,n-1})$. But in the case of $det(G)$, the signs of the corresponding terms in the summation of $det(F_{1,n-1})$ and $det(G)$ are of different parity. Thus $det(F_{1,n-1}) = -det(G)$ ■

Theorem 3.8: Let $G \in G_f = PCG(1,p-1;1,q-1)$ be a partial chain graph such that $\langle V_2 \rangle = K_{1,q-1}$, with the central vertex being the full degree vertex of V_2. Then

$$rank(G) = 4$$
$$det(G) = per(G) = 0 \text{ except when } p = q = 2\ .$$

Proof: After relabeling the vertices of G, the adjacency matrix A of G can be written as

$$A = \left(\begin{array}{c|c} 0_{(p\times p)} & M_{(p\times q)} \\ \hline M^T_{(q\times p)} & A(K_{1,q-1})_{(q\times q)} \end{array}\right)$$

where $A(K_{1,q-1}) = \left(\begin{array}{c|c} 0_{(1\times 1)} & \mathbf{1}_{(1\times q-1)} \\ \hline \mathbf{1}^T_{(p-1\times 1)} & O_{(p-1\times q-1)} \end{array}\right)$ and $M = \left(\begin{array}{c|c} 1_{(1\times 1)} & \mathbf{1}_{(1\times q-1)} \\ \hline \mathbf{1}^T_{(p-1\times 1)} & O_{(p-1\times q-1)} \end{array}\right)$ (**1** being the row vector of one's).

Consider $AX = 0$ where $X^T = \begin{pmatrix} x_1 & x_2 & \ldots x_p & x_{p+1} & \ldots & x_{p+q} \end{pmatrix}$. This is equivalent to

$$\sum_{j=1}^{q} x_{p+j} = 0 \tag{6}$$
$$x_{p+1} = 0 \tag{7}$$
$$\sum_{i=1}^{p} x_i + \sum_{j=2}^{q} x_{p+j} = 0 \tag{8}$$
$$x_1 + x_{p+1} = 0 \tag{9}$$

From 7 and 9, $x_1 = x_{p+1} = 0$. Let $x_{p+i} = k_i$ $(2 \le i \le q-1)$ for some arbitrary constants k_i. From Equation 6, we get $x_{p+q} = -\sum_{i=2}^{q-1} k_i$. Similarly, let $x_i = c_i(2 \le i \le (p-1))$ for some arbitrary constants c_i. Then by 8, $x_p = -\sum_{i=2}^{p-1} c_i$. Thus, $nullity(A) = (p+q-4)$. This implies $rank(A) = 4$.

It is noted that V_1 has at east two pendant vertices except when $p = q = 2$. When $p = q = 2$, (Figure 3) we get $det(G) = per(G) = 1$. ■

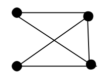

Fig. 3. The graph $G \in G_f = PCG(1,1;1,1)$ where $\langle V_2 \rangle = K_{1,1}$

Theorem 3.9: Let $G \in G_f = PCG(p;q)$ be a partial chain graph such that $\langle V_2 \rangle = P_q$. Then

$$rank(G) = \begin{cases} q & if\ q \equiv 1\ (mod\ 4) \\ q+1 & else \end{cases}$$

$$det(G) = \begin{cases} det(F_{1,q}) & if\ p = 1 \\ 0 & else \end{cases}$$

Proof: After relabeling the vertices of G, the adjacency matrix A of G can be written as

$$A = \left(\begin{array}{c|c} 0_{(p\times p)} & J_{(p\times q)} \\ \hline J_{(q\times p)} & A(P_q)_{(q\times q)} \end{array}\right)$$

where $A(P_q)$ is the adjacency matrix of the path P_q and J is the matrix in which every entry is one.

Consider $AX = 0$ where $X^T = \begin{pmatrix} x_1 & x_2 & \ldots x_p & x_{p+1} & \ldots & x_{p+q} \end{pmatrix}$. This is equivalent to

$$\sum_{j=1}^{q} x_{p+j} = 0 \tag{10}$$

$$\sum_{i=1}^{p} x_i + x_{p+2} = 0 \tag{11}$$

$$\sum_{i=1}^{p} x_i + x_{p+j} + x_{p+j+2} = 0 \; for \; i = 1, 2, \ldots, (q-2) \tag{12}$$

$$\sum_{i=1}^{p} x_i + x_{p+1} + x_{p+q-1} = 0 \tag{13}$$

From Equation 11, we get $\sum_{i=1}^{p} x_i = -x_{p+2}$. When q is even, from 12 and 13, we get $x_{p+j} = 0$ for all $j (1 \le j \le q)$. When q is odd, Equations 12, 13 results in $x_{p+j} = \begin{cases} k & j \equiv 1 \ (mod \ 4) \\ -k & j \equiv 3 \ (mod \ 4) \\ 0 & else \end{cases}$ for some arbitrary constant k. But, whenever $q \equiv 3 \ (mod \ 4)$, the Equation 12 results in $k = 0$, which implies $x_{p+j} = 0$ for all $j (1 \le i \le q)$. Further, for the the remaining variables, let $x_i = c_i \ (1 \le i \le p-1)$ for some arbitrary constants c_i. From 10, we get $x_p = -\sum_{i=1}^{p-1} c_i$. Thus

