Iteration and Existence of Positive Solutions for Fractional Integral Problems involved $p(t)-$Laplacian Operator

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***Abstract*—A new $p(t)-$ Laplacian problems with fractional integration in boundary conditions is investigated in this paper. Monotone iterative technique is adopted. On the basis of the results of positive solution, we also obtain the approximation sequence for it.**

***Index Terms*—Positive solutions; the operator involved $p(t)-$Laplacian; fractional integral boundary value problem; iterative monotone technique.**

I. Introduction

FRACTIONAL differential equations occur more frequently in different research areas, for example control theory, biophysics, signal and image processing and economics, etc [1-5]. Let's take the controller which is called $PI^{\lambda}D^{\mu}-$ as an example. The following form is the heat function of the above controller

$$G_c(S)=\frac{U(S)}{E(S)}=Q_p+Q_IW^{-\rho}+Q_DW^{\nu},\ (\rho,\nu>0).$$

The expression for the $PI^{\lambda}D^{\mu}-$ controller's output is

$$u(t)=Q_pe(t)+Q_ID^{-\rho}e(t)+Q_DD^{\nu}e(t).$$

Taking $\rho=1$ and $\nu=1$, we obtain a classical $PID-$ controller. $\rho=1$ and $\nu=0$ give a $PI-$ controller. $\rho=0$ and $\nu=1$ give a $PD-$ controller. $\rho=0$ and $\nu=0$ give a gain.

All these classical types of $PID-$ controllers are the particular cases of the fractional $PI^{\rho}D^{\nu}-$ controller. The $PI^{\rho}D^{\nu}-$ controller can better characterize the fractional system.

Motivated by the wide application of the equation which is involved in fractional derivative, in the last few years, some scholars have come up with useful results [6-12]. In [12], the existence result of the solution is obtained for

$$D^{\alpha}u(t)=f(t,u(t),u'(t)),\quad t\in(0,1)$$
$$D_{0+}^{\alpha-2}u(0)=0,\quad u(1)=\eta u(\xi).$$

It is well known that the non-Newtonian fluid theory can produce the $p-$ Laplacian equations[13]. So many scholars discuss $p-$ Laplacian equations of fractional-order. For instance, Wang et al [14] studied the following questions

$$D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t)))+f(t,u(t))=0,\quad 0<t<1,$$
$$u(0)=0,\quad u(1)=au(\xi),\quad D_{0+}^{\alpha}u(0)=0.$$

In [15], the author investigates the following questions according to the idea of upper and lower solution

$$D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t)))+f(t,u(t))=0,\quad 0<t<1,$$
$$u(0)=0,\ u(1)=au(\xi),\ D_{0+}^{\alpha}u(0)=0,$$
$$D_{0+}^{\alpha}u(1)=bD_{0+}^{\alpha}u(\eta).$$

$p(t)-$ Laplacian operator, as a generalization of $p-$ Laplacian operator, represents a nonhomogencity. Its nonlinearity is more complex. The problem involved $p(t)-$ Laplacian is seldom studied in the literature, mainly because it is more difficult than the problem involved $p-$ Laplacian, and the results about $p(t)-$ Laplacian operator equations are very few.

For example, the following problem

$$D_{0+}^{\beta}(\varphi_{p(\tau)}(D_{0+}^{\alpha}y(\tau)))+f(\tau,y(\tau))=0,\quad 0<\tau<1,$$
$$y'(0)=y(1)=y''(0)=0,\quad D_{0+}^{\alpha}y(0)=0.$$

was discussed by Shen et al [16].

The question

$${}^cD_{0+}^{\beta}(\varphi_{p(\tau)}(D_{0+}^{\alpha}v(\tau)))=f(\tau,v(\tau)),\quad 0<\tau<1,$$
$$v(0)=0,\ D_{0+}^{\alpha-1}v(0)=\gamma I_{0+}^{\delta}v(1),\ D_{0+}^{\alpha}v(0)=0,$$

here $1<\alpha\le 2,\ 0<\beta,\delta\le 1,\ \gamma>0$ was investigated by Tang et al [17]. The Caputo Derivative is represented by ${}^cD_{0+}^{\beta}$ and the Riemann-Liouville fractional derivative is represented by D_{0+}^{α}.