$X^T = \begin{pmatrix} c_1 & c_2 & \ldots & c_{p-1} & -\sum_{i=1}^{p-1} c_i & k & 0 & -k & 0 & \ldots \end{pmatrix}$

if $q \equiv 1 (mod \ 4)$ and

$X^T = \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_{p-1} & -\sum_{i=1}^{p-1} c_i & 0 & 0 & \ldots & 0 \end{pmatrix}$

otherwise. Thus, $nullity(A) = \begin{cases} p & if \ q \equiv 1 (mod \ 4) \\ p-1 & else \end{cases}$ and the proof follows. Further, the graph G has non-zero determinant only when $q = 1$ as A is of full rank. But, when $q = 1$, G is a fan graph. ■

Theorem 3.10: Let $G \in G_f = PCG(p; q)$ be a partial chain graph such that $\langle V_2 \rangle = C_q$. Then

$$rank(G) = \begin{cases} (q-1) & if \ q \equiv 0 \ (mod \ 4) \\ (q+1) & else \end{cases}$$

$$det(G) = \begin{cases} 2q & if \ p = 1 \ and \ q \equiv 2 \ (mod \ 4) \\ -q & if \ p = 1 \ and \ q \equiv 1 \ (mod \ 4) \\ & or \ p = 1 \ and \ q \equiv 3 \ (mod \ 4) \\ 0 & if \ p = 1 \ and \ q \equiv 0 \ (mod \ 4) \end{cases}$$

Proof: After relabeling the vertices of G, the adjacency matrix A of G can be written as

$$A = \left(\begin{array}{c|c} 0_{(p \times p)} & J_{(p \times q)} \\ \hline J_{(q \times p)} & A(C_q)_{(q \times q)} \end{array} \right)$$

where $A(C_q)$ is the adjacency matrix of the cycle C_q (as given in Theorem 3.5). Consider $AX = 0$ where $X^T = \begin{pmatrix} x_1 & x_2 & \ldots x_p & x_{p+1} & \ldots & x_{p+q} \end{pmatrix}$. This is equivalent to

$$\sum_{j=1}^{q} x_{p+j} = 0 \tag{14}$$

$$\sum_{i=1}^{p} x_i + x_{p+2} + x_{p+q} = 0 \tag{15}$$

$$\sum_{i=1}^{p} x_i + x_{p+j} + x_{p+j+2} + = 0 \; for \; j = 1, 2, \ldots, q-2 \tag{16}$$

$$\sum_{i=1}^{p} x_i + x_{p+q-1} + x_{p+1} = 0 \tag{17}$$

From Equations 15, 16 and 17 we get $x_{p+j} = \begin{cases} k_1 & j \equiv 0 \ (mod \ 4) \\ -k_1 & j \equiv 1 \ (mod \ 4) \\ k_2 & j \equiv 2 \ (mod \ 4) \\ -k_2 & j \equiv 3 \ (mod \ 4) \end{cases}$ for some constants k_1, k_2. But, from Equation 14, we get the constants $k_1 = k_2 = 0$ except when $q \equiv 0 (mod \ 4)$. Also, let $x_i = c_i \ (1 \le i \le p-1)$ for some arbitrary constants c_i. From 15, we get $x_p = -\sum_{i=1}^{p-1} c_i$. Thus

$X^T = \begin{pmatrix} c_1 & \ldots & c_{p-1} & -\sum_{i=1}^{p-1} c_i & k_1 & k_2 & -k_1 & -k_2 & \ldots \end{pmatrix}$

if $q \equiv 0 (mod \ 4)$ and

$X^T = \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_{p-1} & -\sum_{i=1}^{p-1} c_i & 0 & 0 & \ldots \end{pmatrix}$

otherwise. Thus $nullity(A) = \begin{cases} (p+1) & if \ q \equiv 0 (mod \ 4) \\ (p-1) & else \end{cases}$ and hence the rank.