When the first derivative is included in the nonlinear term of the differential equation, the change of the selected Banach space and so on brings many difficulties to the problem under discussion. Therefore, when the first derivative is included in the nonlinear term of the differential equation, There are very little literature on it.

When the f in the equation contains the first-order derivative, there will be many difficulties. Therefore, few articles discuss this kind of problem. Taking for example

$$(\varphi_p(u'))'(t)+q(t)f(t,u(t),u'(t))=0,\quad 0<t<1,$$
$$u(0)-\beta u'(\xi)=0,\quad u(\xi)-\delta u'(\eta)=u(1)+\delta u'(1+\xi-\eta).$$

was studied by the authors in [19].

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As far as the author knows, no one has discussed the following problem

$$^cD^{\beta}_{0+}\varphi_{p(t)}(D^{\alpha}_{0+}u(t)) + q(t,u(t),u'(t)) = 0, \quad t \in [0,1], \tag{1}$$

$$u(0) = 0,\ u'(0) = 0,\ D^{\alpha-1}_{0+}u(1) = \lambda I^{\varepsilon}_{0+}u(\sigma),\ D^{\alpha}_{0+}u(0) = 0, \tag{2}$$

was discussed by us. Here $2 < \alpha \leq 3,\ 0 < \beta, \varepsilon \leq 1,\ \lambda > 0,\ 0 < \sigma < 1,\ q$ is a positive continuous function defined on an interval $[0,1]$. The Caputo Derivative is represented by $^cD^{\beta}_{0+}$, in this place $p(t) \in C'[0,1], p(t) > 1$.

II. Some Useful Basics

Definition 2.1 [20] For a function $k : (0,+\infty) \to R$,

$$I^{\beta}_{0+}k(\tau) = \frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau - s)^{\beta-1}k(s)ds,$$

represents the $\beta(\beta > 0)$ order fractional integration.

Definition 2.2 [20]For a function $k : (0,+\infty) \to R$,

$$^cD^{\beta}_{0+}k(\tau) = \frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau - s)^{n-\beta-1}k^n(s)ds,$$

represents the $\beta(\beta > 0)$ order fractional derivative in Caputo form, in this place $n = [\beta] + 1$.

Definition 2.3 [20] For a function $k : (0,+\infty) \to R$,

$$D^{\beta}_{0+}k(\tau) = \frac{1}{\Gamma(n-\beta)}(\frac{d}{d\tau})^n\int_0^{\tau}(\tau - s)^{n-\beta-1}k(s)ds,$$

represents the $\beta(\beta > 0)$ order fractional derivative in Riemann-Liouville form, in this place $n = [\beta] + 1$.

Lemma 2.1 [6] Allow $n - 1 < \beta \leq n$, the functional expression

$$v(\tau) = c_0 + c_1\tau + c_2\tau^2 + \cdots + c_{n-1}\tau^{n-1},$$

in this place $c_i \in R,\ \ i = 0, 1, 2, \cdots, n-1, n = [\beta] + 1$ is the solution of an equation

$$^cD^{\beta}_{0+}v(\tau) = 0.$$

Lemma 2.2 [6] Allow $n - 1 < \beta \leq n$, suppose $^cD^{\beta}_{0+}v(\tau) \in C[0,1]$, so that

$$I^{\beta}_{0+}\,{}^cD^{\beta}_{0+}v(\tau) = v(\tau) + c_0 + c_1\tau + c_2\tau^2 + \cdots + c_{n-1}\tau^{n-1},$$

in this place $c_i \in R,\ \ i = 0, 1, 2, \cdots, n-1, n = [\beta] + 1$.

Lemma 2.3 [6] Allow $n - 1 < \beta \leq n$, the functional expression

$$v(\tau) = c_1\tau^{\beta-1} + c_2\tau^{\beta-2} + \cdots + c_n\tau^{\beta-n},$$

in this place $c_i \in R,\ \ i = 1, 2, \cdots, n, n = [\beta] + 1$ is the solution of an equation

$$D^{\beta}_{0+}v(\tau) = 0.$$

Lemma 2.4 [6] Allow $n-1 < \beta \leq n$, suppose $D^{\beta}_{0+}v(\tau) \in C[0,1]$, so that

$$I^{\beta}_{0+}D^{\beta}_{0+}v(\tau) = v(\tau) + c_1\tau^{\beta-1} + c_2\tau^{\beta-2} + \cdots + c_n\tau^{\beta-n},$$

in this place $c_i \in R,\ \ i = 1, 2, \cdots, n-1, n = [\beta] + 1$.

Lemma 2.5 [21] In regard to $(\tau, z) \in [0,1] \times R$,when we fix $\tau \in [0,1]$, $\varphi_{p(\tau)}(z) = |z|^{p(\tau)-2}z$ is strictly incremented. In addition,

$$\varphi^{-1}_{p(\tau)}(z) = |z|^{\frac{2-p(\tau)}{p(\tau)-1}}z, \quad z \in R\backslash\{0\},$$

$$\varphi^{-1}_{p(\tau)}(0) = 0, \qquad z = 0,$$

represents the invertible operator of $\varphi_{p(\tau)}(z) = |z|^{p(\tau)-2}z$. Lets write it down as $\varphi^{-1}_{p(\tau)}(\cdot)$

Proposition 2.6 [20] Suppose that $\beta \geq 0, \nu > -1$, then

$$D^{\beta}_{0+}t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\beta+1)}t^{\nu-\beta}$$

holds.

Proposition 2.7 [20] Suppose that $\beta \geq 0, \nu > -1, \nu \neq \beta - j, j = 1, 2, \cdots, [\beta] + 1$, so

$$I^{\beta}_{0+}t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\beta+1)}t^{\nu+\beta}$$

holds.

Lemma 2.8 Allow $\lambda\sigma^{\alpha+\varepsilon-1} < \Gamma(\alpha+\varepsilon)$ and $h(t)$ is a continuous function, the functional expression

$$u(t) = \int_0^1 G(t,s)\varphi^{-1}_{p(s)}(I^{\beta}_{0+}h(s))ds, \tag{3}$$

is the solution of

$$^cD^{\beta}_{0+}\varphi_{p(t)}(D^{\alpha}_{0+}u(t)) + h(t) = 0, \quad t \in [0,1], \tag{4}$$

$$u(0) = 0,\ u'(0) = 0,\ D^{\alpha-1}_{0+}u(1) = \lambda I^{\varepsilon}_{0+}u(\sigma),\ D^{\alpha}_{0+}u(0) = 0, \tag{5}$$

in this place

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}\Gamma(\alpha+\varepsilon) - \lambda t^{\alpha-1}(\sigma - s)^{\alpha+\varepsilon-1}}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})} \\ -\dfrac{(t-s)^{\alpha-1}(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})}, \\ \qquad 0 \leq s \leq t \leq 1, \quad s \leq \sigma, \\ \dfrac{t^{\alpha-1}\Gamma(\alpha+\varepsilon) - \lambda t^{\alpha-1}(\sigma - s)^{\alpha+\varepsilon-1}}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})}, \\ \qquad 0 \leq t \leq s \leq \sigma \leq 1, \\ \dfrac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})} \\ -\dfrac{(t-s)^{\alpha-1}(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})}, \\ \qquad 0 \leq \sigma \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon) - \lambda\sigma^{\alpha+\varepsilon-1})}, \\ \qquad 0 \leq t \leq s \leq 1, \sigma \leq s. \end{cases} \tag{6}$$

Proof: By (4) and in view of the content of Lemma 2.2, the following relationship holds:

$$\varphi_{p(t)}(D^{\alpha}_{0+}u(t)) = -I^{\beta}_{0+}h(t) + c. \tag{7}$$

In view of boundary conditions $D^{\alpha}_{0+}u(0) = 0$, (7) takes the form

$$\varphi_{p(t)}(D^{\alpha}_{0+}u(t)) = -I^{\beta}_{0+}h(t),$$

this equation is equivalent to

$$D^{\alpha}_{0+}u(t) = -\varphi^{-1}_{p(t)}(I^{\beta}_{0+}h(t)). \tag{8}$$

In view of (8) and Lemma 2.4, one has

$$\begin{aligned} u(t) &= -\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}\varphi^{-1}_{p(s)}(I^{\beta}_{0+}h(s))ds \\ &+c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3}. \end{aligned} \tag{9}$$

Well, considering $u(0)=0,\ u'(0)=0$, one has $c_2=c_3=0$, thus, (9) goes like this