We note that the graph G has full rank only when $p = 1$. Further, the graph has no elementary spanning subgraph whenever $q \equiv 0 (mod \ 4)$. Thus, the graph has non-zero determinant only when $p = 1$ and $q \not\equiv 0 (mod \ 4)$. Let $q \not\equiv 0 (mod \ 4)$ and $p = 1$. We note that, elementary spanning subgraphs of G contains elementary spanning subgraphs of $F_{1,q}$ and $F_{1,q-2} \cup K_2$ and $(k-2)$ copies of $C_k \cup \left(\frac{q-k+1}{2}\right) K_2$ for each odd number k such that $3 \le i \le (q+1)$ if q is even and for each even number k such that $4 \le i \le (q+1)$ if q is odd. The sign corresponding to $C_k \cup \left(\frac{q-k+1}{2}\right) K_2$ is given by $(-1)^{(p+1)-1-\frac{(p-k+1)}{2}} = (-1)^{\frac{(p+k-1)}{2}}$. When $q \equiv 3 \ (mod \ 4)$, for each even number $4 \le k \le (p+1)$, the sign corresponding to $C_k \cup \left(\frac{q-k+1}{2}\right) K_2$ is $-(-1)^{\frac{k}{2}}$. Thus, the determinant is given by

$$\begin{aligned} det(G) &= det(F_{1,q}) - det(F_{1,q-2}) + \sum_{\substack{k \ is \ even \\ 4 \le k \le q+1}} -2(-1)^{\frac{k}{2}} (k-2) \\ &= 0 + 1 + 2\left(-2 + 4 - 6 + \cdots - (q-1)\right) \\ &= 0 + 1 + 4\left(-1 + 2 - 3 + \cdots - \frac{(q-1)}{2}\right) \\ &= 0 + 1 + 4\left(-1 + \underbrace{(-1 - 1 - \cdots - 1)}_{\frac{q-3}{4} \text{ times}}\right) \end{aligned}$$

$det(G) = -q.$

When $q \equiv 1 (mod \ 4)$, for each even number $4 \le k \le (q+1)$, the sign corresponding to $C_k \cup \left(\frac{q-k+1}{2}\right) K_2$ is $(-1)^{\frac{k}{2}}$. Thus

$$
\begin{aligned}
det(G) &= det(F_{1,q}) - det(F_{1,q-2}) + \sum_{\substack{k\ is\ even \\ 4\leq k\leq q+1}} 2(-1)^{\frac{k}{2}}(k-2) \\
&= -1 + 0 + 2\left(2 - 4 + \cdots - (q-1)\right) \\
&= -1 + 0 + 4\left(1 - 2 + 3 - \cdots - \frac{(q-1)}{2}\right) \\
&= -1 + 0 + 4\left(\underbrace{-1 - 1 - \cdots - 1}_{\frac{q-1}{4}\text{ times}}\right) \\
det(G) &= -q.
\end{aligned}
$$

When $q \equiv 2 (mod\ 4)$, for each even number $3 \leq k \leq (q+1)$, the sign corresponding to $C_k \cup \left(\frac{q-k+1}{2}\right) K_2$ is $-(-1)^{\frac{(k-1)}{2}}$. Thus

$$
\begin{aligned}
det(G) &= det(F_{1,q}) - det(F_{1,q-2}) + \sum_{\substack{k\ is\ odd \\ 3\leq k\leq q+1}} -2(-1)^{\frac{(k-1)}{2}}(k-2) \\
&= \frac{q+2}{2} - \frac{2-q}{2} + 2\left(1 - 3 + 5 - 7 + \cdots + (q-1)\right) \\
&= \frac{q+2}{2} - \frac{2-q}{2} + 2\left(1 + \underbrace{(2 + 2 + \cdots + 2)}_{\frac{q-2}{4}\text{ times}}\right) \\
&= \frac{q+2}{2} - \frac{2-q}{2} + q \\
det(G) &= 2q.
\end{aligned}
$$

Hence the proof. ■

Corollary 3.11: Let $W_{1,n}$ be a wheel graph on $(n+1)$ vertices. Then

$$
det(W_{1,n}) = \begin{cases} 0 & if\ n \equiv 0\ (mod\ 4) \\ 2n & if\ n \equiv 2\ (mod\ 4) \\ -n & else \end{cases}
$$

$$
per(W_{1,n}) = n^2
$$

Proof: The proof follows from that fact that the wheel graph $W_{1,n} \in G_f = PCG(1;n)$. ■