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds+c_1t^{\alpha-1}. \tag{10}$$

Applying the Propositions 2.6 and 2.7, we obtain

$$D_{0^+}^{\alpha-1}u(t)=-\int_0^t\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds+c_1\Gamma(\alpha)$$

and

$$\begin{aligned}I_{0^+}^{\varepsilon}u(t)\ &=-\frac{1}{\Gamma(\alpha+\varepsilon)}\int_0^t(t-s)^{\alpha+\varepsilon-1}\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds\\&+c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\varepsilon)}t^{\alpha+\varepsilon-1},\end{aligned}$$

based on the boundary condition $D_{0^+}^{\alpha-1}u(1)=\lambda I_{0^+}^{\varepsilon}u(\sigma)$, we get

$$\begin{aligned}&-\textstyle\int_0^1\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds+c_1\Gamma(\alpha)=\\&\frac{-\lambda}{\Gamma(\alpha+\varepsilon)}\int_0^{\sigma}(\sigma-s)^{\alpha+\varepsilon-1}\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds\\&+c_1\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\varepsilon)}\sigma^{\alpha+\varepsilon-1},\end{aligned}$$

it further follows that

$$\begin{aligned}c_1\ &=\frac{\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\Bigg(\int_0^1\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))\\&ds-\frac{\lambda}{\Gamma(\alpha+\varepsilon)}\int_0^{\sigma}(\sigma-s)^{\alpha+\varepsilon-1}\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds\Bigg).\end{aligned}$$

Put c_1 into the upper form

$$\begin{aligned}&u(t)=-\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds\\&+\frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\Bigg(\int_0^1\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds\\&-\frac{\lambda}{\Gamma(\alpha+\varepsilon)}\int_0^{\sigma}(\sigma-s)^{\alpha+\varepsilon-1}\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}h(s))ds\Bigg).\end{aligned}$$

This implies that (3) stands. ■

Lemma 2.9 Suppose $0<\lambda\sigma^{\alpha+\varepsilon-1}<\Gamma(\alpha+\varepsilon)$. The properties of the functions defined by formula (6) are like following:

(i) $0<G(t,s)<\dfrac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\ \forall s,t\in(0,1)$;

(ii) $\dfrac{\partial G(t,s)}{\partial t}<\dfrac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\ \forall s,t\in(0,1)$.

Proof: (i) The expression with respect to the upper form $G(t,s)$,

$$G(t,s)<\frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\ \forall s,t\in(0,1).$$

Moreover, if $0\le s\le t\le 1,\quad s\le\sigma$, denote

$$\begin{aligned}k(t,s)\ &=t^{\alpha-1}\Gamma(\alpha+\varepsilon)-\lambda t^{\alpha-1}(\sigma-s)^{\alpha+\varepsilon-1}\\&-(t-s)^{\alpha-1}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1}).\end{aligned}$$

We can see that

$$\begin{aligned}k(t,s)\ &\ge t^{\alpha-1}\Gamma(\alpha+\varepsilon)-\lambda t^{\alpha-1}\sigma^{\alpha+\varepsilon-1}\\&-(t-s)^{\alpha-1}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})\\&=t^{\alpha-1}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})\\&-(t-s)^{\alpha-1}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})\\&>(t-s)^{\alpha-1}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})\\&-(t-s)^{\alpha-1}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})=0,\end{aligned}$$

from this relation, we can deduce that $G(t,s)$ is a positive function. In other cases, the same is true.

(ii)

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}\dfrac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\-\dfrac{\lambda(\alpha-1)t^{\alpha-2}(\sigma-s)^{\alpha+\varepsilon-1}}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\-\dfrac{(\alpha-1)(t-s)^{\alpha-2}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\\\qquad 0\le s\le t\le 1,\quad s\le\sigma,\\\dfrac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\-\dfrac{\lambda(\alpha-1)t^{\alpha-2}(\sigma-s)^{\alpha+\varepsilon-1}}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\\\qquad 0\le t\le s\le\sigma\le 1,\\\dfrac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\\-\dfrac{(\alpha-1)(t-s)^{\alpha-2}(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\\qquad 0\le\sigma\le s\le t\le 1,\\\dfrac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\\\qquad 0\le t\le s\le 1,\sigma\le s.\end{cases}$$

From the expression of $\frac{\partial G(t,s)}{\partial t}$, we can easily derive

$$\frac{\partial G(t,s)}{\partial t}<\frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\ \forall s,t\in(0,1).$$

■

III. Theorems

The Banach space $C'[0,1]$ is recorded as X. The norm is defined as follows

$$\|\omega\|=\max\{\max_{0\le\tau\le1}|\omega(\tau)|,\max_{0\le\tau\le1}|\omega'(\tau)|\}. \tag{11}$$

We give the expression of cone $P\subset X$ by

$$P=\{u\in X:\ u(t)\ge0,\quad 0\le t\le1\}, \tag{12}$$

The operator expression looks like

$$Su(t)=\int_0^1G(t,s)\varphi_{p(s)}^{-1}(I_{0^+}^{\beta}q(s,u(s),u'(s)))ds. \tag{13}$$

The fixed point of the integral transform S happens to satisfy the equation (1) and the boundary condition(2).

Lemma 3.1 The operator $S:\ P\to P$ is continuous and it is compact.

Proof: The expression $G(t,s),\ \varphi_{p(t)}^{-1}(\cdot)$ and q are continuous which directly causes the operator S to be continuous. We choose an arbitrarily bounded open subset Ω from P. We can choose $M>0$ to make the following relation hold true

$$|\varphi_{p(t)}^{-1}(I_{0^+}^{\beta}q(t,u(t),u'(t)))|\le M$$

because of the continuity of $\varphi_{p(t)}^{-1}(\cdot)$ and q. The following relationship hold for $u \in \Omega$,

$$\begin{aligned}&|(Su)(t)|\\&\le \int_0^1 G(t,s)|\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))|ds\\&\le \frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\int_0^1 Mds\\&\le \frac{\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\int_0^1 Mds\\&= \frac{M\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})},\end{aligned}$$

$$\begin{aligned}&|(Su)'(t)|\\&\le \int_0^1 \tfrac{\partial G(t,s)}{\partial t}|\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))|ds\\&\le \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\int_0^1 Mds\\&\le \frac{(\alpha-1)\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\int_0^1 Mds\\&= \frac{M(\alpha-1)\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}.\end{aligned}$$

So we get the boundedness of $S(\Omega)$. Take a function $u \in \Omega$ and two points t_1, t_2 on $[0,1]$ and it requires $t_1 < t_2$, let's do the calculation

$$\begin{aligned}&|Su(t_2)-Su(t_1)|\\&\le \Bigg|-\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\\&+\tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\Bigg|\\&+\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\&\Bigg|\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\\&-\frac{\lambda}{\Gamma(\alpha+\varepsilon)}\int_0^{\sigma}(\sigma-s)^{\alpha+\varepsilon-1}\\&\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\Bigg|\\&\le \tfrac{M}{\Gamma(\alpha)}\left(\tfrac{|t_1^{\alpha}-t_2^{\alpha}|}{\alpha}\right)\\&+\frac{\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\&\left(M+\frac{\lambda M\sigma^{\alpha+\varepsilon}}{\Gamma(\alpha+\varepsilon+1)}\right)|t_2^{\alpha-1}-t_1^{\alpha-1}|.\end{aligned}$$

Let's continue with the calculations

$$\begin{aligned}&(Su)'(t)\\&=-\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\\&\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))|ds\\&+\frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\&\Bigg(\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\\&-\frac{\lambda}{\Gamma(\alpha+\varepsilon)}\int_0^{\sigma}(\sigma-s)^{\alpha+\varepsilon-1}\\&\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\Bigg),\end{aligned}$$

so,

$$\begin{aligned}&|(Su)'(t_2)-(Su)'(t_1)|\\&\le \Bigg|-\tfrac{1}{\Gamma(\alpha-1)}\int_0^{t_2}(t_2-s)^{\alpha-2}\\&\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))|ds\\&+\tfrac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}\\&\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))|ds\Bigg|\\&+\frac{|t_2^{\alpha-2}-t_1^{\alpha-2}|(\alpha-1)\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\&\Bigg|\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\\&-\frac{\lambda}{\Gamma(\alpha+\varepsilon)}\int_0^{\sigma}(\sigma-s)^{\alpha+\varepsilon-1}\\&\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds\Bigg|\\&\le \frac{M}{\Gamma(\alpha-1)}\left(\frac{|t_1^{\alpha-1}-t_2^{\alpha-1}|}{\alpha-1}\right)\\&+\frac{(\alpha-1)\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}\\&\left(M+\frac{\lambda M\sigma^{\alpha+\varepsilon}}{\Gamma(\alpha+\varepsilon+1)}\right)|t_2^{\alpha-2}-t_1^{\alpha-2}|.\end{aligned}$$