Theorem 3.12: Let $G \in G_f = PCG(1, p-1; 1, q-1)$ be a partial chain graph. Let $\langle V_2 \rangle = \cup K_2's$. Then,

$$
rank(G) = q + 2
$$

$$
det(G) = \begin{cases} (-1)^{\frac{q}{2}} + 1 & if\ p = 2 \\ 0 & else \end{cases}
$$

$$
per(G) = \begin{cases} 1 & if\ p = 2 \\ 0 & else \end{cases}
$$

Proof: Since $\langle V_2 \rangle = \cup K_2's$, q is even and $\langle V_2 \rangle$ has $(\frac{q}{2})$ $K_2's$. After relabeling the vertices of G, the adjacency matrix of G can be rewritten as

$$
A = \left(\begin{array}{c|c} 0_{(p\times p)} & M_{(p\times q)} \\ \hline M^T_{(q\times p)} & N_{(q\times q)} \end{array}\right)
$$

where $N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}_{q\times q}$

and $M = \left(\begin{array}{c|c} 1_{(1\times 1)} & \mathbf{1}_{(1\times q-1)} \\ \hline \mathbf{1}^T_{(p-1\times 1)} & O_{(p-1\times q-1)} \end{array}\right)$ (**1** being the row vector of one's).

Consider $AX = 0$ where $X^T = \begin{pmatrix} x_1 & x_2 & \dots x_p & x_{p+1} & \dots & x_{p+q} \end{pmatrix}$. This is equivalent to

$$
\sum_{j=1}^{q} x_{p+j} = 0 \tag{18}
$$

$$
x_{p+1} = 0 \tag{19}
$$

$$
x_1 + x_{p+j} = 0 \text{ for } j = 1, 3, 4, 5, 6, \dots, q \tag{20}
$$

$$
\sum_{i=1}^{p} x_i + x_{p+2} = 0 \tag{21}
$$

From 19 and 21 with $j = 1$, we get $x_1 = x_{p+1} = 0$. Further, from Equation 21 with all possible values of j, we get $x_{p+j} = 0$ for all $2 \leq j \leq q$. For the remaining variables, let $x_j = c_{j-1}$ for $2 \leq j \leq p-1$. Then, $x_p = -\sum_{j=1}^{p-2} c_j$.

Thus, $X^T = \left(0\ c_1\ c_2\ \dots\ c_{p-2}\ -\sum_{j=1}^{p-2} c_j\ \underbrace{0\ 0\ \dots 0}_{q\text{ times}}\right)$ and $nullity(A) = p-2$. Hence it follows that, $rank(A) = q+2$.

From the rank, it follows that $det(G) > 0$ if and only if $p = 2$, i.e., when $G \in G_f = PCG(1,1;1,q-1)$. But for all the graphs $G \in G_f = PCG(1,1;1,q-1)$, there is only one elementary spanning subgraph given by union of $\left(\frac{q+2}{2}\right) K_2's$. Hence, $det(G) = (-1)^{\frac{q}{2}+1}$.

Similarly, $per(G) = 1$ as G has only one elementary spanning subgraph given by union of $K_2's$ if and only if $p = 2$. ■

As a result of Theorem 3.12, one can easily get the rank, determinant and permanent of friendship graph. That is, when $G \in G_f = PCG(1;2n)$, and $\langle V_2 \rangle = \cup K_2's$ we get the friendship graph $F_n \in G_f = PCG(1;2n)$. The friendship graph F_n is a graph with $2n+1$ vertices and $3n$ edges, which can be constructed by joining n copies of the cycle graph C_3 with a common vertex. The friendship graph $F_3 \in G_f = PCG(1;2n)$ is shown in Figure 4.

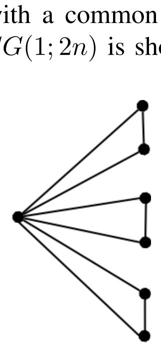

Fig. 4. The friendship graph $F_3 \in G_f = PCG(1;6)$

Corollary 3.13: Let F_n be a friendship graph on $2n+1$ vertices. Then

$$rank(F_n) = 2n + 1$$
$$det(F_n) = per(F_n) = 0$$

IV. Conclusion and scope for future work

With the influence of nesting of neighborhoods in chain and threshold graph, the generalized version, partial chain graphs are defined. The current article provides results on linear algebraic tools like rank, permanent, and determinant of partial chain graphs. Essentially like chain/threshold graphs, we further intend to study the significance of this class of graphs in the field of spectral graph theory.

In contrast to the chain formed by the neighbourhood of vertices, a new class of bipartite graphs named antichain graphs is defined by the authors of the article [17]. In particular, in antichain graphs the neighborhood of vertices in each partite sets form antichain with respect to set inclusion. A similar approach can be extended for partial chain graphs. In other words, when the neighbourhood of vertices of the independent set V_1 forms an antichain, a question regarding the structure and relevance of graphs is raised. This could be a goal for future work on partial chain graphs.

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2. In the authors affiliation we have replaced “of Mathematics Department” by “in the Department of Mathematics”.