We get results $\|(Su)(t_2)-(Su)(t_1)\| \to 0$ when $t_1 \to t_2$, $u \in \Omega$. The conclusion which $S: P \to P$ is continuous and compact can be deduced from the Arzela-Ascoli theorem. ■

Theorem 3.2 Suppose we can find a positive number c

(H_1) $q(t,m_1,n_1) \le q(t,m_2,n_2)$ for any $0 \le t \le 1$, $0 \le m_1 \le m_2 \le (\alpha-1)c$, $0 \le |n_1| \le |n_2| \le (\alpha-1)c$;

(H_2) $\int_0^1 \left(\max\limits_{0\le\tau\le1} q(\tau,(\alpha-1)c,(\alpha-1)c)\right)^{\frac{1}{p(s)-1}} ds < \dfrac{c\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha+\varepsilon)}$;

(H_3) $q(t,0,0) \ne 0$ for $\forall\ 0 \le t \le 1$.

We can calculate that there is a positive function $\delta^* \in P$ satisfying $0 < \delta^* \le (\alpha-1)c$, $0 < |(\delta^*)'| \le (\alpha-1)c$ and $\lim\limits_{n\to\infty}\delta_n = \lim\limits_{n\to\infty}S^n\delta_0 = \delta^*$, $\lim\limits_{n\to\infty}(S^n\delta_0)' = (\delta^*)'$, in this place $\delta_0(t) = ct^{\alpha-1}$, $0 \le t \le 1$. And this positive function δ^* satisfying fractional order equations and integral boundary conditions (1), (2).

Proof: We are going to write

$$P_{(\alpha-1)c} = \{u \in P|\ \ \|u\| < (\alpha-1)c\},$$

thus

$$\overline{P_{(\alpha-1)c}} = \{u \in P|\ \ \|u\| \le (\alpha-1)c\}.$$

So lets try to figure out S mapping $\overline{P_{(\alpha-1)c}}$ to $\overline{P_{(\alpha-1)c}}$. Let $u \in \overline{P_{(\alpha-1)c}}$, We can see the following relationship

$$0 \le u(t) \le \max_{0\le t\le1}|u(t)| \le \|u\| \le (\alpha-1)c,$$

$$|u'(t)| \le \max_{0\le t\le1}|u'(t)| \le \|u\| \le (\alpha-1)c.$$

From the condition (H_1) and (H_2) and Lemma 2.9,

$$\begin{aligned}
&|(Su)(t)| \\
&= |\int_0^1 G(t,s)\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds| \\
&\leq \frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}q(\tau,u(\tau),u'(\tau))d\tau ds \\
&\leq \frac{\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}q(\tau,(\alpha-1)c,(\alpha-1)c)d\tau ds \\
&\leq \frac{\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta+1)}\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)ds \\
&= \frac{\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \left(\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)\right)^{\frac{1}{p(s)-1}} ds \\
&< \frac{\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\frac{c\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha+\varepsilon)} \\
&= c < (\alpha-1)c,
\end{aligned}$$

$$\begin{aligned}
&|(Su)'(t)| \\
&= |\int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,u(s),u'(s)))ds| \\
&\leq \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}q(\tau,u(\tau),u'(\tau))d\tau ds \\
&\leq \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}q(\tau,(\alpha-1)c,(\alpha-1)c)d\tau ds \\
&\leq \frac{(\alpha-1)\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta+1)}\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)ds \\
&= \frac{(\alpha-1)\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \left(\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)\right)^{\frac{1}{p(s)-1}} ds \\
&< \frac{(\alpha-1)\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\frac{c\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha+\varepsilon)} \\
&= (\alpha-1)c.
\end{aligned}$$

Thus, we have the following result

$$\|Su\| \leq (\alpha-1)c.$$

From this we can deduce $S:\ \overline{P_{(\alpha-1)c}} \to \overline{P_{(\alpha-1)c}}$. Choose

$$\delta_0(t) = ct^{\alpha-1},\ 0\leq t\leq 1.$$

We write $\delta_1 = S\delta_0$, then $\delta_1 \in \overline{P_{(\alpha-1)c}}$. Let

$$\delta_{n+1} = S\delta_n = S^{n+1}\delta_0,\ n = 0,1,2,3,\cdots, \tag{14}$$

$S:\ \overline{P_{(\alpha-1)c}} \to \overline{P_{(\alpha-1)c}}$ implies that

$$\delta_n \in S\overline{P_{(\alpha-1)c}} \subseteq \overline{P_{(\alpha-1)c}},\ n = 0,1,2,3,\cdots.$$

Since $S:\ P \to P$ is continuous and compact, the sequence $\{\delta_n\}_{n=0}^{\infty}$ is compact. Moreover,

$$\begin{aligned}
&\delta_1(t) = S\delta_0(t) \\
&= \int_0^1 G(t,s)\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,\delta_0(s),\delta_0'(s)))ds \\
&\leq \frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}q(\tau,\delta_0(\tau),\delta_0'(\tau)))d\tau ds \\
&\leq \frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}q(\tau,(\alpha-1)c,(\alpha-1)c)d\tau ds \\
&\leq \frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta+1)}\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)ds \\
&= \frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \left(\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)\right)^{\frac{1}{p(s)-1}} ds \\
&< \frac{t^{\alpha-1}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\frac{c\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha+\varepsilon)} \\
&= ct^{\alpha-1} = \delta_0(t),
\end{aligned}$$

$$\begin{aligned}
&|\delta_1'(t)| = |(S\delta_0)'(t)| \\
&\leq |\int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}q(s,\delta_0(s),\delta_0'(s)))ds| \\
&\leq \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}q(\tau,\delta_0(\tau),\delta_0'(\tau)))d\tau ds \\
&\leq \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} \\
&q(\tau,(\alpha-1)c,(\alpha-1)c)d\tau ds \\
&\leq \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \varphi_{p(s)}^{-1}\frac{1}{\Gamma(\beta+1)}\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)ds \\
&= \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\int_0^1 \left(\max_{0\leq\tau\leq 1} q(\tau,(\alpha-1)c,(\alpha-1)c)\right)^{\frac{1}{p(s)-1}} ds \\
&< \frac{(\alpha-1)t^{\alpha-2}\Gamma(\alpha+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})} \\
&\frac{c\Gamma(\alpha)\Gamma(\beta+1)(\Gamma(\alpha+\varepsilon)-\lambda\sigma^{\alpha+\varepsilon-1})}{\Gamma(\alpha+\varepsilon)} \\
&= (\alpha-1)ct^{\alpha-2} = |\delta_0'(t)|.
\end{aligned}$$

And that's where we can get the idea $\delta_0(t) \geq \delta_1(t),\ \ |\delta_0'(t)| \geq |\delta_1'(t)|,\ 0\leq t\leq 1$. Therefore,

$$\delta_1(t) = S\delta_0(t) \geq S\delta_1(t) = \delta_2(t), \quad 0\leq t\leq 1,$$

$$|\delta_1'(t)| = |(S\delta_0)'(t)| \geq |(S\delta_1)'(t)| = |\delta_2'(t)|, \quad 0\leq t\leq 1.$$

Using recursive thinking,

$$\begin{aligned}
&\delta_n(t) \geq \delta_{n+1}(t), \quad |\delta_n'(t)| \geq |\delta_{n+1}'(t)|, \quad 0\leq t\leq 1, \\
&(n = 0,1,2\cdots).
\end{aligned}$$

So, We can calculate that there is a positive function $\delta^* \in \overline{P_{(\alpha-1)c}}$ satisfying $\delta_n \to \delta^*$. In (14), let us take $n\to\infty$, the equation $T\delta^* = \delta^*$ holds. The fractional order equations

(1) and integral boundary condition (2) has not the zero solution because of the condition (H_3). Therefore, function δ^* which is positive satisfying fractional order equations (1) and integral boundary condition (2). ■

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